International Journal of Pure and Applied Mathematics

Volume 30 No. 3 2006, 411-423

LONGTIME BEHAVIOUR OF SOLUTIONS OF A GENERALIZED CAHN-HILLIARD EQUATION WITH ORDER PARAMETER DEPENDENT MOBILITY

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Abstract: We consider a generalized Cahn-Hilliard equation based on constitutive equations derived by M. Gurtin and show the existence and uniqueness of solutions and also the existence of the finite-dimensional global attractor via the existence of an exponential attractor.

AMS Subject Classification: 35A05, 35B45
Key Words: generalized Cahn-Hilliard equation, existence and uniqueness of solutions, global attractor, exponential attractor

1. Introduction

Many dissipative nonlinear partial differential equations arising from mechanics and physics possess the global attractor, which is a compact and invariant set lying in the phase space, and which uniformly attracts the trajectories starting from bounded sets when time goes to infinity, and thus appears as a suitable object for the description of the asymptotic behaviour of the system. If the global attractor is of finite fractal or Hausdorff dimension, then the system has an asymptotic behaviour determined by a finite number of degrees of freedom. A remarkable improvement compared to the a priori infinite-dimensional dynamics. The most current method of estimating these dimensions is based on the Lyapunov exponents. Unfortunately, this method is not systematically applicable to all the equations and requires the semigroup to be Fréchet differentiable (see [10]). An alternative to this approach is to show that the semigroup

Received: June 29, 2006

possesses an exponential attractor. An exponential attractor is a compact and positively invariant set which contains the global attractor, has finite fractal dimension and attracts all the trajectories starting from bounded sets with a uniform exponential rate (see [6]).

The aim of this paper is to study the existence and the uniqueness of solutions and also the existence of the global attractor and exponential attarctors for a generalized Cahn-Hilliard equation, completing and extending, in the case of a polynomial nonlinearity, the results of [3] and [2], respectively.

The Cahn-Hilliard equation is central to materials science. It is a conservation law (in the sense that the average of the order parameter is conserved) and describes very important qualitative features of two-phase systems, namely the transport of atoms between unit cells (see [5] and the references therein). Several generalizations of this equation have been introduced by Gurtin in [7]; they are based on constitutive equations that take into account the work of the internal microforces, the anisotropy, and also the deformations of the material (see [8]).

This paper is organized as follows. In Section 2, we set the problem. Section 3 is devoted to the study of the uniqueness of solutions; the existence of solutioins has been proven in [3]. In Section 4, we prove the existence of the global attractor. Finally, in Section 5, we show the existence of an exponential attractor.

Throughout this paper, the same letter c (and sometimes c_i, $i = 0, 1, 2, ...$) denotes positive constants which may change from line to line.

2. Setting of the Problem

We set $\Omega = \prod_{i=1}^{n}]0, L_i[$, $L_i > 0$, $i = 1, ..., n$, $n = 2$ or 3, and consider the following system:

$$\begin{cases} \frac{\partial \rho}{\partial t} - a.\nabla \frac{\partial \rho}{\partial t} = \operatorname{div}(\kappa(\rho)\nabla\mu), \\ \mu - b.\nabla\mu = \beta\frac{\partial \rho}{\partial t} - \alpha\Delta\rho + f'(\rho), \\ \rho|_{t=0} = \rho_0, \\ \rho \text{ and } \mu \text{ are } \Omega - \text{periodic}; \end{cases} \tag{2.1}$$

where $\alpha, \beta > 0$, $a, b \in \mathbb{R}^n$, and ρ is the order parameter (corresponding to a density of atoms). The mobility κ is a function such that

$$\kappa \in \mathcal{C}^1(\mathbb{R}), \quad 0 < \kappa_0 \le \kappa(s) \le \kappa_1, \ \forall s \in \mathbb{R}, \quad |\kappa'(s)| \le C, \ \forall s \in \mathbb{R}. \tag{2.2}$$

For example, $\kappa(s) = 1 - \frac{s^2}{2}$ if $|s| \leq 1$ and $\frac{1}{4}[1 + e^{-2(s^2-1)}]$ if $|s| \geq 1$. The coarse-grain free energy is a nonlinear function which satisfies the following conditions:

$$f \in \mathcal{C}^2;\ \ f(s) \geq -c_0,\ \ c_0 \geq 0,\ \ \forall s \in \mathbb{R}, \tag{2.3}$$

$$|f'(s)| \leq c_1|s|^{2p+1} + c_2,\ \ c_1, c_2 \geq 0,\ \ \forall s \in \mathbb{R}, \tag{2.4}$$

$$|f''(s)| \leq c_3|s|^{2p} + c_4,\ \ c_3, c_4 \geq 0,\ \ \forall s \in \mathbb{R}, \tag{2.5}$$

where $p \in \mathbb{N}$ is arbitrary when $n = 2$ and $p = 0$, 1 or 2 when $n = 3$,

$$\forall \gamma \in \mathbb{R},\ \ \exists c_5 = c_5(\gamma) > 0\ \text{ and }\ c_6 = c_6(\gamma) \geq 0$$
$$\text{such that}\ \ (s-\gamma)f'(s) \geq c_5 f(s) - c_6,\ \ \forall s \in \mathbb{R}, \tag{2.6}$$

where c_5 and c_6 are bounded when γ is bounded (with $c_5 \geq \tilde{c}_5 > 0$),

$$f''(s) \geq -c_7,\ \ c_7 \geq 0,\ \ \forall s \in \mathbb{R}. \tag{2.7}$$

For instance, polynomials of degree $2p+2$ with strictly positive leading coefficients (and with a double-well structure, e.g. $f(s) = (s^2-1)^2$) satisfy (2.3)-(2.7).

For the mathematical setting of the problem, we denote by $\|.\|$ and $(.,.)$ the usual norm and scalar product in $L^2(\Omega)$ (and also in $L^2(\Omega)^n$). For each $\rho \in L^1(\Omega)$, $m(\rho)$ denotes the average of ρ, that is, $m(\rho) = \frac{1}{|\Omega|}\int_\Omega \rho(x)\,dx$. For a space X, we denote by $\dot{X}$ the space $\{q \in X, m(q) = 0\}$. We define by $Nq = -\Delta q$ the linear, self-adjoint, strictly positive operator with compact inverse N^{-1} on $\dot{H}^2_{per}(\Omega)$. We set $\overline{q} = q - m(q)$ and $\Omega_T = \Omega \times]0,T[$. There exists a constant $c_1 > 0$ such that $\|\overline{q}\| \leq c_1\|\nabla q\|$, $\forall q \in H^1_{per}(\Omega)$.

We introduce a weak formulation of the problem: Find $(\rho,\mu) : [0,T] \to H^1_{per}(\Omega) \times H^1_{per}(\Omega)$ ($T > 0$ given) such that $\rho(0) = \rho_0$, and for a.e. $t \in [0,T]$,

$$\frac{d}{dt}(\rho,q) + (\frac{\partial\rho}{\partial t}, a.\nabla q) = -(\kappa(\rho)\nabla\mu, \nabla q),\ \ \forall q \in H^1_{per}(\Omega), \tag{2.8}$$

$$(\mu,q) - (b.\nabla\mu, q) = \alpha(\nabla\rho,\nabla q) + (f'(\rho),q) + \beta(\frac{\partial\rho}{\partial t}, q),$$
$$\forall q \in H^1_{per}(\Omega). \tag{2.9}$$

We note that $(a.\nabla p, q) = -(p, a.\nabla q)$, $\forall p,q \in H^1_{per}(\Omega)$. We first take $q = 1$ in (2.8) and observe that the average of ρ is conserved, that is,

$$m(\rho(t)) = m(\rho_0),\ \ \forall t \geq 0. \tag{2.10}$$

We now take $q = 1$ in (2.9) and obtain

$$m(\mu) = m(f'(\rho)). \tag{2.11}$$

As in [3], we assume, due to thermodynamical considerations, that

$$\beta x^2 + (a+b).yx + \kappa(s)|y|^2 \geq c(x^2 + |y|^2), \quad c > 0, \qquad \forall x \in \mathbb{R},\ \forall y \in \mathbb{R}^n,\ \forall s \in \mathbb{R}. \tag{2.12}$$

3. Existence and Uniqueness of Solutions

We have the following result.

Theorem 3.1. *We assume that (2.2)-(2.7) and (2.12) hold. Then, there exists a pair of functions (ρ, μ) solution of (2.8) and (2.9) such that $\rho \in \mathcal{C}([0,T]; L^2(\Omega)) \cap L^\infty(0,T; H^1_{per}(\Omega)) \cap L^2(0,T; H^2_{per}(\Omega))$, $\mu \in L^2(0,T; H^1_{per}(\Omega))$ and $\frac{\partial \rho}{\partial t} \in L^2(\Omega_T)$, $\forall T > 0$. If furthermore $\rho_0 \in H^2_{per}(\Omega)$, then the solution is unique and also satisfies $\rho \in \mathcal{C}([0,T_0]; H^1_{per}(\Omega)) \cap L^\infty(0,T_0; H^2_{per}(\Omega)) \cap L^2(0,T_0; H^3_{per}(\Omega))$, $\mu \in L^\infty(0,T_0; H^1_{per}(\Omega)) \cap L^2(0,T_0; H^2_{per}(\Omega))$ and $\frac{\partial \rho}{\partial t} \in L^2(0,T_0; H^1_{per}(\Omega)) \cap L^\infty(0,T_0; L^2(\Omega))$, where $T_0 = T$ if $n = 2$ and $T_0 \leq T$ if $n = 3$.*

Proof. The existence of a solution (ρ, μ) satisfying $\rho \in \mathcal{C}([0,T]; L^2(\Omega)) \cap L^\infty(0,T;\ H^1_{per}(\Omega))$, $\mu \in L^2(0,T; H^1_{per}(\Omega))$ and $\frac{\partial \rho}{\partial t} \in L^2(\Omega_T)$ has been proved in [3]. If we take $q = -\Delta\rho$ in (2.8), then we deduce, noting (2.7), that

$$\|\Delta\rho\|^2_{L^2(\Omega_T)} \leq c_1(\|\mu\|^2_{L^2(\Omega_T)} + \|\nabla\mu\|^2_{L^2(\Omega_T)} + \|\nabla\rho\|^2_{L^2(\Omega_T)} + \|\frac{\partial \rho}{\partial t}\|^2_{L^2(\Omega_T)}) \leq c_2, \tag{3.1}$$

which shows that $\rho \in L^2(0,T; H^2_{per}(\Omega))$, observing that

$$\|\rho - m(\rho_0)\|^2_{L^2(0,T;H^2_{per}(\Omega))} \leq C\|\Delta\rho\|^2_{L^2(\Omega_T)}.$$

We now take $q = -\Delta\mu$ in (2.8) and $q = -\Delta\frac{\partial \rho}{\partial t}$ in (2.9), and then obtain

$$(\frac{\partial \rho}{\partial t}, -\Delta\mu) - (a.\nabla\frac{\partial \rho}{\partial t}, -\Delta\mu) = (\kappa(\rho)\Delta\mu, -\Delta\mu) + (\kappa'(\rho)\nabla\rho.\nabla\mu, -\Delta\mu), \tag{3.2}$$

and

$$(\mu, -\Delta\frac{\partial\rho}{\partial t}) - (b.\nabla\mu, -\Delta\frac{\partial\rho}{\partial t}) \\ = \frac{\alpha}{2}\frac{d}{dt}\|\Delta\rho\|^2 + (f'(\rho), -\Delta\frac{\partial\rho}{\partial t}) + \beta\|\nabla\frac{\partial\rho}{\partial t}\|^2. \quad (3.3)$$

We deduce from (3.2) and (3.3) that

$$\frac{\alpha}{2}\frac{d}{dt}\|\Delta\rho\|^2 + \beta\|\nabla\frac{\partial\rho}{\partial t}\|^2 + ((a+b).\nabla\frac{\partial\rho}{\partial t}, \Delta\mu) + (\kappa(\rho)\Delta\mu, \Delta\mu) \\ = -(f''(\rho)\nabla\rho, \nabla\frac{\partial\rho}{\partial t}) - (\kappa'(\rho)\nabla\rho.\nabla\mu, \Delta\mu); \quad (3.4)$$

and, therefore, using (2.12),

$$\frac{d}{dt}\|\Delta\rho\|^2 + c_1(\|\nabla\frac{\partial\rho}{\partial t}\|^2 + \|\Delta\mu\|^2) \leq c_2\|\nabla\rho\|_{L^4(\Omega)^n}\|\nabla\mu\|_{L^4(\Omega)^n}\|\Delta\mu\| \\ + \int_\Omega |f''(\rho)||\nabla\rho||\nabla\frac{\partial\rho}{\partial t}|dx. \quad (3.5)$$

We deduce from classical Sobolev interpolation results, (2.10), (2.11) and noticing $|m(f'(\rho))| \leq c(1 + \|\rho\|^{2p+1}_{H^1(\Omega)})$, that

$$\|\nabla\rho\|_{L^4(\Omega)^n} \leq c\|\nabla\rho\|^{1-\mu}(1 + \|\Delta\rho\|^{\mu}), \quad (3.6)$$

and

$$\|\nabla\mu\|_{L^4(\Omega)^n} \leq c\|\nabla\mu\|^{1-\mu}(1 + \|\rho\|^{2p+1}_{H^1(\Omega)} + \|\Delta\mu\|)^{\mu}, \quad (3.7)$$

where $\mu = \frac{n}{4}$.

By taking $q = \mu$ in (2.8) and $q = \frac{\partial\rho}{\partial t}$ in (2.9) and using (2.12), we deduce that

$$\|\frac{\partial\rho}{\partial t}\|^2 + \|\nabla\mu\|^2 \leq c\| - \alpha\Delta\rho + f'(\rho)\|\|\frac{\partial\rho}{\partial t}\|. \quad (3.8)$$

We can note that $\|f'(\rho)\| \leq c(1 + \|\rho\|^{2p+1}_{H^1(\Omega)})$ when $n = 2$, and $\|f'(\rho)\| \leq c(1 + \|\rho\|_{H^2(\Omega)}\|\rho\|^4_{H^1(\Omega)})$ when $n = 3$. We get from (3.8) that

$$\|\frac{\partial\rho}{\partial t}\| + \|\nabla\mu\| \leq c(1 + \|\rho\|^{2p+1}_{H^1(\Omega)} + \|\Delta\rho\|)(1 + \|\rho\|^4_{H^1(\Omega)}). \quad (3.9)$$

Thus, inequality (3.7) becomes

$$\|\nabla\mu\|_{L^4(\Omega)^n} \leq c(1 + \|\rho\|^4_{H^1(\Omega)})^{1-\mu}(1 + \|\rho\|^{(1-\mu)(2p+1)}_{H^1(\Omega)} + \|\Delta\rho\|^{1-\mu})$$

$$\times (1 + \|\rho\|_{H^1(\Omega)}^{\mu(2p+1)} + \|\Delta\mu\|^{\mu}), \quad (3.10)$$

and we finally have the estimates

$$\begin{aligned}\|\nabla\rho\|_{L^4(\Omega)^n}\|\nabla\mu\|_{L^4(\Omega)^n}\|\Delta\mu\| &\le c(1+\|\rho\|_{H^1(\Omega)}^5)^{1-\mu}(1+\|\rho\|_{H^1(\Omega)}^{(1-\mu)(2p+1)})\\ &\quad\times(1+\|\Delta\rho\|)(1+\|\rho\|_{H^1(\Omega)}^{\mu(2p+1)}+\|\Delta\mu\|^{\mu+1})\\ &\le \epsilon\|\Delta\mu\|^2 + c(\epsilon)(1+\|\rho\|_{H^1(\Omega)}^5)^2(1+\|\rho\|_{H^1(\Omega)}^{2p+1})^2(1+\|\Delta\rho\|^2)^{\frac{1}{1-\mu}}, \quad (3.11)\end{aligned}$$

$\forall\epsilon > 0$.

On the other hand, we have

$$\int_\Omega |f''(\rho)||\nabla\rho||\nabla\frac{\partial\rho}{\partial t}|\,dx \le c\int_\Omega(|\rho|^{2p}+1)|\nabla\rho||\nabla\frac{\partial\rho}{\partial t}|\,dx. \quad (3.12)$$

When $n = 2$, we have

$$\begin{aligned}\int_\Omega(|\rho|^{2p}+1)|\nabla\rho||\nabla\frac{\partial\rho}{\partial t}|\,dx &\le c(1+\|\rho\|_{L^{8p}(\Omega)}^{2p})\|\nabla\rho\|_{L^4(\Omega)^n}\|\nabla\frac{\partial\rho}{\partial t}\|\\ &\le c(1+\|\rho\|_{H^1(\Omega)}^{2p})\|\nabla\rho\|_{L^4(\Omega)^n}\|\nabla\frac{\partial\rho}{\partial t}\|\\ &\le \epsilon\|\nabla\frac{\partial\rho}{\partial t}\|^2 + c(\epsilon)(1+\|\rho\|_{H^1(\Omega)}^{2p})^2\|\nabla\rho\|^{2(1-\mu)}(1+\|\Delta\rho\|^2)^{\mu}, \quad (3.13)\end{aligned}$$

$\forall\epsilon > 0$. When $n = 3$, we use Agmon's inequality and have (for the sake of simplicity, we take $p = 2$)

$$\begin{aligned}\int_\Omega |\rho|^4|\nabla\rho||\nabla\frac{\partial\rho}{\partial t}|\,dx &\le c\|\rho\|_{L^\infty(\Omega)}^2\|\rho\|_{L^6(\Omega)}^2\|\nabla\rho\|_{L^6(\Omega)^n}\|\nabla\frac{\partial\rho}{\partial t}\|\\ &\le c\|\rho\|_{H^1(\Omega)}^3\|\rho\|_{H^2(\Omega)}\|\nabla\rho\|_{L^6(\Omega)^n}\|\nabla\frac{\partial\rho}{\partial t}\|. \quad (3.14)\end{aligned}$$

We also have

$$\|\nabla\rho\|_{L^6(\Omega)^n} \le c\|\nabla\rho\|^{1-\tau}(1+\|\Delta\rho\|^{\tau}), \quad (3.15)$$

where $\tau = \frac{n}{3}$. Here, $\tau = 1$, since $n = 3$. Hence,

$$\begin{aligned}\int_\Omega |\rho|^4|\nabla\rho||\nabla\frac{\partial\rho}{\partial t}|\,dx &\le c\|\rho\|_{H^1(\Omega)}^3(1+\|\Delta\rho\|^2)\|\nabla\frac{\partial\rho}{\partial t}\|\\ &\le \epsilon\|\nabla\frac{\partial\rho}{\partial t}\|^2 + c(\epsilon)\|\rho\|_{H^1(\Omega)}^6(1+\|\Delta\rho\|^2)^2, \quad (3.16)\end{aligned}$$

$\forall \epsilon > 0$.

We finally deduce from (3.5) an inequality of the form

$$\frac{d}{dt}\|\Delta\rho\|^2 + c_1(\|\nabla\frac{\partial\rho}{\partial t}\|^2 + \|\Delta\mu\|^2) \leq M(t)(1+\|\Delta\rho\|^2)^{\frac{1}{1-\mu}}, \tag{3.17}$$

where $t \mapsto M(t) \in L^\infty(0,T)$. This differential equation is resolved in [2], noting that $\frac{1}{1-\mu} = 2(1+\frac{n-2}{4-n})$. If $\rho_0 \in H^2(\Omega)$, then the solution (ρ,μ) satisfies $\rho \in L^\infty(0,T_0;H^2_{per}(\Omega))$, $\mu \in L^2(0,T_0;H^2_{per}(\Omega))$ and $\frac{\partial\rho}{\partial t} \in L^2(0,T_0;H^1_{per}(\Omega))$, where $T_0 = T$ if $n=2$ and $T_0 \leq T$ if $n=3$. The other points are easily deduced from (2.1). We also have $\rho \in L^2(0,T_0;H^3_{per}(\Omega))$, with only $p=0$ or 1 in (2.5) when $n=3$. This latter fact results from showing that $\|\nabla f'(\rho)\|_{L^2(\Omega_{T_0})} \leq C$, and by taking $q = \Delta^2\rho$ in (2.9).

Let now prove the uniqueness of solutions. Let (ρ_1,μ_1) and (ρ_2,μ_2) be two solutions of (2.8) and (2.9). Setting $\rho = \rho_1 - \rho_2$ and $\mu = \mu_1 - \mu_2$, we have $\rho(0) = 0$, and

$$\begin{aligned}\frac{\alpha}{2}\frac{d}{dt}\|\nabla\rho\|^2 + \beta\|\frac{\partial\rho}{\partial t}\|^2 + ((a+b).\nabla\mu, \frac{\partial\rho}{\partial t}) + (\kappa(\rho_1)\nabla\mu,\nabla\mu) \\ = -(f'(\rho_1)-f'(\rho_2),\frac{\partial\rho}{\partial t}) - ((\kappa(\rho_1)-\kappa(\rho_2))\nabla\mu_2,\nabla\mu).\end{aligned} \tag{3.18}$$

There exist $\lambda,\sigma \in [0,1]$ such that $f'(\rho_1)-f'(\rho_2) = \rho\, f''(\lambda\rho_1 + (1-\lambda)\rho_2)$ and $\kappa(\rho_1)-\kappa(\rho_2) = \rho\,\kappa'(\sigma\rho_1 + (1-\sigma)\rho_2)$. On the one hand, we have

$$|\int_\Omega (\kappa(\rho_1)-\kappa(\rho_2))\nabla\mu_2\nabla\mu\,dx| \leq c\|\rho\|_{H^1(\Omega)}\|\mu_2\|_{H^2(\Omega)}\|\nabla\mu\|. \tag{3.19}$$

On the other hand, we have

$$|\int_\Omega (f'(\rho_1)-f'(\rho_2))\frac{\partial\rho}{\partial t}\,dx| \leq c\int_\Omega (|\rho_1|^{2p}+|\rho_2|^{2p}+1)|\rho||\frac{\partial\rho}{\partial t}|\,dx. \tag{3.20}$$

When $n=2$, we find

$$\begin{aligned}|\int_\Omega (f'(\rho_1)-f'(\rho_2))\frac{\partial\rho}{\partial t}dx| \leq c(1+\|\rho_1\|^{2p}_{H^1(\Omega)} + \|\rho_2\|^{2p}_{H^1(\Omega)}) \\ \times \|\rho\|_{H^1(\Omega)}\|\frac{\partial\rho}{\partial t}\|.\end{aligned} \tag{3.21}$$

When $n=3$, we get

$$|\int_\Omega (f'(\rho_1) - f'(\rho_2))\frac{\partial \rho}{\partial t}dx| \leq c(1 + \|\rho_1\|^3_{H^1(\Omega)} + \|\rho_2\|^3_{H^1(\Omega)})$$
$$\times (1 + \|\rho_1\|_{H^2(\Omega)} + \|\rho_2\|_{H^2(\Omega)})\|\rho\|_{H^1(\Omega)}\|\frac{\partial \rho}{\partial t}\|. \quad (3.22)$$

Using (2.12) and (3.19)-(3.22), we deduce from (3.18) an inequality of the form

$$\frac{d}{dt}\|\nabla\rho\|^2 + c_1(\|\frac{\partial \rho}{\partial t}\|^2 + \|\nabla\mu\|^2) \leq M(t)\|\rho\|^2_{H^1(\Omega)}, \quad (3.23)$$

where $t \mapsto M(t) \in L^2(0, T_0)$. Hence, the uniqueness of solution by using Gronwall's Lemma.

We now assume that the mobility function κ further satisfies

$$\kappa \in \mathcal{C}^2(\mathbb{R}), \ \ |\kappa''(s)| \leq C, \ \ \forall s \in \mathbb{R}. \quad (3.24)$$

We also assume that the potential f further satisfies

$$f \in \mathcal{C}^3, \ \ |f'''(s)| \leq c_8|s|^q + c_9, \ \ c_8, c_9 \geq 0, \ \ \forall s \in \mathbb{R}; \quad (3.25)$$

where $q \in \mathbb{N}$ is arbitrary when $n = 2$ and $q = 0$ or 1 when $n = 3$.

We take $q = \Delta^2\mu$ in (2.8) and $q = \Delta^2\frac{\partial \rho}{\partial t}$ in (2.9). Proceeding as for (3.2)-(3.16) using again (2.12), it is not difficult to see that

$$\frac{d}{dt}\|\nabla\Delta\rho\|^2 + c_1(\|\Delta\frac{\partial \rho}{\partial t}\|^2 + \|\nabla\Delta\mu\|^2) \leq c_2\|\nabla\rho\|_{L^4(\Omega)^n}\|\Delta\mu\|_{L^4(\Omega)}\|\nabla\Delta\mu\|$$
$$+ c_3\|\Delta\rho\|_{L^4(\Omega)}\|\nabla\mu\|_{L^4(\Omega)^n}\|\nabla\Delta\mu\| + c_4\|\nabla\rho\|^2_{L^6(\Omega)^n}\|\nabla\mu\|_{L^6(\Omega)^n}\|\nabla\Delta\mu\|$$
$$+ c_5\|f''(\rho)\|_{L^4(\Omega)}\|\Delta\rho\|_{L^4(\Omega)}\|\Delta\frac{\partial \rho}{\partial t}\|$$
$$+ c_6\|f'''(\rho)\|_{L^6(\Omega)}\|\nabla\rho\|^2_{L^6(\Omega)^n}\|\Delta\frac{\partial \rho}{\partial t}\|. \quad (3.26)$$

We have $\|\nabla\rho\|_{L^4(\Omega)^n} + \|f'''(\rho)\|_{L^6(\Omega)} + \|f''(\rho)\|_{L^4(\Omega)} \leq c(1 + \|\rho\|^{r_1}_{H^2(\Omega)})$, $r_1 > 0$, $\|\nabla\mu\|_{L^4(\Omega)^n} \leq c\|\nabla\mu\|^{1-\frac{n}{4}}(1 + \|\Delta\mu\|^{\frac{n}{4}})$, $\|\nabla\mu\|_{L^6(\Omega)^n} \leq c\|\nabla\mu\|^{1-\frac{n}{3}}(1 + \|\Delta\mu\|^{\frac{n}{3}})$, $\|\Delta\rho\|_{L^4(\Omega)} \leq c(1 + \|\nabla\Delta\rho\|)$, and $\|\Delta\mu\|_{L^4(\Omega)} \leq c\|\Delta\mu\|^{1-\frac{n}{4}}(1 + \|\nabla\Delta\mu\|^{\frac{n}{4}})$. We then deduce from (3.26) that

$$\frac{d}{dt}\|\nabla\Delta\rho\|^2 + c_1(\|\Delta\frac{\partial \rho}{\partial t}\|^2 + \|\nabla\Delta\mu\|^2) \leq c(1 + \|\rho\|^2_{H^2(\Omega)})^r$$
$$\times (1 + \|\Delta\mu\|^2)(1 + \|\nabla\Delta\rho\|^2), \quad (3.27)$$

$r > 0$. □

4. Existence of the Global Attractor

We first recall the definition of the global attractor. Let E be a Banach space and $\{S(t)\}_{t\geq 0}$ a continuous semigroup on E.

Definition 4.1. A compact set $\mathcal{A}$ is called the global attractor for the semigroup $\{S(t)\}_{t\geq 0}$ if:

(i) $\mathcal{A}$ is strictly invariant, that is, $S(t)\mathcal{A} = \mathcal{A}, \ \forall t \geq 0$;

(ii) $\mathcal{A}$ is an attracting set for $\{S(t)\}_{t\geq 0}$ in the following sense: for any bounded set $B \subset E$, we have

$$\mathrm{dist}_E(S(t)B, \mathcal{A}) \to 0 \ \text{ when } \ t \to \infty,$$

where dist_E is the Hausdorff pseudo-distance for the metric of E:

$$\mathrm{dist}_E(A, B) = \sup_{a\in A} \inf_{b\in B} \|a - b\|_E.$$

Thanks to Theorem 3.1, we can define the semigroup $S(t) : H^2_{per}(\Omega) \to H^2_{per}(\Omega)$, $\rho_0 \mapsto \rho(t)$, $\rho(t)$ being the solution at time t of (2.8)-(2.9). We set $K_\delta = \{\rho \in H^2_{per}(\Omega), |m(\rho)| \leq \delta\}$, $\delta \geq 0$. We can easily prove that $S(t)$ is continuous on K_δ.

We take $q = \mu$ in (2.8) and $q = \frac{\partial \rho}{\partial t}$ in (2.9). Using (2.12), we obtain that

$$\frac{d}{dt}(\frac{\alpha}{2}\|\nabla\rho\|^2 + \int_\Omega f(\rho)dx) + c(\|\frac{\partial\rho}{\partial t}\|^2 + \|\nabla\mu\|^2) \leq 0; \tag{4.1}$$

and, $J(\rho) = \int_\Omega (f(\rho) + \frac{\alpha}{2}|\nabla\rho|^2)\, dx$ is a Lyapunov function for problem (2.1). Taking now $q = \overline{\rho}$ in (2.9), we obtain

$$\frac{\beta}{2}\frac{d}{dt}\|\rho\|^2 + \alpha\|\nabla\rho\|^2 + (f'(\rho), \overline{\rho}) = (\mu, \overline{\rho}) - (b.\nabla\mu, \overline{\rho}), \tag{4.2}$$

and, therefore,

$$\frac{d}{dt}\beta\|\rho\|^2 + \alpha\|\nabla\rho\|^2 + 2(f'(\rho), \overline{\rho}) \leq c\|\nabla\mu\|^2. \tag{4.3}$$

We sum (4.1) and σ(4.3), for a sufficiently small $\sigma > 0$. We get that

$$\frac{d}{dt}(\frac{\alpha}{2}\|\nabla\rho\|^2 + \int_\Omega f(\rho)dx + \sigma\beta\|\rho\|^2) + c(\|\frac{\partial\rho}{\partial t}\|^2 + \|\nabla\mu\|^2 + \|\nabla\rho\|^2) + c_1\int_\Omega f(\rho)dx \leq c_2, \tag{4.4}$$

and

$$\frac{dE_1}{dt} + cE_1 + c_1(\|\frac{\partial\rho}{\partial t}\|^2 + \|\nabla\mu\|^2) \le c_2, \tag{4.5}$$

where $E_1(t) = \frac{\alpha}{2}\|\nabla\rho\|^2 + \int_\Omega f(\rho)dx + \sigma\beta\|\rho\|^2$. We note that the constant c in (4.5) is bounded from bellow by a strictly positive constant that does not depend on $m(\rho_0)$ if $\rho \in K_\delta$; the other constants are bounded independently of $m(\rho_0)$). We deduce from (4.5), and by applying the uniform Gronwall's Lemma to (3.17) and (3.27), the existence of bounded absorbing sets B_2 and B_3 for $S(t)$ on K_δ and on $K_\delta \cap H^3_{per}(\Omega)$, respectively (and for only $n = 2$). We thus have the following result.

Theorem 4.1. *Let $n = 2$ and let the assumptions of Theorem 3.1, (3.24) and (3.25) hold. Then, the semigroup $\{S(t)\}_{t\ge0}$ possesses the global attractor $\mathcal{A}_\delta$ in K_δ, which is bounded in $H^3_{per}(\Omega)$.*

5. Existence of Exponential Attractors

Let X be a compact subset of E and consider a continuous semigroup $\{S(t)\}_{t\ge0}$ mapping X into X.

Definition 5.1. A set $\mathcal{M}$ is called an exponential attractor for $\{S(t)\}_{t\ge0}$ for the topology of E if:

(i) $\mathcal{M}$ is compact and contains the global attractor, that is, $\mathcal{A} \subset \mathcal{M} \subset X$;
(ii) $\mathcal{M}$ is positively invariant under the flow, that is, $S(t)\mathcal{M} \subset \mathcal{M}, \ \forall t \ge 0$;
(iii) the fractal dimension of $\mathcal{M}$ is finite;
(iv) $\forall B \subset X$, B bounded, $\exists\, c_0(B),\, c_1(B) > 0$ such that

$$\mathrm{dist}_E(S(t)B, \mathcal{M}) \le c_0 e^{-c_1 t}, \ \ \forall t \ge 0.$$

Sufficient conditions ensuring the existence of exponential attractors in an Hilbert space is given in [6]. It depends on a dichotomy principle called the squeezing property.

Definition 5.2. The semigroup $\{S(t)\}_{t\ge0}$ verifies the squeezing property on X if, for a real number η belonging to $[0, \frac{1}{4}[$, there exists a projection $P_{N^\star}$ of finite rank $N^\star(\eta)$ and a time $t^\star$ such that

$$\begin{aligned}
&\forall\, (\phi,\psi) \in X^2 \\
&\quad \text{if} \quad \|(I - P_{N^\star})(S(t^\star)\phi - S(t^\star)\psi)\|_E \ge \|P_{N^\star}(S(t^\star)\phi - S(t^\star)\psi)\|_E \\
&\qquad\qquad \text{then} \quad \|S(t^\star)\phi - S(t^\star)\psi\|_E \le \eta\|\phi - \psi\|_E.
\end{aligned} \tag{5.1}$$

Proposition 5.1. *If $\{S(t)\}_{t\geq 0}$ satisfies the squeezing property on X and if $S(t^\star)$ is Lipschitz on X with Lipschitz constant L, then there exists an exponential attractor $\mathcal{M}$ for $\{S(t)\}_{t\geq 0}$ on X such that the fractal dimension of $\mathcal{M}$ is bounded as follows:*

$$d_F(\mathcal{M}) \leq N^\star \max(1, \frac{\ln(16L+1)}{\ln 2}) \tag{5.2}$$

In the order to verify the squeezing property, Babin and Nicolaenko introduced in [1] a method based on a decomposition of the difference of two trajectories. Furthermore, they give a theorem on the existence of exponential attractors which does not use the existence of a compact absorbing set X; only the existence of an asymptotically attracting compact set X is needed. A consequence of their result is stated as follows (see [9]).

Proposition 5.2. *Let E and V be two Hilbert spaces such that the inclusion $V \subset E$ is compact. Let $X \subset E$ be a closed set and $S(t) : X \to X$ be a semigroup. Let us furthermore assume that there exists a projection $P_{N^\star}$ of finite rank $N^\star$ such that $\|(I - P_{N^\star})y\|_E \leq c(N^\star)\|y\|_V$, $\forall y \in V$, where $c(N^\star) \to 0$ as $N^\star \to +\infty$. If there exist φ^1, φ^2 satisfying $S(t)\phi - S(t)\psi = \varphi^1(t) + \varphi^2(t)$ and $\|\varphi^1(t)\|_E^2 \leq d(t)\|\phi - \psi\|_E^2$, $\|\varphi^2(t)\|_V^2 \leq h(t)\|\phi - \psi\|_E^2$, where $d(t)$ is continuous and satisfies $\lim_{t\to+\infty} d(t) = 0$ and $h(t)$ is continuous, then $\{S(t)\}_{t\geq 0}$ enjoys the squeezing property on X for the topology of E.*

We now want to apply this result to study the existence of exponential attractors $\mathcal{M}_\delta$ on the set $X_\delta = \overline{\cup_{t\geq t_1} S(t)B_3}$, where B_3 is an absorbing set for the semigroup $S(t)$ on $K_\delta \cap H^3_{per}(\Omega)$, and t_1 is such that $t \geq t_1$ implies $S(t)B_3 \subset B_3$. The set X_δ is compact and positively invariant by $S(t)$. We then consider the difference (ρ, μ) of two solutions (ρ_1, μ_1) and (ρ_2, μ_2) of (2.8) and (2.9), that is, $\rho = \rho_1 - \rho_2$ and $\mu = \mu_1 - \mu_2$, with $\rho_1|_{t=0} = \rho_{01}$ and $\rho_2|_{t=0} = \rho_{02}$. We set $\rho_0 = \rho_{01} - \rho_{02}$ and assume that $\rho_{01}, \rho_{02} \in X_\delta$. We now introduce the decomposition $\rho = \rho^1 + \rho^2$ and $\mu = \mu^1 + \mu^2$, where (ρ^1, μ^1) and (ρ^2, μ^2) are the solutions of the following problems, respectively,

$$\frac{d}{dt}(\rho^1, q) = -(\kappa(\rho_2)\nabla\mu^1, \nabla q), \ \ \forall q \in H^1_{per}(\Omega), \tag{5.3}$$

$$(\mu^1, q) = (\rho^1, q) + \alpha(\nabla\rho^1, \nabla q) + \beta(\frac{\partial\rho^1}{\partial t}, q), \ \ \forall q \in H^1_{per}(\Omega), \tag{5.4}$$

$$\rho^1(0) = \rho_0, \tag{5.5}$$

and

$$\frac{d}{dt}(\rho^2, q) = -((\kappa(\rho_1) - \kappa(\rho_2))\nabla\mu_1, \nabla q) - (\kappa(\rho_2)\nabla\mu^2, \nabla q) - (\frac{\partial\rho}{\partial t}, a.\nabla q), \quad \forall q \in H^1_{per}(\Omega), \quad (5.6)$$

$$(\mu^2, q) = (\rho^2, q) + \alpha(\nabla\rho^2, \nabla q) + \beta(\frac{\partial\rho^2}{\partial t}, q) + (f'(\rho_1) - f'(\rho_2), q) + (b.\nabla\mu, q) - (\rho, q), \quad \forall q \in H^1_{per}(\Omega), \quad (5.7)$$

$$\rho^2(0) = 0. \quad (5.8)$$

We take $q = -\Delta\mu^1$ in (5.3), $q = \rho^1$, $q = \frac{\partial\rho^1}{\partial t}$ and $q = -\Delta\frac{\partial\rho^1}{\partial t}$ in (5.4), respectively. It is not difficult to find that

$$\|\rho^1\|^2_{H^2(\Omega)} \leq ce^{-c't}\|\rho_0\|^2_{H^2(\Omega)}, \quad \forall t \geq 0 \quad (5.9)$$

$$\int_0^t \|\rho^1\|^2_{H^3(\Omega)} dt \leq c\|\rho_0\|^2_{H^2(\Omega)}, \quad \forall t \geq 0. \quad (5.10)$$

We now take $q = \mu^2$, $q = -\Delta\mu^2$ and $q = \Delta^2\mu^2$ in (5.6) and also $q = \rho^2$, $q = \frac{\partial\rho^2}{\partial t}$, $q = -\Delta\frac{\partial\rho^2}{\partial t}$ and $q = \Delta^2\frac{\partial\rho^2}{\partial t}$ in (5.7), respectively. Proceeding as in the proof Theorem 3.1 and using (5.9) and (5.10), we can check that

$$\|\rho^2\|^2_{H^3(\Omega)} \leq ce^{c't}\|\rho_0\|^2_{H^2(\Omega)}, \quad \forall t \geq 0, \quad (5.11)$$

when $n = 2$. In consequence, the semigroup $S(t)$ enjoys the squeezing property on X_δ. Moreover, we easily prove that $S(t)$ is Lipschitz on X_δ. We consider $P_{N^\star}$ as the projection on the space generated by the $N^\star$ first eigenfunctions of the operator $I + N$ on $H^2_{per}(\Omega)$. We then give the following theorem.

Theorem 5.1. *Let $n = 2$ and let the assumptions of Theorem 4.1 hold. Then, the semigroup $\{S(t)\}$ possesses an exponential attractor $\mathcal{M}_\delta$ on X_δ.*

Since the exponential attractor $\mathcal{M}_\delta$ contains the global attractor $\mathcal{A}_\delta$, we have the following consequence.

Corollary 5.1. *The global attractor $\mathcal{A}_\delta$ obtained in Theorem 4.1 has finite fractal dimension*

Acknowledgments

I am currently an Abdus Salam International Centre for Theoretical Physics post-doctoral research fellow, and I would like to thank the Centre for all its support.

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