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APPROXIMATE SOLUTIONS OF A CLASS OF NONLINEAR DIFFERENTIAL EQUATIONS BY USING DIFFERENTIAL TRANSFORMATION METHOD

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Abstract: This work presents numerical solutions to a class of nonlinear differential equations by applying differential transformation method. To illustrate the capabilities of the presented method, examples are carried out and numerical results with comparison to exact solutions are given.

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1. Introduction

Many problems in applied science and engineering are modelled by ordinary differential equations. Classical methods for obtaining an explicit formula for the solution are limited to certain types of equations. Particularly, in nonlinear differential equations, obtaining such solutions becomes difficult and sometimes it is impossible and therefore one has to resort to numerical methods.

The technique that we used is the differential transformation method, which is based on Taylor series expansion. The concept of differential transformation was first proposed by Zhou [18] in 1986, and it was applied to solve linear and

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nonlinear initial value problems in electric circuit analysis. This method was first applied to eigenvalue problems by Chen and Ho [7]. Since then, researchers started applying this differential transformation method to solve many engineering problems [9], [15], [8], [13], [17], [14], [5], [6], [1], [10], [11], [12], [16], [2], [3], [4].

In this work, differential transformation method is applied to a class of initial value problems which are represented by second order ordinary differential equations of the following form:

$$y'' + a(x)y' + b(x)y = f(y), \tag{1.1}$$

$$y(0) = \alpha, \quad y'(0) = \beta. \tag{1.2}$$

Here $f(y)$ is a nonlinear operator and, $a(x)$ and $b(x)$ are known functions in the underlying function space. We gave two numerical examples to show how accurate and fast the results can be obtained. In Example 3.1, we obtained closed form exact series solution and in Example 3.2, we gave a high order series solution.

2. Differential Transformation Method

The differential transformation of the k-th derivative of the function $f(x)$ in one variable is defined as follows:

$$F(k) = \frac{1}{k!}\left[\frac{d^k f(x)}{dx^k}\right]_{x=x_0} \tag{2.1}$$

and the differential inverse transformation of $F(k)$ is defined as follows:

$$f(x) = \sum_{k=0}^{\infty} F(k)(x-x_0)^k. \tag{2.2}$$

In real applications, the function $f(x)$ is expressed by a finite series and (2.2) can be written as

$$f(x) = \sum_{k=0}^{n} F(k)(x-x_0)^k. \tag{2.3}$$

The following theorems that can be deduced from equation (2.1) and equation (2.2) are given below.

Theorem 1. *If $f(x) = g(x) \pm h(x)$, then $F(k) = G(k) \pm H(k)$.*

Theorem 2. *If $f(x) = cg(x)$, then $F(k) = cG(k)$, where c is a constant.*

Theorem 3. *If* $f(x) = \frac{d^n g(x)}{dx^n}$, *then* $F(k) = \frac{(k+n)!}{k!} G(k+n)$.

Theorem 4. *If* $f(x) = g(x)h(x)$, *then* $F(k) = \sum_{k_1=0}^{k} G(k_1)H(k-k_1)$.

Theorem 5. *If* $f(x) = x^n$, *then* $F(k) = \delta(k-n)$, *where*

$$\delta(k-n) = \begin{cases} 1 & k = n, \\ 0 & k \neq n. \end{cases}$$

Theorem 6. *If* $f(x) = g_1(x)g_2(x)\dots g_{n-1}(x)g_n(x)$, *then*

$$F(k) = \sum_{k_{n-1}=0}^{k} \sum_{k_{n-2}=0}^{k_{n-1}} \dots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} G_1(k_1)G_2(k_2-k_1)\dots G_{n-1}(k_{n-1}-k_{n-2})G_n(k-k_{n-1}).$$

3. Numerical Examples

Differential transformation method, described in Section 2, is applied to some special cases of the class of nonlinear initial value problems given in (1.1) and (1.2).

Example 3.1. Let us consider the following nonlinear initial value problem

$$y'' + (1-x)y' - y = 2y^3, \tag{3.1}$$

$$y(0) = 1, \quad y'(0) = 1. \tag{3.2}$$

The closed form solution is

$$y = \frac{1}{1-x}. \tag{3.3}$$

Applying the above theorems to equation (3.1), we obtain the following recurrence relation

$$Y(k+2) = \frac{1}{(k+1)(k+2)} \times \left(\begin{array}{l} \sum_{k_1=0}^{k} \delta(k_1-1)(k-k_1+1)Y(k-k_1+1) + Y(k) \\ -(k+1)Y(k+1) + 2\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2} Y(k_1)Y(k_2-k_1)Y(k-k_2) \end{array} \right). \tag{3.4}$$

The initial conditions in equation (3.2) can be transformed at $x_0 = 0$ as

$$Y(0) = 1, \quad Y(1) = 1. \tag{3.5}$$

Utilizing the recurrence relation in equation (3.4) and the transformed boundary conditions in equation (3.5), $Y(k)$ for $k \geq 2$ are easily obtained and then using equation (2.3), the following series solution is evaluated up to $n = 10$

$$y(x) = 1 + x + x^2 + x^3 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + O(x^{11}). \tag{3.6}$$

In the limit case $n \to \infty$, one can observe that the series solution obtained by differential transformation method converges to the series expansion of the closed form solution (3.3).

Example 3.2. We will consider the following nonlinear initial value problem

$$y'' + y' = -y^3, \tag{3.7}$$

$$y(0) = 1, \quad y'(0) = 1. \tag{3.8}$$

The initial conditions in equation (3.8) can be transformed at $x_0 = 0$ as

$$Y(0) = 0 \text{ and } Y(1) = 1. \tag{3.9}$$

Using the above theorems, equation (3.7) transforms to

$$Y(k+2) = \frac{1}{(k+1)(k+2)} \times \left(-(k+1)Y(k+1) - \sum_{k_2=0}^{k} \sum_{k_1=0}^{k_2} Y(k_1)Y(k_2-k_1)Y(k-k_2) \right). \tag{3.10}$$

Using equations (3.9), (3.10) and (2.3), for $n = 16$, the following series solution is obtained:

$$\begin{aligned} y_{16}(x) = {} & 1.0000000 + 1.0000000x - 1.0000000x^2 - 0.1666667x^3 + 0.0416667x^4 \\ & + 0.2666667x^5 - 0.0152778x^6 - 0.1061508x^7 - 0.0137401x^8 + 0.0281718x^9 \\ & + 0.0248911x^{10} - 0.0119875x^{11} - 0.0106566x^{12} + 0.0007512x^{13} + 0.0051924x^{14} \\ & + 0.0012088x^{15} - 0.0020914x^{16} + O(x^{17}). \end{aligned} \tag{3.11}$$

Due to the length of expression, the value of $y_{36}(x)$and $y_{56}(x)$ is not given.

x	$n = 16$	$n = 36$	$n = 56$	*Mathematica* solution
0.1	1.0898401	1.0898401	1.0898401	1.0898401
0.2	1.1588163	1.1588163	1.1588163	1.1588163
0.3	1.2064509	1.2064509	1.2064509	1.2064509
0.4	1.2328945	1.2328945	1.2328945	1.2328945
0.5	1.2390538	1.2390538	1.2390538	1.2390538
0.6	1.2265936	1.2265936	1.2265936	1.2265936
0.7	1.1978214	1.1978205	1.1978205	1.1978205
0.8	1.1554882	1.1554821	1.1554821	1.1554821
0.9	1.1025580	1.1025279	1.1025279	1.1025279
1.0	1.0419779	1.0418806	1.0418809	1.0418809

Table 1: Comparison of numerical results with the *Mathematica* solution for different values of n

Comparison of numerical results with the numerical solution of the nonlinear problem given in (3.7) and (3.8) evaluated using *Mathematica* commands for $n = 16$, $n = 36$ and $n = 56$ is reported in Table 1. As one can see from Table 1, the results obtained with differential transform method for $n = 56$ are seven digits accurate.

4. Conclusion

In this study, differential transformation method is applied to a class of nonlinear differential equations of second order. Two nonlinear equations are solved and series solutions are obtained. In all the two examples, it is shown that differential transformation method is a reliable tool for the solution of a class of nonlinear differential equations.

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