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# BOCHER'S THEOREM FOR POLYHARMONIC FUNCTIONS IN $R^n$

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**Abstract:** Our aim in this paper is to give a proof of Bocher's Theorem for Polyharmonic functions in $R^n$, $n \geq 2$.

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**Key Words:** Kelvin trasform, m-harmonic functions, m-potentials

## 1. Preliminaries

A real valued function u on an open set G $\subset R^n$ is said to be m-harmonic function (polyharmonic) on G if u$\in C^{2m}(G)$ and $\triangle^m u = 0$ on G, where m is a positive integer and $\triangle^m u = \triangle^{m-1}(\triangle u)$, where $m > 1$.

Given a locally integrable function $u_m$ on an open set G$\subset R^n$, let u =($u_1$, $u_2, \ldots,$ $u_m$) where $\triangle u_1 = -u_2, \triangle u_2 = -u_3, \ldots, \triangle u_{i-1} = -u_i, 1 \leq i \leq m-1$. Then u is called an m-superharmonic function in G if $u_m$ is superharmonic.

A function q is said to be an m-potential if q is a potential, (- $\triangle$ )q, (- $\triangle)^2$ q,... (- $\triangle)^{m-1}$ q are potentials.

T. Futamura (see [2]) gave a proof of Bocher's Theorem for polyharmonic functions. The purpose of this paper is to give a proof of the same theorem by a different approach.

**Theorem 1.** *Let u be a biharmonic function outside a compact set in $R^n$, $n \geq 5$. Then near* inf *inity u(x)=*$q_1(x) - q_2(x) + p_1(x) - p_2(x) + v(x)$, *where v*

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*is a biharmonic function on $R^n$, $p_1$, $p_2$, $q_1$, $q_2$, are potentials (finite continuous), also $q_1$ and $q_2$ are bipotentials (2-potentials). This representation is uniqe.*

Now assume that u is biharmonic in $0 < |x| < 1$. Then $Ku(x) = |x|^{4-n}\, u(\frac{x}{|x|^2})$ is a biharmonic function in $|x| > 1$. By Theorem 1 $Ku(x) = q_1(x) - q_2(x) + p_1(x) - p_2(x) + v(x)$, $|x| > 1$.

Appling Kelvin transform to both sides, we get

$$u(x) = Kq_1(x) - Kq_2(x) + Kp_1(x) - Kp_2(x) + Kv(x),$$

since $p_1(x)$, $p_2(x)$ are potentials with compact support, near infinity $\mathrm{p}_1$, and $\mathrm{p}_2$ are of the form

$$p_1(x) = \frac{\alpha_1}{|x|^{n-2}} + b_1(x), \quad p_2(x) = \frac{\alpha 2}{|x|^{n-2}} + b_2(x),$$

where $\mathrm{b}_1(x)$ and $\mathrm{b}_2(x)$ are some bounded harmonic functions. Hence $Kp_1(x) = \alpha_1\,|x|^{n-2} + b_1(\frac{x}{|x|^2})$, for $0 < |x| < 1$ and $Kp_2(x) = \alpha_2\,|x|^{n-2} + b_2(\frac{x}{|x|^2})$. Since $q_1(x)$ and $q_2(x)$ are bipotentials in $R^n$, $n \geq 5$ then $q_1(x) = \beta_1\,|x|^{4-n}$, $q_2(x) = \beta_2\,|x|^{4-n}, Kq_1(x) = \beta_1, Kq_2(x) = \beta_2$. Hence $u(x) = \alpha\,|x|^2$ +bounded function, $0 < |x| < 1$, which is Bocher's Theorem for biharmonic functions.

**Theorem 2.** *Let u be an m-harmonic function outside a compact set in $R^n$, $n \geq 2m+1$. Then near infinity*

$$u(x) = (p_m - p'_m)(x) + \cdots + (p_1 - p'_1)(x) + v(x),$$

*where v is an m-harmonic function on $R^n, p_i, p'_i$ are i- potentials (finite continuous ), $i = 1, 2, \ldots, m$.*

*Proof.* We use mathematical induction. This relation is true for m=1**.** Assume it is true for m=k, that is if u is a k-harmonic function out side a compact set

$$u(x) = (p_k - p'_k)(x) + \cdots + (p_1 - p'_1)(x) + v(x)\,, \tag{1}$$

we show it is true for n=k+1. Assume that u is ( k+1) -harmonic function outside a compact set, $\Delta^{K+1}$ u=0, that is $\Delta^K(\Delta u) = 0$. Let h= $\Delta u$, then, $\Delta^K h = 0$, and h is a k-harmonic function, then by (1)

$$h(x) = (p_k - p'_k)(x) + \cdots + (p_1 - p'_1)(x) + v(x),$$

where $p_i$ are potentials, v is k-harmonic function in $R^n$, $n \geq 2m+1$,

$$\Delta u(x) = (p_k - p'_k)(x) + \cdots + (p_1 - p'_1)(x) + v(x),$$

since p $_i$ are potentials with compact harmonic support, and there exist potentials q $_1, q_2, \ldots, q_k,$ such that $\Delta q_1 = -p_1, \Delta q_2 = -p_2, \ldots \Delta q_k = -p_k, \Delta q_1' = -p_1', \Delta q_2' = -\Delta p_2' \ldots, \Delta q_k' = -p_k', \Delta B = -v$, then $\Delta u = -\Delta q_k' + \Delta q_k' + \cdots - \Delta q_1 + \Delta q_1' + v$. Therefore

$$\Delta(u - q_k + q_k' + \cdots - q_1 + q_1' + B) = 0,$$

and $u - q_k + q_k' + \cdots - q_1 + q_1' + B$ is a harmonic function outside a compact set, hence $u - q_k' + q_k' + \cdots - q_1 + q_1' + B =$p $_1 - p_2 + H$ or u $= (q_k - q_k') + \ldots (p_1 - p_2) + H$ out side a compact set. □

## 2. Bocher's Theorem for m-Harmonic Functions

**Theorem 3.** *Assume that u is an m-harmonic function in* $0 < |x| < 1$, *then* u(x)=( $\alpha_m - \alpha_m'$)+ ( $\alpha_{m-1} - \alpha_{m-1}'$) $|x|^2 + \ldots .$ ( $\alpha_1 - \alpha_1'$) $|x|^{2m-2} + v(x)$, *where v is an m- harmonic function in* $R^n$, $n \geq 2m+1$.

*Proof.* Since u is an m-harmonic function then Ku$(\frac{x}{|x|^2})$ is an m-harmonic function in $|x| \succ 1$, by Theorem 2, K $u(x) = (p_m - p_m')(x) + \cdots + (p_1 - p_1')(x) + v(x)$. Appling Kelvin Trasform to both sides, we get

$$u(x) = K(p_m - p_m')(x) + \cdots + K(p_1 - p_1')(x) + Kv(x), p_i$$

is an i-potential means that p $_i$ is a potential and (- $\Delta)p_i$ is a potential (- $\Delta)^2 p_i, \ldots,$ (- $\Delta)^{i-1} p_i$ is a potential, then p $_i(x) = \frac{\alpha_i}{|x|^{n-2i}}$ and K$p_i(x) = K(\frac{\alpha_i}{|x|^{n-2i}}) = |x|^{2m-n} \frac{\alpha_i}{\left|\frac{x}{|x|^2}\right|^{n-2i}} = \alpha_i |x|^{2m-2i}$, $i = 1, 2, \ldots.$ Hence

$$u(x) = (\alpha_m - \alpha_m') + (\alpha_{m-1} - \alpha_{m-1}')|x|^2 + \cdots + (\alpha_1 - \alpha_1')|x|^{2m-2} + Kv(x), \quad 0 < |x| < 1.$$ □

## 3. Properties of m-Harmonic Functions Near $x = 0$

**Theorem 4.** *Let u be an m-harmonic function in* $0 < |x| < 1$, *in* $\Re^n, n \geq 2m+1$. *Then u extends as an m-harmonic function in* $|x| < 1$ *if* $u(x) = o(|x|^{2m-n})$, *when* $|x| \longrightarrow 0$.

*Proof.* By Theorem 3 u(x)=( $\alpha_m - \alpha_m'$)+ ( $\alpha_{m-1} - \alpha_{m-1}'$) $|x|^2 + \ldots (\alpha_1 - \alpha_1')|x|^{2m-2} + v(x)$. If $u(x) = o(|x|^{2m-n})$, when $|x| \longrightarrow 0$, then $v(x) = o(|x|^{2m-n})$, when $|x| \longrightarrow 0$. Hence $\alpha_i' = 0$, $i = 1, 2, m$, and $u(x) = v(x)$ in $|x| < 1$.

In particular, if u is a bounded m-harmonic function in $0 < |x| < 1$ in in $\Re^n$, $n \geq 2m+1$, then u extends as an m-harmonic function in $|x| < 1$. □

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