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ON DOUBLE SEQUENCE SPACES ${}_0c_2^P(p)$, ${}_0c_2^{PB}(p)$ AND $\ell_2(p)$

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Abstract: In this paper we introduce the double sequence spaces ${}_0c_2^P(p)$, ${}_0c_2^{PB}(p)$ and $\ell_2(p)$. We give some inclusion relations between these spaces. Furthermore we show that ${}_0c_2^{PB}(p)$ and $\ell_2(p)$ are complete paranormed spaces under some certain conditions. We also give the $\alpha-,\beta-$ and $\gamma-$ duals of the double sequence spaces ${}_0c_2^{PB}(p)$ and $\ell_2(p)$.

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1. Introduction

We write w^2 for the set of all complex double sequences (x_{mn}) $m,n=1,2,...,w^2$ is a linear space under coordinatewise addition and scalar multiplication.

A double sequence $x=(x_{mn})$ is said to be bounded if $\sup\limits_{m,n\geq 1}|x_{mn}|<\infty$, see [5].

We remind the reader that x is said to be convergent in the Pringsheim's sense if there exists a number ℓ such that x_{mn} converges to ℓ as both m and n tend to ∞ independently of one another: $\lim\limits_{m,n\to\infty} x_{mn}=\ell$.

It is trivial that $x=(x_{mn})$ converges in Pringsheim's sense if and only if for every $\varepsilon>0$ there exists an $n_o=n_o(\varepsilon)$ such that $|x_{mn}-\ell|<\varepsilon$, for

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all $m,n \geq n_o$. The limit ℓ is called the double limit (or Pringsheim limit). A sequence $x = (x_{mn})$ is called double null sequence if it converges to zero. The crucial difference between the convergence of single sequences and the convergence in Pringsheim's sense of double sequences is that the latter does not imply the boundedness of the terms of the double sequence in question.

A double sequence $x = (x_{mn})$ is said to be Cauchy sequence if and only if for every $\varepsilon > 0$ there exists an $n_o = n_o(\varepsilon)$ such that $|x_{mn} - x_{pq}| < \varepsilon$, for all $m, n, p, q \geq n_o$. It is known that a double sequence (x_{mn}) with complex terms is a *Cauchy sequence* if and only if it is convergent.

The double series $\sum\limits_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence (S_{mn}) is convergent in Prigsheim's sense, where $S_{mn} = \sum\limits_{i,j=1}^{m,n} x_{ij}$ $(m, n = 1, 2, ...)$.

Móricz defined the double sequence spaces ℓ_2^∞, c_2^P, ${}_0c_2^P$, c_2^{PB}, ${}_0c_2^{PB}$ and ℓ_2^1 in [4] and proved that ℓ_2^∞,c_2^{PB} and ${}_0c_2^{PB}$ are Banach spaces endowed with the norm $||x||_\infty = \sup\limits_{m,n\geq 1} |x_{mn}|$. Furthermore, he also proved that c_2^P is complete under the pseudonorm $||x||_P = \lim\limits_{N\to\infty} \sup\limits_{m,n\geq N} |x_{mn}|$.

2. Paranormed Double Sequence Spaces

Gökhan and Çolak defined the following sequence spaces in [1], [2] as

$\ell_2^\infty(p) = \{x = (x_{mn}) \in w^2 \ : \sup\limits_{m,n\geq 1} |x_{mn}|^{p_{mn}} < \infty\}$,

$c_2^P(p) = \{x = (x_{mn}) \in w^2 \ : \lim\limits_{m,n\to\infty} |x_{mn} - \mathrm{L}|^{p_{mn}} = 0 \text{ for some } L\}$,

$c_2^{PB}(p) = \ell_2^\infty(p) \cap c_2^P(p)$,

where $p = (p_{mn})$ is a double sequence of strictly positive real numbers p_{mn}, and showed that $\ell_2^\infty(p)$ is a complete paranormed space with

$$g(x) = \sup_{m,n\geq 1} |x_{mn}|^{p_{mn}/M} \Leftrightarrow \inf p_{mn} > 0,$$

where $M = \max(1, H)$ and $H = \sup p_{mn} < \infty$.

It is also showed that $c_2^P(p)$ is a complete paranormed spaces with

$$g(x) = \lim_{N\to\infty} \sup_{m,n\geq N} |x_{mn}|^{p_{mn}/M} \Leftrightarrow \lim_{N\to\infty} \inf_{m,n\geq N} p_{mn} > 0,$$

where $M = \max(1, \sup_{m,n\geq N} p_{mn})$, $N = \max(N_1, N_2)$,

$$N_1 = \min\{n_o : \sup_{m,n\geq n_o} |x_{mn}|^{p_{mn}} < \infty\}$$

and $N_2 = \min\ \{n_o : \sup_{m,n\geq n_0} p_{mn} < \infty\}$.

3. The Double Sequence Spaces ${}_0c_2^P(p)$, ${}_0c_2^{PB}(p)$ and $\ell_2(p)$

Definition 1. Let $p = (p_{mn})$ be a double sequence of strictly positive real numbers p_{mn}, then we define

$${}_0c_2^P(p) = \{x = (x_{mn}) \in w^2 \ : \lim_{m,n\to\infty} |x_{mn}|^{p_{mn}} = 0\}$$

and

$$\ell_2(p) = \{x = (x_{mn}) \in w^2 \ : \sum_{m,n=1}^{\infty} |x_{mn}|^{p_{mn}} < \infty\}.$$

It is trivial that $\ell_2^\infty(p)$ does not include ${}_0c_2^P(p)$. So we define

$${}_0c_2^{PB}(p) ={}_0 c_2^P(p) \cap \ell_2^\infty(p).$$

It is trivial that ${}_0c_2^P(p) \subset c_2^P(p)$ and ${}_0c_2^{PB}(p) \subset c_2^{PB}(p)$. When all terms of (p_{mn}) are constant and all are equal to $p > 0$, we obtain $\ell_2(p)= \ell_2^p$, ${}_0c_2^{PB}(p) ={}_0 c_2^{PB}$, where $\ell_2^p = \{x = (x_{mn}) \in w^2 \ : \sum_{m,n=1}^{\infty} |x_{mn}|^p < \infty\ \}$ and ${}_0c_2^{PB}(p)= {}_0c_2^P\cap \ell_2^\infty$. Furthermore, when all terms of (p_{mn}), excluding the first finite number of m and n, are constant and all are equal to $p > 0$, we obtain ${}_0c_2^P(p) = {}_0c_2^P$ and ${}_0c_2^{PB}(p) = {}_0c_2^{PB}$. If $x \in \ell_2(p)$ then $x \in {}_0c_2^{PB}(p)$, since $|x_{mn}|^{p_{mn}} = t_{mn} - t_{m-1,n} - t_{m,n-1} + t_{m-1,n-1}$, for every $m, n \in \mathbf{N}$, where $t_{mn} = \sum_{i,j=1}^{m,n} |x_{ij}|^{p_{ij}}$ and $t_{0,n} = t_{m,0} = t_{0,0} = 0$.

We need the following well-known inequalities in the sequel of the paper.

1. Let $p \geq 1$ and $\sum_{m,n=1}^{\infty} |x_{mn}|^p < \infty$, $\sum_{m,n=1}^{\infty} |y_{mn}|^p < \infty$. Then

$$\left[\sum_{m,n=1}^{\infty} |x_{mn} + y_{mn}|^p\right]^{1/p} \leq \left[\sum_{m,n=1}^{\infty} |x_{mn}|^p\right]^{1/p} + \left[\sum_{m,n=1}^{\infty} |y_{mn}|^p\right]^{1/p}. \qquad (1)$$

2. If $0 < p \leq 1$, then the inequality

$$|a+b|^p \leq |a|^p + |b|^p \tag{2}$$

is valid for complex numbers a and b.

Theorem 1. *Let (p_{mn}) be a sequence of strictly positive real numbers. Then:*

(i) ${}_0c_2^P \subset_0 c_2^P(p)$ *if and only if* $\lim_{N\to\infty} \inf_{m,n\geq N} p_{mn} > 0$,

(ii) ${}_0c_2^P(p) \subset_0 c_2^P$ *if and only if* $\lim_{N\to\infty} \sup_{m,n\geq N} p_{mn} < \infty$,

(iii) ${}_0c_2^P(p) =_0 c_2^P$ *if and only if* $0 < \lim_{N\to\infty} \inf_{m,n\geq N} p_{mn} \leq \lim_{N\to\infty} \sup_{m,n\geq N} p_{mn} < \infty$.

Proof. The sufficiency is trivial by Theorem 1 (i) of [1] with $L = 0$. For the necessity let ${}_0c_2^P \subset_0 c_2^P(p)$ and suppose that $\lim_{N\to\infty} \inf_{m,n\geq N} p_{mn} = 0$. Then there exists a strictly increasing sequence $(N(i))$ of positive integers such that

$$p_{m(i),\ \ n(i)} < \frac{1}{i}\,, \tag{3}$$

for all $i \in \mathbf{N}$, where $m(i),\ n(i) \geq N(i) > 1$. Now we define $x_{mn} = n$ $(m = 1$ and $n = 1, 2, ...)$, $\frac{1}{i}$ $(m = m(i)$ and $n = n(i))$, 0 (otherwise). It is clear that $x \in {}_0c_2^P$, but we get $x \notin {}_0c_2^P(p)$ since

$$\lim_{m,n\to\infty} |x_{mn}|^{p_{mn}} = \lim_{i,j\to\infty} (1/i)^{p_{m(i),n(i)}} \geq \lim_{i,j\to\infty} (1/i)^{1/i} = 1\,,$$

by (3). This contradicts the assumption, whence $\lim_{N\to\infty} \inf_{m,n\geq N} p_{mn} > 0$.

(ii) The sufficiency is trivial by Theorem 1 (ii) of [1] with $L = 0$. Therefore we omit it.

For the necessity let ${}_0c_2^P(p) \subset_0 c_2^P$, but $\lim_{N\to\infty} \sup_{m,n\geq N} p_{mn} = \infty$. Then there exists a strictly increasing sequence $(N(i))$ of positive integers such that

$$p_{m(i),n(i)} > i\,, \tag{4}$$

for all $i \in \mathbf{N}$, where $m(i),\ n(i) \geq N(i) > 1$. If we define $x_{mn} = n^{1/p_{mn}}$ $(m = 1$ and $n = 1, 2, ...)$, $(\frac{1}{i})^{1/p_{mn}}$ $(m = m(i)$ and $n = n(i))$, 0 (otherwise). Then it can easily seen that $x \in_0 c_2^P(p)$ but $x \notin_0 c_2^P$ by (4). This is a contradiction.

(iii) It follows from (i) and (ii). □

Theorem 2. ${}_0c_2^{PB}(p) =_0 c_2^{PB}$ *if and only if* $0 < \inf p_{mn} \leq \sup p_{mn} < \infty$.

Proof. It is rather easy to prove this theorem. Therefore we omit it. □

Theorem 3. ${}_0c_2^P(p)$ is a linear space if and only if $M = \lim_{N\to\infty} \sup_{m,n\geq N} p_{mn} < \infty$.

Proof. The sufficiency is obtained by Theorem 3 of [1] with $L = L' = 0$. Since the proof of the necessity is similar to that of the necessity of Theorem 3 in [1], we omit the detail of the proof. □

Theorem 4. *${}_0c_2^{PB}(p)$ is a linear space if and only if $H = \sup_{m,n\geq 1} p_{mn} < \infty$.*

Proof. Let $H < \infty$. Then ${}_0c_2^P(p)$ is a linear space from Theorem 3 since $\lim_{N\to\infty} \sup_{m,n\geq N} p_{mn} < \sup_{m,n\geq 1} p_{mn} < \infty$ and $\ell_2^\infty(p)$ is also a linear space from [2]. Hence ${}_0c_2^{PB}(p)$ is a linear space. □

The necessity can be similarly shown as in the necessity part of the proof of Theorem 4 in [1].

Theorem 5. *$\ell_2(p)$ is a linear space if and only if $\sup_{m,n\geq 1} p_{mn} < \infty$.*

Proof. This is a routine verification. So, we omit it. □

Theorem 6. *Let $H = \sup_{m,n\geq 1} p_{mn} < \infty$, $h = \inf_{m,n\geq 1} p_{mn}$ and $M = max(1, H)$. Then:*

(i) ${}_0c_2^{PB}(p)$ is a paranormed space with $g(x) = \sup_{m,n\geq 1} |x_{mn}|^{p_{mn}/M}$ if and only if $h > 0$.

(ii) ${}_0c_2^{PB}(p)$ is a complete linear metric space with the paranorm g defined in (i).

Proof. Necessity. Let ${}_0c_2^{PB}(p)$ be a paranormed space with $g(x) = \sup_{m,n\geq 1} |x_{mn}|^{p_{mn}/M}$ and suppose that $h = 0$. Let $\lambda \to 0$. If we define $x \in {}_0c_2^{PB}(p)$ such that $x_{mn} = 1$ ($m = 1$ and $n \geq 1$), 0 (otherwise). Then $\sup_{m,n\geq 1} |x_{mn}|^{p_{mn}/M} = 1$ since $|\lambda|^{p_{mn}} \leq |\lambda|^h = 1$ for every $\lambda \in (0, 1]$ and for every $m, n \geq 1$. Therefore we obtain that $g(\lambda x) = 1$. This contradicts our assumption.

Sufficiency. It is easy to see that g is a paranorm. Therefore we omit it.

(ii) Let (x^{kl}) be a Cauchy sequence in ${}_0c_2^{PB}(p)$, where $x^{kl} = (x^{kl}_{mn})_{m,n\in\mathbf{N}}$. Then for every $\varepsilon > 0$ $(0 < \varepsilon < 1)$, there exists $s \in \mathbf{N}$ such that

$$g(x^{kl} - x^{rt}) = \sup_{m,n\geq 1} \left|x^{kl}_{mn} - x^{rt}_{mn}\right|^{p_{mn}/M} < \varepsilon\,, \tag{5}$$

for all $k, l, r, t > s$. From this, we obtain $\left|x^{kl}_{mn} - x^{rt}_{mn}\right| < \varepsilon$ for each $m, n \geq 1$ and for all positive integers $k, l, r, t > s$. This implies that $(x^{kl}_{mn})_{k,l\in\mathbf{N}}$ is convergent to x_{mn} say, i.e.

$$\lim_{k,l\to\infty} x^{kl}_{mn} = x_{mn}\,, \tag{6}$$

for each fixed $m, n \geq 1$. Getting x_{mn}, we define $x = (x_{mn})$. From (5) we obtain

$$g(x^{kl} - x) = \sup_{m,n\geq 1} \left| x_{mn}^{kl} - x_{mn} \right|^{p_{mn}/M} < \varepsilon\,,$$

as $r, t \to \infty$, for $k, l > s$ by (6). This implies that $\lim_{k,l\to\infty} x^{kl} = x$ since $x^{kl} \in_0 c_2^{PB}(p)$ for each fixed $k, l \in \mathbf{N}$, $\lim_{m,n\to\infty} \left| x_{mn}^{kl} \right|^{p_{mn}/M} = 0$, and hence

$$\lim_{m,n\to\infty} |x_{mn}|^{p_{mn}/M} = 0\,,$$

as $k, l \to \infty$, by (6). Thus we obtain $x \in_0 c_2^{PB}(p)$. □

Theorem 7. *Let* $H = \sup_{m,n\geq 1} p_{mn} < \infty$ *and* $M = \max(1, H)$. *Then:*

(*i*) $\ell_2(p)$ *is a paranormed space with* $g(x) = \left[\sum_{m,n=1}^{\infty} |x_{mn}|^{p_{mn}} \right]^{1/M}$.

(*ii*) $\ell_2(p)$ *is a complete linear metric space with the paranorm* g *define in* (*i*).

Proof. (*i*) It is trivial that $g(\theta) = 0, g(-x) = g(x)$. For any $x, y \in \ell_2(p)$, we may write

$$g(x + y) \leq \left[\sum_{m,n=1}^{\infty} |x_{mn}|^{p_{mn}} \right]^{1/M} + \left[\sum_{m,n=1}^{\infty} |y_{mn}|^{p_{mn}} \right]^{1/M} = g(x) + g(y)$$

by (1). For any complex number λ, we have $g(\lambda x) \leq \max(1, |\lambda|)g(x)$. Therefore the function $(\lambda, x) \to \lambda x$ is continuous at $\lambda = 0, x = \theta$ and that, when λ is fixed, the function $x \to \lambda x$ is continuous at $x = \theta$. If x is fixed, and $\varepsilon > 0$, we can choose $K, N > 1$ such that

$$R(x) = \sum_{m=1}^{K} \sum_{n=N+1}^{\infty} |x_{mn}|^{p_{mn}} + \sum_{m=K+1}^{\infty} \sum_{n=1}^{N} |x_{mn}|^{p_{mn}} + \sum_{m=K+1}^{\infty} \sum_{n=N+1}^{\infty} |x_{mn}|^{p_{mn}} < \varepsilon/2\,.$$

Thus $R(\lambda x) < \varepsilon/2$ since $|\lambda| < 1$ and $\delta > 0$, so that $|\lambda| < \delta$ gives

$$\sum_{m,n=1}^{K,N} |\lambda x_{mn}|^{p_{mn}} < \varepsilon/2.$$

Therefore $|\lambda| < \min\,(1, \delta)$ implies that $g(\lambda x) < \varepsilon$. Thus the function $\lambda \to \lambda x$ is continuous at $\lambda = 0$ and so $\ell_2(p)$ is a paranormed space.

(ii) Let (x^{kl}) be a Cauchy sequence in $\ell_2(p)$, where $x^{kl} = (x^{kl}_{mn})_{m,n\in\mathbf{N}}$. Then for every $\varepsilon > 0$ $(0 < \varepsilon < 1)$, there exists a positive integers $N = N(\varepsilon)$ such that

$$g(x^{kl} - x^{rt}) = \left[\sum_{m,n=1}^{\infty} \left|x^{kl}_{mn} - x^{rt}_{mn}\right|^{p_{mn}}\right]^{1/M} < \varepsilon \tag{7}$$

for all $k, l, r, t > N$. From this we obtain

$$\left|x^{kl}_{mn} - x^{rt}_{mn}\right| \leq g(x^{kl} - x^{rt}) < \varepsilon\,,$$

for every fixed $m, n \in \mathbf{N}$ and for every positive integers $k, l, r, t > N$. Whence $(x^{kl}_{mn})_{k,l\in\mathbf{N}}$ is a Cauchy sequence in $\mathbf{C}$ for every fixed $m, n \in \mathbf{N}$. Since every Cauchy sequence in $\mathbf{C}$ is convergent, the sequence $(x^{kl}_{mn})_{k,l}$ is convergent to x_{mn} say, namely

$$\lim_{k,l\to\infty} x^{kl}_{mn} = x_{mn}\,, \tag{8}$$

for each fixed $m, n \in \mathbf{N}$. Getting x_{mn}, we define $x = (x_{mn})$. We may write $\sum_{m,n=1}^{P,Q} \left|x^{kl}_{mn} - x^{rt}_{mn}\right|^{p_{mn}} < \varepsilon^M$ $(P, Q = 1, 2, ...)$, for $k, l, r, t > N$ by (7). Letting $r, t \to \infty$ and then $P, Q \to \infty$ we obtain

$$\sum_{m,n=1}^{\infty} \left|x^{kl}_{mn} - x_{mn}\right|^{p_{mn}} < \varepsilon^M\,,$$

for $k, l > N$, by (8). Hence we have $g(x^{kl} - x) < \varepsilon$ for $k, l > N$ and this completes the proof. □

4. Dual Spaces

Definition 2. Let X be a nonempty subset of w^2. Then we define:

$$X^\alpha = \left\{(y_{mn}) : \sum_{m,n=1}^{\infty} |x_{mn}y_{mn}| < \infty \text{ for every } x \in X\right\},$$

$$X^\beta = \left\{(y_{mn}) : \sum_{m,n=1}^{\infty} x_{mn}y_{mn} \text{ converges for every } x \in X\right\},$$

$$X^\gamma = \left\{(y_{mn}) : \sup_{M,N\geq 1} \left|\sum_{m,n=1}^{M,N} x_{mn}y_{mn}\right| < \infty \text{ for every } x \in X\right\}.$$

We call that $X^{\alpha}, X^{\beta}, X^{\gamma}$ are α-, β- and γ- dual of X, respectively. α-dual for double sequence spaces is defined by M. Gupta and P.K. Kamptan [3]. It is well known that $X^{\alpha} \subset X^{\beta}$ and $X^{\alpha} \subset X^{\gamma}$, but $X^{\beta} \subset X^{\gamma}$ does not hold since the sequences of partial sums of a double convergent series need not to be bounded.

Lemma 1. *Let $X, Y \subset w^2$. If $X \subset Y$, then $Y^{\eta} \subset X^{\eta}$, where $\eta = \alpha$, β or γ.*

Proof. The proof is trivial. Therefore, we omit it. □

The space $M_0^2(p)$ is defined by

$$M_0^2(p) = \bigcup_{N \in \mathbf{N}-\{1\}} \left\{ x = (x_{mn}) : \sum_{m,n=1}^{\infty} |x_{mn}| \, N^{-1/p_{mn}} < \infty \right\}.$$

Furthermore, it was shown that $M_0^2(p) = \ell_2^1$ if and only if $\inf_{m,n \geq 1} p_{mn} > 0$ [1].

Theorem 8. *(i) $M_0^2(p) \subset [{}_0c_2^{PB}(p)]^{\beta}$ if and only if $h = \inf_{m,n \geq 1} p_{mn} > 0$,*

(ii) $[{}_0c_2^{PB}(p)]^{\beta} \subset M_0^2(p)$,

(iii) $[{}_0c_2^{PB}(p)]^{\beta} = M_0^2(p)$ if and only if $h = \inf_{m,n \geq 1} p_{mn} > 0$.

Proof. (i) Let $h > 0$ and $x \in M_0^2(p)$. Using Lemma 1, we obtain from Theorem 7 in [1] that $[c_2^{PB}(p)]^{\beta} = M_0^2(p) \subset [{}_0c_2^{PB}(p)]^{\beta}$.

Let $M_0^2(p) \subset [{}_0c_2^{PB}(p)]^{\beta}$ and $h = 0$. Then there are two cases:

(a) There exist strictly increasing sequence $(m(i))$ of positive integers and $n(1) < n(2) < \ldots < n(k_0)$ for some fixed $k_0 \in \mathbf{N}$ such that $p_{m(i),n(j)} < 1/i$ for all positive integers i and for $1 \leq j \leq k_0$ (or there exist strictly increasing sequence $(n(j))$ of positive integers and $m(1) < m(2) < \ldots < m(k_0)$ for some fixed $k_0 \in \mathbf{N}$ such that $p_{m(i),n(j)} < 1/j$ for all positive integers j and for $1 \leq i \leq k_0$) or

(b) There exist strictly increasing sequences $(m(i))$and $(n(j))$ of positive integers such that

$$p_{m(i),n(j)} < (i+j)^{-1} \tag{9}$$

for all positive integers i, j.

Now suppose that (9) holds for the strictly increasing sequences $(m(i))$ and $(n(j))$. If we consider the sequence (x_{mn}), where $x_{mn} = i^{-2}.N^{1/p_{mn}}$ $(m = m(i)$ and $n = n(1))$, 0 (otherwise) for any $N \in \mathbf{N} - \{1\}$, then it is clear that $x \in M_0^2(p)$. Let us define the sequence (y_{mn}), where $y_{mn} = i^2$ $(m = m(i)$ and $n = n(1))$, 0 (otherwise). It is easy to see that $y \in {}_0c_2^{PB}(p)$ by (9). By using (9) we have

$$\left| \sum_{m,n=1}^{\infty} x_{mn} y_{mn} \right| = \sum_{i=1}^{\infty} N^{1/p_{m(i),n(1)}} > \sum_{i=1}^{\infty} N^{1+i} = \infty,$$

whence contrary to $M_0^2(p) \subset [{}_0c_2^{PB}(p)]^{\beta}$.

The proof of (a) is similar to that of (b). Therefore we omit it.

(ii) Let $x\in [{}_0c_2^{PB}(p)]^\beta$ but $x\notin M_0^2(p)$. Then there exist strictly increasing sequences $(l\,(i))$ of positive integers and $k(1) < k(2) < ... < k(j_0)$ for some fixed $j_0 \in \mathbf{N}$ such that

$$M_{ij} = \sum_{m=l(i-1)+1}^{l(i)} \sum_{n=k(j-1)+1}^{k(j)} |x_{mn}|\,(i+j)^{-1/p_{mn}} > 1 \tag{10}$$

for all positive integers i and for $1 \le j \le j_0$, for some $j_0 \in \mathbf{N}$, where $l(0) = 0$ and $k(0) = 0$ (or there exist strictly increasing sequence $(k(j))$ of positive integer and $l(1) < l(2) < ... < l(i_0)$ for some fixed $i_0 \in \mathbf{N}$ such that (10) holds for all positive integers j and $1 \le i \le i_0$ for some $i_0 \in \mathbf{N}$).

Let (y_{mn}) be the sequence defined by $y_{mn}= (i+j)^{-1/p_{mn}}\ sgn\ x_{mn}$ $(l(i-1)+1 \le m \le l\,(i)$ and $k(j-1)+1 \le n \le k(j)), 0$ (otherwise) for all positive integers i, when $1 \le j \le j_0$ for some fixed $j_0 \in \mathbf{N}$ (or for all positive integers j, when $1 \le i \le i_0$ for some $i_0\in\mathbf{N}$). Then it is easy to see that $y\in {}_0c_2^{PB}(p)$ but

$$\left| \sum_{m,n=1}^{\infty} x_{mn} y_{mn} \right| > j_0 \sum_{i=1}^{\infty} 1$$

from (10). Hence we obtain that $x\notin[{}_0c_2^{PB}(p)]^\beta$, which contradicts to the assumption that $x\in[{}_0c_2^{PB}(p)]^\beta$.

(b) Then there exists strictly increasing sequences $(l\,(i))$ and $(k(j))$ such that (10) holds for all positive integers $i, j > 0$, where $l(0) = 0$ and $k(0) = 0$.

The proof of this case is similar to that of (a).

(iii) Follows from *(i)* and *(ii)*. □

Theorem 9. *(i)* $[{}_0c_2^{PB}(p)]^\alpha = M_0^2(p)$ *if and only if* $\inf p_{mn} > 0$.

(ii) $[{}_0c_2^{PB}(p)]^\gamma = M_0^2(p)$ *if and only if* $\inf p_{mn} > 0$.

Proof. The proof of *(i)* and *(ii)* is similar to that of Theorem 9. Therefore we omit it. □

Corollary. $[{}_0c_2^{PB}(p)]^\alpha = [{}_0c_2^{PB}(p)]^\beta = [{}_0c_2^{PB}(p)]^\gamma = \ell_2^1$ *if and only if* $\inf p_{mn} > 0$ *from Lemma 1 in [1].*

Theorem 10. *Let* $p_{mn} > 1$ *for every* $m, n \in \mathbf{N}$ *and* $\frac{1}{p_{mn}} + \frac{1}{q_{mn}} = 1$. *Then:*

(i) $[\ell_2(p)]^\beta = M_2(p)$,

(ii) $[\ell_2(p)]^\alpha = M_2(p)$,

(iii) $[\ell_2(p)]^\gamma = M_2(p)$,

where $M_2(p) = \bigcup_{N\in\mathbf{N}-\{1\}} \left\{ x = (x_{mn}) : \sum_{m,n=1}^{\infty} |x_{mn}|^{q_{mn}} N^{-q_{mn}/p_{mn}} < \infty \right\}$.

Proof. Let $x \in M_2(p)$ and $y \in \ell_2(p)$. Using the inequality $|x_{mn}y_{mn}| \leq |x_{mn}|^{q_{mn}} + |y_{mn}|^{p_{mn}}$, we obtain

$$\left|\sum_{m,n=1}^{\infty} x_{mn}y_{mn}\right| \leq \sum_{m,n=1}^{\infty} |x_{mn}|^{q_{mn}} N^{-q_{mn}/p_{mn}} + N \sum_{m,n=1}^{\infty} |y_{mn}|^{p_{mn}} .$$

This gives us that $x \in [\ell_2(p)]^{\beta}$.

Let $x \in [\ell_2(p)]^{\beta}$ but $x \notin M_2(p)$. Then there are two cases:

(a) There exist strictly increasing sequences $(l\,(i))$ and $(k(j))$ such that

$$M_{ij} = \sum_{m=l(i-1)+1}^{l(i)} \sum_{n=k(j-1)+1}^{k(j)} |x_{mn}|^{q_{mn}} (i.j)^{-q_{mn}/p_{mn}} > 1\,, \tag{11}$$

where $l(0) = 0$ and $k(0) = 0$. Now, let us define the sequence $y = (y_{mn}\ )$, by

$$y_{mn} = M_{ij}^{-1}(i.j)^{-q_{mn}} |x_{mn}|^{q_{mn}^{-1}} \operatorname{sgn} x_{mn}$$

for $l(i-1)+1 \leq m \leq l\,(i)$ and $k(j-1)+1 \leq n \leq k(j)(i,j > 0)$. Using (11), we obtain that

$$\sum_{m=l(i-1)+1}^{l(i)} \sum_{n=k(j-1)+1}^{k(j)} |y_{mn}|^{p_{mn}} \leq 1/(i.j)^2\,,$$

for all $i, j > 0$. Hence $(y_{mn}) \in \ell_2(p)$. But $\sum\limits_{m,n=1}^{\infty} x_{mn}y_{mn}$ is divergent, since

$$\sum_{m=l(i-1)+1}^{l(i)} \sum_{n=k(j-1)+1}^{k(j)} x_{mn}y_{mn} = \frac{1}{i.j}\,,$$

i.e. $x \notin [\ell_2(p)]^{\beta}$. This contradicts the assumption. Thus $[\ell_2(p)]^{\beta} = M_2(p)$.

(b) There exist strictly increasing sequences $(l\,(i))$ of positive integers and $k(1) < k(2) < \ldots < k(j_0)$ such that (11) holds for all positive integers i, when $1 \leq j \leq j_0$ for some fixed $j_0 \in \mathbf{N}$, where $l(0) = 0$ and $k(0) = 0$ (or there exist strictly increasing sequence $(k(j))$ of positive integer and $l(1) < l(2) < ... < l(i_0)$ such that (11) holds for all positive integers j, when $1 \leq i \leq i_0$ for some $i_0 \in \mathbf{N}$, where $l(0) = 0$ and $k(0) = 0$).

The proof of this case is similar to that of (a).

(ii) We omit it.

(iii) We omit it. □

Theorem 11. *Let* $0 < h = \inf\limits_{m,n\geq 1} p_{mn} \leq p_{mn} \leq 1$ *for every* $m, n \in \mathbf{N}$*. Then:*

(*i*) $[\ell_2(p)]^\beta = \ell_2^\infty(p)$,
(*ii*) $[\ell_2(p)]^\alpha = \ell_2^\infty(p)$,
(*iii*) $[\ell_2(p)]^\gamma = \ell_2^\infty(p)$.

Proof. (*i*) To prove that $[\ell_2(p)]^\beta \subset \ell_2^\infty(p)$, let $x \in [\ell_2(p)]^\beta$ and suppose that $x \notin \ell_2^\infty(p)$. Then there are two cases:

(a) There exist strictly increasing sequences $(m(i))$ and $(n(j))$ such that

$$\left|x_{m(i),n(j)}\right|^{p_{m(i)n(j)}} > (i+j)^3 \,, \tag{12}$$

for every $i, j > 0$. If we consider the sequences $y = (y_{mn})$, defined by $y_{mn} = (x_{m(i),n(j)})^{-1}$ ($m = m(i)$ and $n = n(j)$), 0 (otherwise). Then $y \in \ell_2(p)$ since

$$\sum_{m,n=1}^{\infty} |y_{mn}|^{p_{mn}} < \sum_{i,j=1}^{\infty} (i+j)^{-3} < \infty \,,$$

from (12). But $x \notin [\ell_2(p)]^\beta$ since

$$\sum_{m,n=1}^{\infty} x_{mn} y_{mn} = \sum 1.$$

This contradicts to our assumption. Hence $x \in \ell_2^\infty(p)$.

b) There exist strictly increasing sequences $(m(i))$ of positive integers and $n(1) < n(2) < ... < n(j_0)$ such that (12) holds for all positive integers i, when $1 \le j \le j_0$ for some fixed $j_0 \in \mathbf{N}$ (or there exist strictly increasing sequences $(n(j))$ of positive integers and $m(1) < m(2) < ... < m(i_0)$ such that (12) holds for all positive integers j, when $1 \le i \le i_0$ for some fixed $i_0 \in \mathbf{N}$).

The proof of this case is similar to that of (a).

Now let $x \in \ell_2^\infty(p)$ and $y \in \ell_2(p)$. Thus there exists some positive real number K such that $|x_{mn}|^{p_{mn}} \le K$ for every $m, n \in \mathbf{N}$. Furthermore we can find an integer $N \in \mathbf{N} - \{1\}$ such that

$$\sum_{m,n=1}^{\infty} \left|N^{-1} y_{mn}\right|^{p_{mn}} \le K^{-1} \,,$$

since $\ell_2(p)$ is a linear space. Hence we can write $\left|N^{-1} y_{mn}\right|^{p_{mn}} \le K^{-1}$ for every $m, n \in \mathbf{N}$. Using these, we obtain $\left|N^{-1} y_{mn} x_{mn}\right|^{p_{mn}} \le 1$ for every $m, n \in \mathbf{N}$. Hence, we obtain $\left|N^{-1} y_{mn} x_{mn}\right| \le \left|N^{-1} y_{mn} x_{mn}\right|^{p_{mn}} \le 1$ for every $m, n \in \mathbf{N}$. Therefore $\sum\limits_{m,n=1}^{\infty} x_{mn} y_{mn}$ converges, since

$$\left|N^{-1}\sum_{m,n=1}^{\infty} x_{mn}y_{mn}\right| \leq \sum_{m,n=1}^{\infty} \left|N^{-1}y_{mn}x_{mn}\right|^{p_{mn}}$$

$$< \sup_{m,n\geq 1} |x_{mn}|^{p_{mn}} \sum_{m,n=1}^{\infty} \left|N^{-1}y_{mn}\right|^{p_{mn}} < \infty.$$

This implies that $x \in [\ell_2(p)]^{\beta}$ which completes the proof of (i).

The proof of (ii) and (iii) are similar to that of (i). Therefore we omit it. □

Remark. If we take $p_{mn} = 1$ for all $m, n \in \mathbf{N}$, then we obtain $[\ell_2]^{\eta} = \ell_2^{\infty}$ and $[{}_0c_2^{PB}]^{\eta} = \ell_2^1$, where $\eta = \alpha$, β or γ.

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