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L^2 POTENTIAL SPACES IN $\mathbb{R}^n$

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Abstract: Given an elliptic differential operator L of order 2 with constant coefficients and a domain Ω in $\mathbb{R}^n$. We obtain some sufficient conditions for the existence of solutions for $(-L)^i u \geq 0$ on Ω, $0 \leq i \leq 2$. When such solutions exist on Ω, we prove some interesting results, including the representation of the solutions h of $L(Lh) = 0$ outside a compact set in Ω, by means of certain special functions defined on the whole of Ω.

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1. Introduction

Let L be an elliptic differential operator of order 2 with constant coefficients defined on $\mathbb{R}^n$, $n \geq 1$. Let Ω be a domain in $\mathbb{R}^n$. The existence of positive solutions u in Ω for the inequality $Lu \leq 0$ is an important situation, leading to the proofs of many interesting results connected with the operator L.

Equally interesting is the search for solutions u for inequalities $u \geq 0, Lu \leq 0$ and $L(Lu) \geq 0$ on Ω. Such solutions form the basis of the study of biharmonic functions in Ω (when L is taken as the special operator Δ, the Laplacian), along with the Almansi representation and the Riquier solution. We obtain in this paper some sufficient conditions for the existence of such solutions u on Ω; also we give some necessary and sufficient conditions also.

The existence of such solutions u on Ω implies some very interesting properties of the solutions $L(Lu) \geq 0$ in Ω.

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2. Preliminaries

Let $L = \sum\limits_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum\limits_{i=1}^{n} b_i \frac{\partial}{\partial x_i}$ be an elliptic differentials operator of order 2 with constant coefficients in $\mathbb{R}^n$, $n \geq 1$. Let Ω be a domain in $\mathbb{R}^n$. Then the $C^{\infty}-$solutions of $Lu = 0$ in Ω satisfy locally the sheaf property, solve the local Dirichlet problem and have the Harnack property, commonly known as the Axioms 1, 2, and 3 of Brelot in the axiomatc potential theory (see Brelot [4, pp. 13-15 and p. 26]). Let us call such solutions u as $L-$harmoinc functions on Ω. Notice that the constants are $L-$harmoinc on Ω.

With the above remarks, it is clear that we are in a position to use the defintions and the results of the harmoinc function theory developed by Brelot [4] in the context of axiomatic potential theory.

Definition 2.1. A locally (Lebesgue) integrable function s on an open set ω in Ω is called an $L-$superharmoinc function, if there exists a Radon measure $\mu \geq 0$ on ω such that $Ls = -\mu$ in the sense of distributions; A locally integrable function t on ω is an $L-$subharmoinc function if and only if $(-t)$ is $L-$superharmoinc on ω.

Definition 2.2. An $L-$superharmoinc function $p > 0$ on an open set ω in Ω is called an $L-$potential, if for any $L-$subharmoinc function t on ω such that $t \leq p$, we have $t \leq 0$ on ω.

We shall base our further investigation in this paper on the following lemma (see Anandam [3], Theorem 2.1]) mentioned without proof.

Lemma 2.3. *Let $\mu \geq 0$ be a Radon measure defined on an open set ω in Ω. Then there exists a locally integrable function u on Ω such that $Lu = -\mu$ in the sense of distibutions.*

Notice that if $Lv = -\mu$ is another solution, then $L(u - v) = 0$ on ω. Hence by the theory of distributions, there exists a $C^{\infty}-$function h on ω such that $Lh = 0$ and $u - v = h$, a.e. Thus $u = v + h$ in the sense of distributions.

Since every locally integrable function f on an open set ω in Ω coresponds to a signed measure λ on ω given by $d\lambda(x) = [f^{+}(x)\,dx] - [f^{-}(x)\,dx]$, it is easy to see that there exists a locally integrable function u on ω such that $Lu = -f$. Note that by Definition 2.1, u is the difference of two superharmonic functions on ω; we say that u is a $\delta-$superharmonic function generated by f.

Now given any Radon measure $\mu \geq 0$ on Ω, there exists an $L-$superharmonic funtion u on Ω such that $Lu = -\mu$ on Ω. But it is not necessary that u should be positive on Ω. In fact, it may happen that Ω be such that for any

$\mu \geq 0$, $Lu = -\mu$, u is not lower bounded on Ω.

Definition 2.4. A domain Ω is called an $L-$potential space if and only if there exists $u > 0$ on ω such that $Lu = -\mu$, where $\mu \geq 0$ is a Radon measure on Ω ($\mu \not\equiv 0$).

It is shown in Hervé [5] that Ω is an $L-$poential domain if and only if it is a $\Delta-$potential domain (the Laplacian Δ is a special case of the operator L considered here). Hence any domain in $\mathbb{R}^n$, $n \geq 3$, is an $L-$potential domain and in $\mathbb{R}^n$ ($n = 1, 2$), Ω is an $L-$potential domain if and only if $\mathbb{R}^n \backslash \Omega$ is not locally polar (recall that e is locally polar in $\mathbb{R}^n$ if and only if there exists a $\Delta-$superharmonic function s on $\mathbb{R}^n$ such that $e \subset \{x : s(x) = \infty\}$).

3. L^2-Potential Spaces

Let us write (u, f) to denote a pair of locally integrable functions u and f on Ω, such that $Lu = -f$.

Definition 3.1. (u, h) is called an L^2-harmonic pair on an open set ω in Ω if h is $L-$harmonic on ω and $Lu = -h$ on ω. (v, s) is called an L^2-superharmonic pair if s is $L-$superharmonic on ω and $Lv = -s$.

Note since h is C^∞ and L is an ellipic operator, we can always take u as a $C^\infty-$function when (u, h) is L^2-harmonic.

Theorem 3.2. (Riquier Problem) *Let ω be a relatively compact domain in Ω. Suppose ω is regular for the Dirichlet problem with respect to the Laplacian operator Δ. Then given two finite continuous functions f and g on $\partial\omega$, there exists a unique L^2-harmonic function (u, h) on ω such that $u \to g$ on $\partial\omega$ and $h \to f$ on $\partial\omega$.*

Proof. Since ω is $\Delta-$regular, it is regular for the operator L also (Hervé [5]). Hence there exists an $L-$harmonic function h on ω, such that $h \to f$ on $\partial\omega$. Let

$$h_1 = \begin{cases} h & \text{on} \quad \omega\,, \\ f & \text{on} \quad \partial\omega. \end{cases}$$

Then h, is continuous on $\overline{\omega}$. Let h_2 be a continuous function on Ω such that $h_2 = h_1$ on $\overline{\omega}$.

Since h_2 is locally integrable, by Lemma 2.3, there exists a locally integrable function v on Ω such that $Lv = -h_2$ on Ω. Since h_2 is continuous, v is continuous on Ω. Let v_1 be $L-$harmonic on ω such that $v_1 \to v - g$ on $\partial\omega$. Write

$u = v - v_1$. Then $u \to g$ on $\partial\omega$ and $Lu = Lv = -h_2 = -h$ on ω.

Consequently, (u, h) is an L^2-harmonic function on ω such that $u \to g$ on $\partial\omega$ and $h \to f$ on $\partial\omega$.

For the uniqueness, suppose (s, t) is an L^2-harmonic function on ω such that $s \to g$ on $\partial\omega$ and $t \to f$ on $\partial\omega$.

Then $h - t$ is $L-$harmonic on ω and tends to 0 on $\partial\omega$. Hence by the maximum principle for $L-$harmonic function s, $h - t \equiv 0$ on ω. Hence $L(u - s) = 0$ on ω, which means that $u - s$ is $L-$harmonic on ω and also has the limit 0 on $\partial\omega$. Again by the maximum principle, $u - s \equiv 0$. Consequently, $(u, h) = (s, t)$.

The theorem is proved. □

Notation. We shall write $(u, f) \geq 0$ if $u \geq 0$ and $f \geq 0$.

Definition 3.3. (q, p) is called an L^2-potential on an open set ω in Ω if p and q are $L-$potential > 0 on ω, $Lq = -p$. Ω is called an L^2-potential space if there exists an L^2-potential $(q, p) > 0$ on Ω.

Theorem 3.4. *Let Ω be an L^2-potential space. Then, given any $y \in \Omega$, there exists a unique potential $Q_y(x)$ on Ω such that $L^2 Q_y(x) = \delta_y(x)$, and $(Q_y, -LQ_y)$ is an L^2-potential on Ω.*

Proof. Any L^2-potential space is necessarily an $L-$potential space. Hence given $y \in \Omega$, we can construct the $L-$Green potential $Q_y(x) > 0$ on Ω such that $LQ_y(x) = -\delta_y(x)$.

Since Ω is an L^2-potential space, there exists an L^2-potential $(q, p) > 0$ on Ω (by definition). Remark that if k is a compact set in Ω, containing y in its interior $\mathring{k}$, then $Q_y(x) \leq \alpha p(x)$ for $x \in \Omega \backslash k$ and for a suitable constant $\alpha > 0$.

Let now $Ls(x) = G_y(x)$, where $s(x)$ is an $L-$superharmonic function (Lemma 2.3). Note

$$L[s(x) - \alpha q(x)] = -Q_y(x) - \alpha(-p(x)) = -[G_y(x) - \alpha p(x)] \geq 0$$

for $x \in \Omega \backslash k$. This means that $s(x) - \alpha q(x) = t(x)$ for $x \in \Omega \backslash k$, where t is an $L-$subharmonic function on $\Omega \backslash k$. But then, we can find an $L-$subharmonic function $f(x)$ on Ω such that $|t(x) - f(x)|$ is bounded outside a compact set in Ω (see Anandam [2]) consequently, we can choose a constant β such that on $\Omega \backslash k$, $s(x) - \alpha q(x) = t(x) \geq f(x) - \beta$. Since $q(x) > 0$, we conclude that $s(x) > f(x) - \beta$ on $\Omega \backslash k$. Thus, $s(x)$ is an $L-$subharmonic function on Ω, majorizing an $L-$subharmonicfunction outside a compact set. Hence by Anandam [3], $s(x)$ has an $L-$harmonic minorant on Ω. write $s(x) = Q_y(x) + h(x)$ on Ω, where $Q_y(x)$ is an $L-$potential on Ω and $h(x)$ is an $L-$harmonic function on Ω.

Remark now that $LQ_y(x) = Ls(x) = -G_y(x)$ on Ω. Since $LG_y(x) = -\delta_y(x)$, we have $L^2Q_y(x) = \delta_y(x)$ on Ω; also $(Q_y, -LQ_y) = (Q_y, G_y)$ is an L^2-potential on Ω.

As for the uniqueness, suppose $L^2u(x) = \delta_y(x)$ on Ω for an $L-$potential $u(x)$, and $(u(x), -Lu(x))$ is an L^2-potential on Ω. Then $L(-Lu(x)) = -\delta_y(x) = L(G_y(x))$ on Ω. Hence, $-Lu(x) = G_y(x) + h(x)$, where $h(x)$ is an $L-$harmonic function on Ω. Since $-Lu(x)$ and $G_y(x)$ are $L-$potentials on Ω, by the uniqueness of Riesz representation, we should have $h \equiv 0$.

Hence $-Lu(x) = G_y(x) = -LQ_y(x)$ on Ω, so that $u(x) = Q_y(x) +$ (an $L-$ harmonic funtion) on Ω. Since $u(x)$ and $Q_y(x)$ are $L-$potentials, again by the uniqueness of representation, we have $u(x) = Q_y(x)$. Thus the uniqueness of $Q_y(x)$ is proved. □

It is clear that Ω should be an $L-$potential space, if it is to be an L^2-potential space. But it is not a sufficient condition for Ω to be an L^2-potential space. To obtain such a sufficient condition, we intoduce the following definition.

Definition 3.5. A pair $(q, h) > 0$ in Ω is called an $L-$harmonic potential if q is an $L-$potential on Ω and $Lq = -h$ is an $L-$harmonic function on Ω.

Theorem 3.6. *Let Ω be a domain on which an $L-$harmonic potential $(q, h) > 0$ exists. Then Ω is an L^2-potential space.*

Proof. Clearly Ω is an $L-$potential space. Let p an $L-$potential on an L^2-potential space Ω. Choose a compact set k with nonempty interior. Then choose a constant λ such that $h \leq \lambda p$ on k.

Let $\mathcal{H}$ be the family of all positive $L-$superharmonic functions s on Ω such that $h \leq s$ on k. Define $u(x) = \inf_{s\in\mathcal{H}} s(x)$. $u(x)$ may not be lower semi continuous on Ω. Define therefore $\widehat{u} = \lim_{y\to x} u(y)$ for every $x \in \Omega$. Then $\widehat{u}(x)$ is semicontinuous and it is easy to see that $\widehat{u}$ is an $L-$superharmonic function > 0. By definition, $\widehat{u} \leq \lambda p$ on Ω. Since a positive $L-$superharmonic function majorized by an $L-$potential is itself an $L-$potential, we conclude that $\widehat{u}$ is an $L-$potential on Ω.

Now, let $Lv = -\widehat{u}$ on Ω. Then by Lemma 2.3, v is an $L-$superharmonic function. Let $Lv_1 = -(h - \widehat{u})$. Since $h - \widehat{u} \geq 0$, v_1 also is an $L-$superharmonic function and $L(v + v_1) = -h = Lq$ because $(q, h) > 0$ is an $L-$harmonic potential on Ω.

Hence $v + v_1 = q +$ (an $L-$ harmnic function H) on Ω. Since $q > 0$, this implies that $v > -v_1 + H$ on Ω. Since $-v_1 + H$ is $L-$subharmonic function on Ω, by the Riesz decomposition, $v =$ (an $L -$ potential Q)+(an $L -$ harmonic function H_1) on Ω.

Clearly, $LQ = Lv = -\widehat{u}$ on Ω.

Since $\widehat{u}$ and Q are $L-$potentials and $LQ = \widehat{u}$, we have $(Q, \widehat{u})$ as an L^2-potential on Ω. This means that Ω is an L^2-potential space. □

Theorem 3.7. *Suppose there exists an $L-$superharmonic function $(s, t) > 0$ on Ω. Then Ω is an L^2-potential spase.*

Proof. First note that s and t are two positive $L-$superharmonic functions on Ω. Write $t = p + h$, where p is an $L-$potential and h is an $L-$harmonic function on Ω.

a) suppose $p = 0$. Then write $s =$(an $L-$potential q) + (an $L-$harmonic function) on Ω. Clearly $Lq = Ls = -t = -h$, so that $(q, h) > 0$ is an $L-$harmonic potential on Ω. Then by Theorem 3.5, Ω is an L^2-potential space.

b) Suppose $p > 0$. Let $Lu = -p$, $Lv = -h$ on Ω. Then u and v are $L-$superharmonic functions on Ω. Moreover, $Ls = -t = -(p + h) = L(u + v)$. Hence $u + v = s+$(an $L-$harmonic function H) on Ω. This means that $u > -v + H$ since $s > 0$. But $-v + H$ is an $L-$subharmonic function, so that u majorizes an $L-$subharmonic function on Ω. Hence $u =$(an $L-$potential Q)+(an $L-$harmonic function) on Ω, so ther $LQ = Lu = -p$. Since Q and p are $L-$potentials on Ω, $(Q, p) > 0$ is an L^2-potential on Ω. Hence, Ω is an L^2-potintial space. This proves the theorem. □

Now, we shall give a necessary and sufficient condition for Ω to be an L^2-potential space; that is, on Ω there exist functions u satsfying the conditions $(-L)^i u \geq 0$, $i = 0, 1, 2$. For Ω to be an L^2-potential space, it is essential that Ω should be an $L-$potential space which implies the existence on Ω of the Green function $G(x, y)$ associated with L; in particular, for any y in Ω, if $G_y(x) = G(x, y)$ then $L[G_y(x)] = -\delta_y(x)$.

Recall thar, in an $L-$potential space Ω, when we write $Lu = -\mu$ and $u > 0$, it means that $u(x) = \int_\Omega G(x, y)\, d\mu(y) + h(x)$, where $h(x)$ is an $L-$harmonic funcion on Ω; and $h = 0$ if and only if u is an $L-$potential. Remark that u should be finite q.e. (that is, u is finite except possibly on a locally polar set associated with the Laplacian Δ).

Similarly $Lu = -v$ and $v \geq 0$ means that $u(x) = \int_\Omega G(x, y)\, v(y)\, dy + h(x)$.

Theorem 3.8. *Ω is an L^2-potential space if and only if*

$$\iint_{\Omega\times\Omega} G(x,z)\,G(z,y)\,dz d\lambda(y)$$

is finite at some point x in Ω, and for some nonzero Radon measure $\lambda \geq 0$.

Proof. a) Let Ω be an L^2-potential space. Then by Definition 3.3, there exist $L-$potentials p and q on Ω such that $Lq(x) = -p(x)$. Hence

$$q(x) = \int_\Omega G(x,z)\,p(z)\,dz = \int_\Omega G(x,z)\left[\int_\Omega G(z,y)\,d\lambda(y)\right] dz$$
$$\text{(for some Radon measure } \lambda \geq 0,\ \lambda \not\equiv 0).$$

Since $q(x)$ is an $L-$potential on Ω, it is finite q.e. Hence, by Fubini-Tonelli Theorem

$$\iint_{\Omega\times\Omega} G(x,z)\,G(z,y)\,dz d\lambda(y) < \infty$$

for $x \in \Omega\backslash e$, where e is a locally polar set.

b) Conversely suppose $u(x) = \iint_{\Omega\times\Omega} G(x,z)\,G(z,y)\,dz d\lambda(y) < \infty$ for some $x \in \Omega$ and some nonzero Radon measure $\lambda \geq 0$. Then $p(z) = \int_\Omega G(z,y)\,d\lambda(y) \not\equiv \infty$; this means that $p(z)$ is an $L-$potential on Ω. Again

$$u(x) = \int_\Omega G(x,z)\,p(z)\,dz \not\equiv \infty.$$

This means that $u(x)$ is an $L-$potential and $Lu(x) = -p(x)$. Consequently, (u,p) is an L^2-potential on Ω. This proves the theorem. □

We conclude by pointing out how the existence of an L^2-potential on Ω leads to a global representation of an L^2-harmonic function defined outside a compact set in Ω.

Theorem 3.9. *Let (u,h) be an L^2-harmonic function defined outside a compact set in Ω. Then outside a compact set, $u = q_1 - q_2 + p_1 - p_2 + v$, where $(q_1, (-L)\,q_1)$ and $(q_2, (-L)\,q_2)$ are L^2-potentials on Ω; p_1 and p_2 are $L-$potentials on Ω, harmonic outside a compact set; and $(v, (-L)\,v)$ is an L^2-harmonic function on Ω.*

Proof. Since h is $L-$harmonic outside a compact set in Ω, $h = s - t + H$ outside a compact set, where s and t are $L-$potentials with compact harmonic support on Ω (that is, harmonic outside a compact set) and H is a uniquely fixed $L-$harmonic function on Ω (see Anandam [1, Théorème 3]).

Write $Lq_1 = -s_k$ and $Lq_2 = -t$. Since s and t have compact harmonic support, we can assume that q_1 and q_2 are $L-$potentials (as in the proof of Theorem 3.4). Let $LH_1 = -H$ on Ω. Then $(q_1, (-L)\, q_1)$ and $(q_2, (-L)\, q_2)$ are L^2-potentials on Ω and $(H_1, (-L)\, H_1)$ is an L^2-harmonic function on Ω.

Moreover,

$$\begin{aligned} L\,(q_1 - q_2 + H_1) &= -\,(s - t + H) \qquad \text{on } \Omega \\ &= -h \qquad \text{outside a compact set in } \Omega \\ &= Lu \qquad \text{outside a compact set in } \Omega. \end{aligned}$$

Consequently, $u = (q_1 - q_2 + H_1) + h_1$ outside a compact set, where h_1 is an $L-$harmonic function outside a compact set. Use again [1, Théorème 3] to write $h_1 = p_1 - p_2 + h_2$, where p_1 and p_2 are $L-$potentials with compact harmonic support on Ω and h_2 is $L-$harmonic on Ω.

Write now $v = H_1 + h_2$, so that $(v, (-L)\, v)$ is L^2-harmonic on Ω. Then we have also, $u = q_1 - q_2 + p_1 - p_2 + v$ outside a compact set in Ω. The theorem is proved. □

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