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THE NON-EXISTENCE OF RADIALLY ACTING FREDHOLM INTEGRAL OPERATORS ON THE ANALYTIC FUNCTION SPACE $\mathbf{A_2(D)}$ OF THE OPEN UNIT DISC D

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Abstract: It is herein demonstrated that radially acting Fredholm integral operators $(Kf)(re^{i\theta}) = \int_0^1 K(r,r',\theta)f(r'e^{i\theta})e^{i\theta}dr'$ on the analytic function space $A_2(D)$, $K(r,r',\theta)$ having uniformly bounded double norm $|||K(\cdot,\cdot,\theta)|||$ on the unit square of $\mathbb{R}^2$ in the sense of $L_2[0,1]$, can only be given by a θ-parameter family of L_2-Volterra kernels $K(r,r',\theta)$ - i.e. $K(r,r',\theta) = 0$ $(0 \le r' \le r \le 1,\ -\pi \le \theta \le \pi)$ - and hence determine a quasi-nilpotent operator $K \in \mathcal{L}(A_2(D))$. Representation in terms of the sesquilinear tensor product $A_{s2}(D) \oslash L_2[0,1]$ - i.e. $K(r,r',\theta) = \sum_{n=0}^{\infty} (\cdot^n \oslash k_n)(r,r',\theta)$ - is shown with $|||\ \ |||_s$-convergence if and only if $K(r,r',\theta)$ is uniformly square integrable on the triangle $\Delta \equiv \{(r,r') : 0 \le r' \le r \le 1\}$ or equivalently $(\theta \mapsto K(\cdot,\cdot,\theta)) \in C([-\pi,\pi], L_2(\Delta))$. Uniform square integrability in terms of "when do boundary values of $H(D)$-functions belong to the Banach algebra Λ_α or Λ_*" is discussed. Radial Fredholm and Hammerstein integral equations are solved in $A_{s2}(D)$ and $A_2(D)$. Finally, the holomorphic extension from $[0,1]$ to all of $\overline{D}$ of the solutions of ordinary linear differential equations, whose coefficients are restrictions of $H(\overline{D})$-functions to the interval $[0,1]$, is explicitly given.

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1. Introduction

The topic dealt within this article is the explicit form assumed by Lebesgue measurable kernels $K : [0,1]^2 \times [-\pi,\pi) \to \mathbb{C}$ satisfying

$$|||K|||_s \equiv \sup_{-\pi\leq\theta\leq\pi} |||K(\cdot,\cdot,\theta)||| \quad (|||\cdot||| \equiv [\int_0^1\int_0^1 |\cdot(r,r')|^2 dr' dr]^{1/2}) \tag{1}$$

and inducing a radially acting Fredholm integral operator $K \in \mathcal{L}(A_2(D))$ with

$$(Kf)(re^{i\theta}) = \int_0^1 K(r,r',\theta) f(r'e^{i\theta}) e^{i\theta} dr' \text{ a.e. in } r \text{ for each } \theta \tag{2}$$

θ $(-\pi \leq \theta \leq \pi)$, where D is the open unit disc of $\mathbb{C}$ and $\mathcal{L}(X,Y)$ stands for the Banach space of continuous linear transformations from the normed linear space X into the Banach space Y with $\mathcal{L}(X) \equiv \mathcal{L}(X,X)$. The set of these kernels shall henceforth be designated $(\mathfrak{V}\mathfrak{K}_2)(D)$.

Radially acting Fredholm integral operators on the Banach space $H_p(\Pi_+)$ for $1 < p < \infty$ with $(Kf)(re^{i\theta}) = \int_0^\infty K(r,r',\theta) f(r'e^{i\theta}) e^{i\theta} dr'$ a.e. in r for each θ $(0 < \theta < \pi)$, $K(r,r',\theta)$ Lebesgue measurable on $(0,\infty)^2 \times (0,\pi)$ and $\| K(r,\cdot,\theta) \|_{L'_p(0,\infty)}$ defining a θ-parameter family of $L_p(0,\infty)$-functions with uniformly bounded $L_p(0,\infty)$-norm $(p' = p(p-1)^{-1})$ have been studied by Clasine van Winter [13] for $p = 2$ and [7] for $1 < p < \infty$. Originating from scattering theory $(p = 2)$, they have been investigated by means of: interpreting the upper half plane Π_+ of $\mathbb{C}$ as a θ-parameter family of rays $(0,\infty)e^{i\theta}(0 < \theta < \pi)$ emanating from the origin, whose boundary is the disjoint union $\partial\Pi_+ =$

$(0,\infty)e^{i\pi} \ \cup \ \{0\} \ \cup \ (0,\infty)e^{i0}$, utilizing the unique angular limits of $H_p(\Pi_+)$-functions in the $L_p(0,\infty)$-sense as $\theta \to 0^+$ and π^- and deriving from $K(r,r',\theta)$ two unique Fredholm integral operators on $L_p(0,\infty)$ with kernels of finite double norm ([5], Abschnitt 11; [15], Chapters 9 and 13) as $\theta \to 0^+$ and π^-. The Mellin transform version of the Paley-Wiener Theorem and properties of the Poisson kernel for the cases $p = 2$ and $1 < p < \infty$ respectively lead to a trace of kernel $K(r,r',\theta)$, which trace is independent of θ. This trace enables the construction of the modified Fredholm determinant and minors of $K(r,r',\theta)$ and hence, the $H_p(\Pi_+)$-solution of the radial Fredholm integral equation $f(re^{i\theta}) = g(re^{i\theta}) + \lambda \int_0^\infty K(r,r',\theta) f(r'e^{i\theta}) e^{i\theta} dr'$ a.e. in r for each $\theta (0 < \theta < \pi)$ for unknown f.

Radially acting Fredholm integral operators on the analytic function space $A_2(D)$, or for that matter, on the Banach space $A_{s2}(D)$ of $H(D)$-functions f satisfying

$$\| f \|_s \equiv \sup_{-\pi \le \theta \le \pi} \| f(\cdot e^{i\theta}) \|_{L_2[0,1]} \ < \ \infty, \tag{3}$$

as defined by relations (1) and (2) pose a different problem, see [8]. ∂D, namely the unit circle of $\mathbb{C}$, is not the angular limit of rays emanating from the origin and even more disheartening, $A_{s2}(D)$- as well as $A_2(D)$-functions generally do not posses boundary values. Consequently, the aforementioned techniques for the radial $H_p(\Pi_+)$-"Fredholm integral equation" fail for the radial $A_2(D)$-"Fredholm integral equation"

$$f(re^{i\theta}) = g(re^{i\theta}) + \lambda \int_0^1 K(r,r',\theta) f(r'e^{i\theta}) e^{i\theta} dr' \text{ a.e. in } r \text{ for each } \theta \tag{4}$$

$(-\pi \le \theta \le \pi)$ for unknown f.

The simplest kinds of $(\mathfrak{V}\mathfrak{K}_2)(D)$-kernels are those arising from the sesquilinear tensor product $A_{s2}(D) \oslash L_2[0,1]$ with

$$(g \oslash k)(r,r',\theta) \equiv g(re^{i\theta}) \overline{k}(r'/r) \chi_{[0,r]}(r') \ (|||g \oslash k||| \le \| g \|_s \| k \|), \tag{5}$$

where χ_M is the characteristic function of the set M. It is established in this manuscript that $(\mathfrak{V}\mathfrak{K}_2)(D)$-kernels assume the form

$$K(r,r',\theta) = \sum_{n=0}^{\infty} (\cdot^n \oslash k_n)(r,r',\theta) \ \ (\sum_{n=0}^{\infty} (2n+2)^{-1} \| k_n \|^2 < \infty) \tag{6}$$

a.e. in (r,r') on $\square \equiv [0,1]^2$. $||| \ \ |||_s$-approximation of $K(r,r',\theta)$ by means of these $A_{s2}(D) \oslash L_2[0,1]$-kernels is possible if and only if $K(r,r',\theta)$ is uniformly square integrable on triangle $\Delta \equiv \{(r,r') : 0 \le r' \le r \le 1\}$ in $\theta(-\pi \le \theta \le \pi)$. Otherwise, it can only be concluded that

$$\lim_{N\to 0} \| K(\cdot,\cdot,\theta) - \sum_{n=0}^{N} (\cdot^n \oslash k_n)(\cdot,\cdot,\theta) \|_{L_2(E)} = 0 \text{ uniformly in } \theta(-\pi \le \theta \le \pi)$$

for Lebesgue-measurable subsets $E \subset \Delta$ satisfying $\overline{E} \subseteq \Delta \setminus (\{1\} \times [0,1])$. Conditions as to when an $H(D)$-function has its boundary value function in the Banach spaces Λ_α and Λ_*, yield conditions as to when a $(\mathfrak{V}\mathfrak{K}_2)(D)$-kernel is uniformly square integrable. Estimates of iterates of $(\mathfrak{V}\mathfrak{K}_2)(D)$-kernels are obtained, whereby the quasi-nilpotency ([5], p. 49) of $(\mathfrak{V}\mathfrak{K}_2)(D)$-kernels is established. In conclusion, radial Fredholm as well as Hammerstein integral equations involving $(\mathfrak{V}\mathfrak{K}_2)(D)$-kernels are solved and the holomorphic extension of the solution of boundary value problems on $[0,1]$ to all of $\overline{D}$ is explicitly given, provided the coefficients of the linear differential operator are $H(\overline{D})$-functions restricted to $[0,1]$.

2. Preliminary Remarks

The analytic function space $A_2(D)$ of all $H(D)$-functions having finite $L_2(D)$-norm is a Hilbert space with the λ-parameter family of inner products

$$< f|g >_\lambda \equiv \int_0^1 r^\lambda \int_{-\pi}^{\pi} f(re^{i\theta})\overline{g(re^{i\theta})}d\theta dr$$

$$= (2\pi)^{-1} \sum_{n=0}^{\infty} (2n+1+\lambda)^{-1} c_n(f)\overline{c_n(g)}, \quad (7)$$

where $c_n(f)$ is the n-th McClaurin series coefficient of f. $< | >_\lambda$ ($\| \quad \|_\lambda$) for $\lambda = 0$ and 1 are designated as the alternate and conventional inner products (norms) of $A_2(D)$ respectively. They become unitarily and topologically equivalent under the isomorphism $U(Uf \equiv \sum_{n=0}^{\infty} (n+1)^{1/2}(n+1/2)^{-1/2}c_n(f)\cdot^n)$ and norm inequalities

$$\| f \|_1 \le \| f \|_0 \le 2^{1/2} \| f \|_1 \quad (f \in A_2(D)). \quad (8)$$

Forthcoming, it shall be necessary to deal with the expression $\| \cdot^{1/2} f \|_s$ for $f \in A_{s2}(D)$. Taking the inner product of the McClaurin series of f with $\cdot^n$ yields $c_n(f) < \cdot^n|\cdot^n >_1 = \int_{-\pi}^{\pi} \int_0^1 r^{1/2} f(re^{i\theta}) r^{n+1/2} e^{in\theta} dr d\theta$, and thus $|c_n(f)| \le$

$(2n+2)^{1/2} \parallel \cdot^{1/2} f \parallel_s$. For any β $(0<\beta<1)$ however,

$$\parallel f(\cdot e^{i\theta}) \parallel \leq \parallel \sum_{n=0}^{\infty} c_n(f)(\cdot e^{i\theta})^n \parallel_{L_2[0,\beta]} + \beta^{1/2} \parallel \cdot^{1/2} f(\cdot^{i\theta}) \parallel_{L_2[\beta,1]}$$

$$< \gamma_\beta \parallel \cdot^{1/2} f \parallel_s,$$

where $\gamma_\beta = \beta^{-1/2}[1+\sum_{n=0}^{\infty}(n+1)^{1/2}\ (n+1/2)^{-1/2}\beta^{n+1}](0<\beta<1)$; therefore, $\gamma \equiv \inf_{0<\beta<1} \gamma_\beta$ yields

$$\parallel \cdot^{1/2} f \parallel_s \leq \parallel f \parallel_s \leq \gamma \parallel \cdot^{1/2} f \parallel_s \quad (f \in A_{s2}(D)). \tag{9}$$

Linear manifold $A_{c2}(D) \equiv \{f \in H(D) : (\theta \mapsto f(\cdot e^{i\theta})) \in C([-\pi,\pi], L_2[0,1])\}$ is a closed subspace of $A_{s2}(D)$, because: if f_n is a $\parallel \cdot \parallel_s$-Cauchy sequence of $A_{c2}(D)$-functions, then $\parallel f_n - f_m \parallel_0 \leq (2\pi)^{1/2} \parallel f_n - f_m \parallel_s$ ((3) and (7)) entails $f_n \to f \in A_2(D)$ and further, $\parallel f(\cdot e^{i\theta}) - f(\cdot e^{i\psi}) \parallel \leq 2 \parallel f - f_n \parallel_s + \parallel f_n(\cdot e^{i\theta}) - f_n(\cdot e^{i\psi}) \parallel$ - i.e. $\theta \mapsto f(\cdot e^{i\theta})$ is continuous. $A_{c2}(D)$ is actually a proper subspace of $A_{s2}(D)$, as shall follow from the counter-example four sections hence. Since polynomials belong to $A_{c2}(D)$, $A_{c2}(D)$ as well as $A_{s2}(D)$ are dense linear manifolds of $A_2(D)$. Moreover, the Müntz-Szász Theorem ([2], Band I., Seite 532) ensures that $\mathfrak{M}_\theta = \{f(\cdot e^{i\theta}) : f \in A_{c2}(D)\}$ is dense in $L_2[0,1]$ for each $\theta(-\pi \leq \theta \leq \pi)$.

The set of functions $P_n(\cdot) \equiv (2n+1)^{1/2}(n!)^{-1}[\cdot^n(1-\cdot)^n]^{(n)}(n \geq 0)$ and $(2\pi)^{-1/2}\ e^{in\cdot}(n \in \mathbb{N})$ are orthonormal bases of the Hilbert spaces $L_2[0,1]$ and $L_2[-\pi,\pi]$ respectively. By means of them, the following three functions of two and three variables

$$Q_{\mu\nu}(r,r') \equiv P_\mu(r)P_\nu(r')\ (\mu,\nu \geq 0), \quad R_{\mu n}(r,\theta) \equiv (2\pi)^{-1/2}P_\mu(r)e^{in\theta}$$

$$(\mu \geq 0, n \in \mathbb{N}) \text{ and } S_{\mu\nu n}(r,r',\theta) \equiv (2\pi)^{-1/2}Q_{\mu\nu}(r,r')e^{in\theta}\ (\mu,\nu \geq 0; n \in \mathbb{N})$$

become orthonormal bases of the Hilbert spaces $L_2(\Box), L_2([0,1]\times[-\pi,\pi])$ and $L_2(\mathcal{R})$ respectively, where $\mathcal{R} \equiv \Box \times [-\pi,\pi]$. $(\mathfrak{V}\mathfrak{K}_2)(D)$-kernels $K(r,r',\theta)$ have, in terms of the inner product $< K|L > \equiv \iiint_{\mathcal{R}} K(r,r',\theta)\overline{L(r,r',\theta)}dr'dr\ d\theta$, Fourier-expansions $K(r,r',\theta) = \sum_{\mu,\nu,n} < K|S_{\mu\nu n} > S_{\mu\nu n}(r,r',\theta)$ a.e. on $\mathcal{R}$, with $\sum_{\mu,\nu,n} | < K|S_{\mu\nu n} > |^2 = {\parallel K(\cdot,\cdot,\cdot) \parallel_{L_2(\mathcal{R})}}^2 \leq 2\pi{|||K|||_s}^2$, because $(\mathfrak{V}\mathfrak{K}_2)(D)$-kernels also belong to $L_2(\mathcal{R})$. Rearranging this summation results in

$$K(r,r',\theta) \quad = \quad \sum_{n=-\infty}^{\infty} K_n(r,r')(2\pi)^{-1/2}e^{in\theta} \text{ with} \tag{10}$$

$$K_n(r,r') \quad = \quad \sum_{\mu,\nu\geq 0} < K|S_{\mu\nu n} > Q_{\mu\nu}(r,r') \text{ and}$$

$\| K(\cdot,\cdot,\cdot) \|_{L_2(\mathcal{R})}{}^2 = \sum_{n=-\infty}^{\infty} |||K_n|||^2$ (Bessel equality), where $||| \cdot ||| \equiv \| \cdot \|_{L_2(\Box)}$. If $K(r,r',\theta)$ is momentarily expressed as $K_{r,r'}(\theta)$, then in terms of convolution ([1], Abschnitte 31. sund 32.) with $e^{iN\cdot}$ and the Dirichlet kernels $D_N(\cdot) = (\sin \cdot/2)^{-1} \sin(N+1/2)(\cdot)$ the N-th term and N-th partial sum of "trigonometric series (10) with $L_2(\Box)$-coefficients" are given, respectively by the formulas

$$(e^{iN\cdot} * K_{r,r'})(\theta) \quad = \quad (2\pi)^{-1} \int_{-\pi}^{\pi} e^{iN(\theta - t)} K_{r,r'}(t)dt, \tag{11}$$

$$(D_N * K_{r,r'})(\theta) \quad = \quad (2\pi)^{-1} \int_{-\pi}^{\pi} D_N(\theta - t) K_{r,r'}(t)dt$$

and also $\| K(r,r',\cdot) - (D_N * K_{r,r'})(\cdot) \| \to 0$ a.e. on $\Box$ as $N \to 0$. Invoking the Lebesgue Dominated Convergence Theorem ($\| D_N * K_{r,r'} \| \leq \| K(r,r',\cdot) \|$ a.e. on $\Box$) leads to $\lim_{N\to\infty} \| (D_N * K_{\cdot,\cdot})(\cdot) - K(\cdot,\cdot,\cdot) \|_{L_2(\mathcal{R})} = 0$.

3. The Form of $(\mathfrak{V}\mathfrak{K}_2)(D)$-Kernels

The ascertainment of the form of a $(\mathfrak{V}\mathfrak{K}_2)(D)$-kernel is accomplished by means of the interaction of the Hilbert spaces $L_2[0,1]$, $L_2(\Box)$, $L_2(\mathcal{R})$ and $A_2(D)$. Given arbitrary u, f and $\psi \in L_2[0,1]$, $A_2(D)$ and $L_\infty[-\pi,\pi]$, respectively and $\psi_m \equiv e^{-im\cdot}\psi$, the function $(u\overline{f}\psi)(r,r',\theta) \equiv u(r)\overline{f(re^{i\theta})}\ \psi(\theta)$ belongs to $L_2[0,1]\widehat{\otimes}L_2[0,1]\widehat{\otimes}L_2[-\pi,\pi] = L_2(\mathcal{R})$ ([14], Abschnitt 3.4) since $\| u\overline{f}\psi - \sum_{m=0}^{N} \overline{c_m(f)} u \otimes \cdot^m \otimes \psi_m \| \leq \| u \| \| \| f - \sum_{m=0}^{N} c_m(f) u \cdot^m \|_0 \| \psi \|_\infty \to 0$ as $N \to \infty$. If $\psi = 1_{-n}$, where $1(\theta) \equiv \chi_{[-\pi,\pi]}(\theta)$, in the inner product relation $< K|u\overline{f}\psi > = < Kf|u\otimes\psi >$, then the a.e. valid equation (10) allows each of these inner products to be expressed as

$$< K|u\overline{f}\psi > = \sum_{m=0}^{\infty} c_m(f) [\sum_{n=-\infty}^{\infty} < K_n|u \otimes \cdot^m >< 1_{-n}|\psi_m >]$$

$$\text{and } < Kf|u \otimes \psi > = \sum_{m=0}^{\infty} c_m(Kf) < \cdot^m|u >< 1_{-m}|\psi >$$

and further, the integral formula for the m-th Laurent series coefficient $c_m(Kf)$, which is zero for $m < 0$, yields

$$< \sum_{m=0}^{\infty} c_m(f) \int_0^1 K_{n-m-1}(\cdot, r')r'^m dr' | u >$$
$$= [\oint_{|z|=s} (Kf)(z) z^{-n-1} dz] < \cdot^n | u > \text{ counterclockwise with } 0 < s < 1. \quad (12)$$

Herein $f(z)$ is replaced by $z^m (c_n(\cdot^m) = \delta_{mn})$, thereafter equation (12) is multiplied by $\chi_E(s)$ (E a Lebesgue measurable subset of $(0,1)$ with $\overline{E} \subset (0,1)$) and integrated in the Lebesgue sense with respect to s on $[0,1]$. This results in

$$< \int_0^1 K_n(\cdot, r')r'^m dr' | u >$$
$$= [m_1(E)]^{-1} \int_E < \int_0^1 K_n(s, r')r'^m (\cdot/s)^{n+m+1} dr' | u > ds \quad (m \geq 0), \quad (13)$$

provided E has positive Lebesgue measure $m_1(E)$. The inherent reasons behind equation (13) are: equation (10) with its accompanying Bessel equality and the Tonelli-Hobson Theorem.

The term on the left of equation (13) is independent of all admissible subsets E of $(0,1)$, whereas the one on the right is the average of the integrand over each admissible subsets E of $(0,1)$. This integrand as a function of s is consequently a.e. equal to the expression on the left. Hence, to every $u \in L_2[0,1]$ correspond Lebesgue-measurable subsets $S_{nm}(u)$ of $(0,1)$ satisfying $m_1(S_{nm}(u)) = 1$ and

$$< s^{n+1} \int_0^1 K_n(\cdot, r')(sr')^m dr' | u > = < \int_0^1 \cdot^{n+1} K_n(s, r')(\cdot r')^m dr' | u > \quad (14)$$

for all $s \in S_{nm}(u) (m \geq 1)$. It should be brought out however, that conclusion (14) was achieved by factoring s^{-n-m-1} out of the integrand on the right of equality (13) and transferring it to the term on the left. Further, replacement of the variable of integration r' respectively with r'/s and r'/r on the left and right of equality (14) combined with sesquilinearity results in

$$< s^n K_n(\cdot, \cdot/s)\chi_{[0,s]} | u \otimes \overline{P} > = < \cdot^n K_n(s, \cdot/\cdot)\chi_{[o,\cdot]} | u \otimes \overline{P} > \quad (15)$$

for all polynomials P (over $\mathbb{C}$), provided $s \in S_n(u) = \bigcap_{m=0}^{\infty} S_{nm}(u)$. $S_n(u)$ is a Lebesgue measurable subset of $(0,1)$ with $m_1(S_n(u)) = 1$, where

$$(K_n(\cdot, \cdot/s)\chi_{[0,s]})(r, r') \text{ and } (\cdot^n K_n(s, \cdot/\cdot)\chi_{[0,\cdot]})(r, r')$$

stand for $K_n(r, r/s)\chi_{[0,s]}(r')$ and $r^n K_n(s, r'/r)\chi_{[o,r]}(r')$ respectively.

Continuing in similar way, namely choosing $u = P_\mu$ (μ-th Legendre polynomial) and letting $S_n = \bigcap_{\mu=0}^{\infty} S_n(P_\mu)$, has for the Lebesgue-measurable subsets S_n of $(0,1)$ the following consequences: $m_1(S_n) = 1$ and

$$< s^n K_n(\cdot, \cdot/s)\chi_{[0,s]}|P_\mu \otimes P_\nu >=< \cdot^n K_n(s, \cdot/\cdot)\chi_{[0,\cdot]}|P_\mu \otimes P_\nu > \quad (s \in S_n; \mu, \nu \geq 0).$$

Thus for the orthonormal basis $Q_{\mu\nu} = P_\mu \otimes P_\nu (\mu, \nu \geq 0)$ of the Hilbert space $L_2[0,1]\widehat{\otimes} L_2[0,1] = L_2(\Box)$

$$< s^n K_n(\cdot, \cdot/s)\chi_{[0,s]}|Q_{\mu\nu} >=< \cdot^n K_n(s, \cdot/\cdot)\chi_{[0,\cdot]}|Q_{\mu\nu} > (s \in S_n) \tag{16}$$

and writing the Bessel equality for the respective $L_2(\Box)$-functions in terms of these Fourier coefficients entails

$$|||s^n K_n(\cdot, \cdot/s)\chi_{[0,s]}|||^2 = |||\cdot^n K_n(s, \cdot/\cdot)\chi_{[0,\cdot]}|||^2 \quad (s \in S_n). \tag{17}$$

Substitution for the variable of integration r' respectively sr' and rr' in the integrals expressing the $L_2(\Box)$-norms of relation (17) reduces it to

$$s^{2n+1}|||K_n|||^2 = [\int_0^1 r^{2n+1} dr]||K_n(s, \cdot)||^2 \quad (s \in S_n), \tag{18}$$

which with the aid of notation (5) leads to the proof of the following result.

Theorem 1. *To every $(\mathfrak{V}\mathfrak{K}_2)(D)$-kernel $K(r, r', \theta)$ correspond unique $k_n \in L_2[0,1]$ $(n \geq 0)$ such that*

$$K(r, r', \theta) = \sum_{n=0}^{\infty} (\cdot^n \oslash k_n)(r, r', \theta) \; (\sum_{n=0}^{\infty} (2n+2)^{-1} \parallel k_n \parallel^2 \leq [|||K|||_s]^2) \tag{19}$$

a.e. in $(r, r') \in \Box$ for each θ $(-\pi \leq \theta \leq \pi)$.

Proof. Statement (18) for $n < 0$ implies $K_n(r, r') = 0$ on $\Box$, whereas for $n \geq 0$ it guarantees that $\{K_n(s, \cdot) : s \in S_n\}$ is an s-parameter family of $L_2[0,1]$-functions, whose norms are uniformly bounded by $(2n+2)^{1/2}|||K_n|||$. Moreover, out of $m_1(S_n) = 1$ follows that 1 is not an isolated point of $\overline{S_n}$, and thus there exists a unique $L_2[0,1]$-function k_n such that $K_n(s, \cdot) \xrightarrow{\omega} (2\pi)^{1/2}\overline{k_n}$ as $s \to 1^-$ through values $s \in S_n$, where $\xrightarrow{\omega}$ denotes weak convergence in the Hilbert space $L_2[0,1]$.

The uniqueness of k_n is established by assuming the existence of two strictly increasing sequences s_{lp} of elements from S_n satisfying: $\lim_{p\to\infty} s_{lp} = 1 (l = 1, 2)$ and $K_n(s_{lp}, \cdot) \xrightarrow{\omega} (2\pi)^{1/2}\overline{k_{ln}}$ in $L_2[0,1]$ as $p \to \infty (l = 1, 2)$. After setting $s = s_{lp} (l = 1, 2)$ in equation (16), noticing that both terms therein (16) have limit $< K_n|Q_{\mu\nu} >$ as $s \to 1^-(s \in S_n)$ and finally letting $p \to \infty$, it immediately follows that

$$< K_n|Q_{\mu\nu} >= (2\pi)^{1/2} < \cdot^n\overline{k_{ln}}(\cdot/\cdot)\chi_{[0,\cdot]}|Q_{\mu\nu} > (\mu,\nu \geq 0; l = 1,2), \qquad (20)$$

where $(\cdot^n\overline{k_{ln}}(\cdot/\cdot)\chi_{[0,\cdot]})(r,r') \equiv r^n\overline{k_{ln}}(r/r')\chi_{[0,r]}(r')(l = 1,2)$. Subtraction of the two expressions $(l = 1,2)$ in relation (20) from each other gives the identities $< Q_{\mu\nu}| \cdot^n (k_{2n} - k_{1n})(\cdot/\cdot)\chi_{[0,\cdot]} >= 0 \ \ (\mu,\nu \geq 0)$ or equivalently, through the use of $Q_{\mu\nu} = P_\mu \otimes P_\nu$,

$$\int_0^1 r^{n+1} < P_\nu(r\cdot)|k_{2n} - k_{1n} > P_\mu(r)dr = 0 \ \ (\mu,\nu \geq 0). \qquad (21)$$

Completeness of the Legendre polynomials P_n as an orthonormal basis of $L_2[0,1]$ as well as the Müntz-Szász Theorem ([2], Band I, Seite 532) imply that $< P_\nu(r\cdot)|k_{2n} - k_{1n} > (\nu \geq 0)$ are a.e. vanishing $C[0,1]$-functions of variable r. One has for $r = 1$ particularly that $< P_\nu|k_{2n} - k_{1n} >= 0$ for all $\nu \geq 0$, and consequently $k_{1n} = k_{2n}$.

Finally, the Riesz-Fischer Theorem in terms of the complete orthonormal set $Q_{\mu\nu}$ $(\mu,\nu \geq 0)$ of $L_2(\Box)$ and Fourier coefficients (20) (k_{ln} replaced with k_n) yield $K_n(r,r') = r^n\overline{k_n}(r'/r) \ \chi_{[0,r]}(r')$ in $L_2(\Box)$. These results are substituted into equation (10) and the Bessel equality ensuing therefrom, thus completing the proof. □

This theorem establishes $(\mathfrak{V}\mathfrak{K}_2)(D)$-kernels $K(r,r',\theta)$ as a θ-parameter family of Volterra kernels - i. e. $K(r,r',\theta) = 0$ for $0 \leq r < r' \leq 1 \ \ (-\pi \leq \theta \leq \pi)$, whence the "$\mathfrak{V}$" in the designation $(\mathfrak{V}\mathfrak{K}_2)(D)$ - and thereby reduces the action (2) of the radially acting linear integral operator K to

$$(Kf)(re^{i\theta}) = \int_0^r K(r,r',\theta)f(r'e^{i\theta})e^{i\theta}dr'$$

$$= \sum_{n=0}^\infty (re^{i\theta})^{n+1} < f(r\cdot e^{i\theta})|k_n > (re^{i\theta} \in D, f \in A_2(D)) \qquad (22)$$

with equality on D, instead of the somewhat awkward appendage of "a.e. in r for each θ $(-\pi \leq \theta \leq \pi)$", as well as

$$|||K(\cdot,\cdot,\theta)||| = \| K(\cdot,\cdot,\theta) \|_{L_2(\Delta)} \ (\Delta = \{(r,r') : 0 \leq r' \leq r \leq 1\}). \qquad (23)$$

4. Topological Structure of $(\mathfrak{V}\mathfrak{K}_2)(D)$

Through action (22) $(\mathfrak{V}\mathfrak{K}_2)(D)$-kernels $K(r,r',\theta)$ define a bounded linear operator K on $A_2(D)$ because of $\| Kf \|_0 \leq |||K|||_s \| f \|_0$ and equation (7). Therefore, the Banach algebra $\mathcal{L}(A_2(D))$ imposes on $(\mathfrak{V}\mathfrak{K}_2)(D)$ the normed algebra structures of

$$\begin{aligned}&(\alpha K+\beta L)(r,r',\theta)=\alpha K(r,r',\theta)+\beta L(r,r',\theta) \quad (\alpha,\beta\in\mathbb{C}),\\ &(KL)(r,r',\theta)=\int\limits_{[r',r]} K(r,r'',\theta)L(r'',r',\theta)e^{i\theta}dr'',\\ &|||KL|||_s\leq |||K|||_s|||L|||_s\,,\end{aligned} \tag{24}$$

for all $K(r,r'',\theta)$ and $L(r,r',\theta)\in(\mathfrak{V}\mathfrak{K}_2)(D)$, and note moreover that $[r',r]=\emptyset$ for $r<r'$. From representations $K=\sum\limits_{n=0}^{\infty}\cdot^n\oslash k_n$ and $L=\sum\limits_{n=0}^{\infty}\cdot^n\oslash l_n$ in the sense of Theorem 1, the following representations ensue:

$$\alpha K+\beta L=\sum_{n=0}^{\infty}\cdot^n\oslash[\overline{\alpha}k_n+\overline{\beta}l_n],\quad KL=\sum_{n=0}^{\infty}\cdot^n\oslash[kl]_n,$$

$$\text{where}\quad [kl]_n(t)=\sum_{\mu=0}^{n-1}<k_\mu\chi_{[t,1]}|\cdot^{n-1-\mu}l_{n-1-\mu}(t/\cdot)>. \tag{25}$$

Furthermore, since representation (19) has its origins in equation (10) with $L_2(\square)$-Fourier coefficient $K_n(r,r')=0$ for $n<0$, the "r,r'" need no longer be suppressed to appear as subscripts as for instance after relation (10), and thus formulas (11) have the handier $(A_{c2}(D)\oslash L_2[0,1])$-format of

$$(\cdot^N\oslash k_N)(r,r',\theta)=(e^{iN\cdot}*K)(r,r',\theta)\equiv(2\pi)^{-1}<K(r,r',\cdot)|e^{-iN(\theta-\cdot)}>,$$

$$\begin{aligned}(D_N*K)(r,r',\theta)&\equiv(2\pi)^{-1}<K(r,r',\cdot)|D_N(\theta-\cdot)>\\ &=\sum_{n=0}^{N}(\cdot^n\oslash k_n)(r,r',\theta).\end{aligned} \tag{26}$$

Theorem 2. *$(\mathfrak{V}\mathfrak{K}_2)(D)$ endowed with norm $|||\cdot|||_s$ is a Banach algebra.*

Proof. Every $|||\cdot|||_s$-Cauchy sequence $K_n(r,r',\cdot)$ of $(\mathfrak{V}\mathfrak{K}_2)(D)$ is such that $|||(K_{n'}-K_n)(\cdot,\cdot,\theta)|||\to 0$ uniformly in $\theta(-\pi\leq\theta\leq\pi)$ as $n,n'\to\infty$. Therefore, by completeness of $L_2(\Delta)$ there exists a θ-parameter family of Volterra kernels $K(r,r',\theta)$, Lebesgue-measurable on $\Delta\times[-\pi,\pi)$, for which $|||K-K_n|||_s\to 0$ as $n\to\infty$. If $f\in A_2(D)$, then $(Kf)(r,\theta)$ defined by equation (22) satisfies

$\| Kf - K_n f \|_0 \leq \| | |K - K_n| | |_s \| f \|_0 \to 0$ as $n \to \infty$. Hence, one clearly sees that $(Kf)(r, \theta) = (Kf)\ (re^{i\theta})$ – i.e. $Kf \in A_2(D)$ – and thereby completing the proof. □

Theorem 1 and Theorem 2 invariably raise the question of the density of $(L.H.)$ $(\{\cdot^n \oslash k_n : k_n \in L_2[0,1], n \geq 0\})$ in $(\mathfrak{V}\mathfrak{K}_2)(D)$, where $(L.H.)$ stands for "Linear Hull of $(\cdot)$". The answer is negative and the rationale hereto stems from the notion of uniform square integrability ([10], pp. 133-134). In particular, the θ-parameter family of $L_2(\Delta)$-functions $K_\theta(r, r')(\theta \in \Theta)$ is said to be uniformly square integrable on Δ, if the θ-parameter family of Borel-measures μ_θ on Δ with $\mu_\theta(E) \equiv \int\int_E |K_\theta(r, r')|^2 dr' dr$ is uniformly (in terms of $\theta \in \Theta$) absolutely continuous with respect to Lebesgue-measure m_2 of Δ ([10], Chapter 6). To make the application of this concept to $(\mathfrak{V}\mathfrak{K}_2)(D)$ practicable, the increasing sequence of triangles $\Delta_n \equiv \{(r, r') : 0 \leq r' \leq 1 - n^{-1}\}(n \geq 1)$, whose union is $\Delta \backslash (\{1\} \times [0, 1])$, is introduced as well as the constants

$$C_{nN} \equiv (1 - n^{-1})^{N+2}[1 - (1 - n^{-1})^2]^{-1/2} \quad \text{and}$$
$$\gamma_N(K) \equiv [\sum_{\nu=N+1}^{\infty} (2\nu + 2)^{-1} \| k_\nu \|^2]^{1/2} \quad (n \geq 1, N \geq 0), \tag{27}$$

which originate in the approximation of $K(r, r', \theta)$ in terms of expression (19) by means of the N-th partial sum (4) in the spaces $L_2(\Delta_n)$. Clearly, $\lim_{N\to\infty} C_{nN} = \lim_{N\to\infty} \gamma_N(K) = 0$ and thereby lays the groundwork for the proof of the following result.

Theorem 3. *$K(r, r', \theta) \in (\mathfrak{V}\mathfrak{K}_2)(D)$ is uniformly square integrable with respect to θ, if and only if $\lim_{n\to\infty} \| K(\cdot, \cdot, \theta) \|_{L_2(\Delta\backslash\Delta_n)} = 0$ uniformly in θ $(-\pi \leq \theta \leq \pi)$.*

Proof. The necessity is self-evident and for sufficiency, let E be any Lebesgue-measurable subset of Δ. $E \subset (E \cap \Delta_n) \cup (\Delta\backslash\Delta_n)$ and therefore the $L_2(E)$-norm of $K(\cdot, \cdot, \theta)$ does not exceed the sum of the $L_2(E \cap \Delta_n)$- and $L_2(\Delta\backslash\Delta_n)$-norms of the same. Replacing further the $L_2(E \cap \Delta_n)$-norm of $K(\cdot, \cdot, \theta)$ by the sums of the $L_2(\Delta_n)$-norm of $(K - D_N * K)(\cdot, \cdot, \theta)$ and $L_2(E)$-norms of $r^\nu k_\nu(r'/r)$ $(0 \leq \nu \leq N)$ results in

$$\| K(\cdot, \cdot, \theta) \|_{L_2(E)} \leq \| (K - D_N * K)(\cdot, \cdot, \theta) \|_{L_2(\Delta_n)} + \sum_{\nu=0}^{N} \| \cdot^n k_n(\cdot/\cdot) \|_{L_2(E)} + \quad \| K(\cdot, \cdot, \theta) \|_{L_2(\Delta\backslash\Delta_n)} .$$

This can be made less than ϵ independent of θ by choosing n sufficiently large so that the third term is less than $3^{-1}\epsilon$ (independent of θ), thereafter selecting a N such that $C_{nN}\gamma_N(K) \leq 3^{-1}\epsilon$ and noting that this $C_{nN}\gamma_N(K)$ is an upper bound of the first expression. The summation of the $L_2(E)$-norms of $r^\nu k_\nu(r'/r)$ $(0 \leq \nu \leq N)$ is less than $3^{-1}\epsilon$ if E is chosen such that $m_2(E) \leq \delta(\epsilon)$. Such $\delta(\epsilon)$ independent of θ exists, because the Lebesgue integrals of $|r^\nu k_\nu(r'/r)|^2$ over any Lebesgue-measurable subset $E \subset \Delta$ determine $N+1$ absolutely continuous Borel-measures on Δ, whereby the proof is completed. $\square$

With the aid of Theorem 3 the uniform square integrability of $(\mathfrak{V}\mathfrak{K}_2)(D)$-kernels can be reformulated without the cumbrous "uniform absolute continuity of a θ -parameter family of Borel measures μ_θ with respect to m_2".

Theorem 4. *$K(r, r', \theta) \in (\mathfrak{V}\mathfrak{K}_2)(D)$ is uniformly square integrable with respect to θ, if and only if the transformation $(\theta \mapsto K(\cdot, \cdot, \theta)) \in C([-\pi, \pi], L_2(\Delta))$.*

Proof. For the necessity one observes that $|||K(\cdot, \cdot, \theta) - K(\cdot, \cdot, \theta_0)|||$ does not exceed the sum of the $L_2(\Delta_n)$-norm of $[K(\cdot, \cdot, \theta) - K(\cdot, \cdot, \theta_0)]$ and the $L_2(\Delta \backslash \Delta_n)$-norms of $K(\cdot, \cdot, \theta)$ and $K(\cdot, \cdot, \theta_0)$ respectively. Utilizing the N-th partial sum (4) to find a further upward estimate of the $L_2(\Delta_n)$-norm of the function of two variables $[K(\cdot, \cdot, \theta) - K(\cdot, \cdot, \theta_0)]$ gives

$$\begin{aligned} |||K(\cdot, \cdot, \theta) - K(\cdot, \cdot, \theta_0)||| &\leq \| (K - D_N * K)(\cdot, \cdot, \theta) \|_{L_2(\Delta_n)} \\ &+ \sum_{\nu=0}^{N} |e^{i\nu\theta} - e^{i\nu\theta_0}| \, \| \cdot^\nu k_\nu(\cdot/\cdot) \|_{L_2(\Delta)} + \| (K - D_N * K)(\cdot, \cdot, \theta_0) \|_{L_2(\Delta_n)} \\ &\times \| K(\cdot, \cdot, \theta) \|_{L_2(\Delta \backslash \Delta_n)} + \| K(\cdot, \cdot, \theta_0) \|_{L_2(\Delta \backslash \Delta_n)}, \end{aligned}$$

where n is picked so large that the last two terms are each less than $5^{-1}\epsilon$ independent of θ. Thereafter, one chooses N so that $C_{nN}\gamma_N(K) < 5^{-1}\epsilon$ and notices that the first and third expressions are both bounded by $C_{nN}\gamma_N(K)$. The uniform continuity of the N functions $e^{i\nu \cdot}(0 < \nu \leq N)$ provides a $\delta(\epsilon)$ (independent of θ) with the property: $|\theta - \theta_0| < \delta(\epsilon)$ implies the summation expression is less than $5^{-1}\epsilon$. Hence, $\theta \mapsto K(\cdot, \cdot, \theta)$ is continuous.

For the sufficiency: the continuity of the map $[-\pi, \pi] \to L_2(\Delta)$ combined with the fact that $|||K(\cdot, \cdot, \theta) - K(\cdot, \cdot, \theta')|||$ exceeds the $L_2(\Delta \backslash \Delta_n)$-norms of $[K(\cdot, \cdot, \theta) - K(\cdot, \cdot, \theta')]$ $(n \geq 1)$ entails that the $L_2(\Delta \backslash \Delta_n)$-norms of $K(\cdot, \cdot, \theta)$ define a decreasing sequence of $C[-\pi, \pi]$-functions with point-wise limit 0. By Dini's Theorem, this convergence is uniform and thus the proof is completed.

□

The theorem just proved characterizes the linear manifold of uniformly square integrable $(\mathfrak{V}\mathfrak{K}_2)(D)$-kernels as

$$(\mathfrak{V}\mathfrak{K}_2)_c(D) = \{K(r,r',\theta) \in (\mathfrak{V}\mathfrak{K}_2)(D) : (\theta \mapsto K(\cdot,\cdot,\theta)) \in C([-\pi,\pi], L_2(\Delta))\},$$

which leads to the following theorem.

Theorem 5. *$(\mathfrak{V}\mathfrak{K}_2)_c(D)$ is a closed subspace of $(\mathfrak{V}\mathfrak{K}_2)(D)$.*

Proof. Let $K_n(r,r',\theta)$ be a $|||\cdot|||_s$-Cauchy-sequence of $(\mathfrak{V}\mathfrak{K}_2)_c(D)$-kernels with $|||\cdot|||_s$-limit $K(r,r',\theta) \in (\mathfrak{V}\mathfrak{K}_2)(D)$. The continuity of $\theta \to K(r,r',\theta)$ is an immediate consequence of $|||K(\cdot,\cdot,\theta) - K(\cdot,\cdot,\theta')||| \le 2|||K - K_n|||_s + |||K_n(\cdot,\cdot,\theta) - K_n(\cdot,\cdot,\theta')|||$ and thereby completes the proof. □

Moreover, since

$$|||K(\cdot,\cdot,\theta)|||^2 = \parallel K(\cdot,\cdot,\theta) \parallel_{L_2(\Delta_n)}{}^2 + \parallel K(\cdot,\cdot,\theta) \parallel_{L_2(\Delta\setminus\Delta_n)}{}^2$$

and the $L_2(\Delta_n)$-norm of $K(\cdot,\cdot,\theta)$ is always continuous, the continuity of the expression $|||K(\cdot,\cdot,\theta)|||$ leads to the conclusion that $(\mathfrak{V}\mathfrak{K}_2)_c(D) = \{K(r,r',\theta) \in (\mathfrak{V}\mathfrak{K}_2)(D) : (\theta \mapsto |||K(\cdot,\cdot,\theta)|||) \in C([-\pi,\pi],[0,\infty))\}$. The $|||\cdot|||_s$-approximation of $(\mathfrak{V}\mathfrak{K}_2)_c(D)$-kernels by finite sums of kernels $(\cdot^\nu \oslash k_\nu)(r.r',\theta)$ $(\nu \ge 0)$ requires the introduction of the t-constriction kernels

$$K_t(r,r',\theta) \equiv K(tr,tr',\theta) = \sum_{n=0}^{\infty} t^n(\cdot^n \oslash k_n)(r.r',\theta) \ (0 \le t < 1) \qquad (28)$$

of $K(r,r',\theta) \in (\mathfrak{V}\mathfrak{K}_2)(D)$. They belong to $(\mathfrak{V}\mathfrak{K}_2)_c(D)(0 \le t < 1)$ and lead to the following result.

Theorem 6. *$K(r,r',\theta) \in (\mathfrak{V}\mathfrak{K}_2)_c(D)$ if and only if $\lim_{t\to 1^-} |||K - K_t|||_s = 0$.*

Proof. Theorem 5 justifies the sufficiency. For the necessity, one notes first that $m_2(tE) = t^2 m_2(E)$ for every Lebesgue-measurable subset E of Δ and second that

$$|||(K-K_t)(\cdot,\cdot,\theta)||| \le \parallel (K-K_t)(\cdot,\cdot,\theta) \parallel_{L_2(\Delta_n)} + \parallel K(\cdot,\cdot,\theta) \parallel_{L_2(\Delta\setminus\Delta_n)} + t^{-1} \parallel K(\cdot,\cdot,\theta) \parallel_{L_2(t(\Delta\setminus\Delta_n))} . \qquad (29)$$

Because $m_2(t(\Delta\setminus\Delta_n)) \le m_2(\Delta\setminus\Delta_n)$ $(0 \le t \le 1)$ and $K(r,r',\theta)$ is uniformly square integrable, the last expression of equality (29) does not exceed $4^{-1}\epsilon$ provided n is suitably large and $t_\epsilon \le t \le 1$ (for some $t_\epsilon < 1$). By writing

$(K-K_t)(r,r',\theta)$ as the summation of $(A_{c2}(D)\oslash L_2[0,1])$-kernels, namely

$$(K-K_t)(r,r',\theta)=\sum_{\nu=0}^{\infty}(1-t^{\nu})(\cdot^{n}\oslash k_{\nu})(r,r',\theta)\ (\nu\geq 0),$$

each of whose $L_2(\Delta_n)$-norm is less that $(1-t^{\nu})(2\nu+2)^{-1/2}(1-1/n)^{\nu}\parallel k_{\nu}\parallel$, one obtains a majorization of the $L_2(\Delta_n)$-norm of $(K-K_t)(r,r',\theta)$ by the summation of $(1-t^{\nu})(2\nu+2)^{-1/2}(1-1/n)^{\nu}\parallel k_{\nu}\parallel\ (\nu\geq 0)$. This majorant has a further upward estimate $[\sum_{\nu=0}^{\infty}(1-t^{\nu})^2(1-1/n)^{2\nu}]^{1/2}\gamma_0(K)$ (Cauchy-Schwarz Inequality), which applied to inequality (29) results in

$$|||(K-K_t)(r,r',\theta)|||\leq[(1-t^{N})n+C_{nN}]\gamma_0(K) + 2^{-1}\epsilon\ (-\pi\leq\theta\leq\pi) \quad (30)$$

with n previously specified. Picking a N in relation (30) such that $C_{nN}\gamma_0(K)<4^{-1}\epsilon$ and subsequently a $T_{\epsilon}<1$ satisfying $(1-T_{\epsilon}^{N})n\gamma_0(K)<4^{-1}\epsilon$ demonstrates that $|||K-K_t|||_s<\epsilon\ (T_{\epsilon}\leq t\leq 1)$ and thereby completes the proof. ☐

An immediate result of Theorem 6 is the $|||\cdot|||_s$-approximation by the $(A_{c2}(D)\oslash L_2[0,1])$-kernels $(\cdot^{n}\oslash k_n)(r,r',\theta)$ of $K(r,r',\theta)\in(\mathfrak{VK}_2)_c(D)$. To this end one invokes $|||K-D_N*K|||_s\leq|||K-K_t|||_s+|||(K-D_N*K)_t|||_s$, appeals to Theorem 6 for a $t<1$ such that $|||K-K_t|||_s<2^{-1}\epsilon$ and utilizes representation (28) of $(K-D_N*K)_t(r,r',\theta)$ to estimate $|||(K-D_N*K)_t|||_s$ from above by the sum of $t^{\nu}(2\nu+2)^{-1/2}\parallel k_{\nu}\parallel\ (\nu\geq N+1)$. $t^{N+1}(1-t^2)^{-1/2}\gamma_N(K)$ is a majorant of this sum, which is less than $2^{-1}\epsilon$ provided N is suitably large. Formulas (4) for such t and N imply $|||K-(D_N*K)_t|||_s\leq\epsilon$ - i.e.

Theorem 7. *$(L.H.)(\{(\cdot^{n}\oslash k_n):k_n\in L_2[0,1],n\geq 0\})$ has the property of*

$$[(L.H.)(\{(\cdot^{n}\oslash k_n):k_n\in L_2[0,1],n\geq 0\})]^{-}=A_{c2}(D)\overline{\oslash}L_2[0,1]=(\mathfrak{VK}_2)_c(D),$$

where $\overline{\oslash}$ denotes the closure in the $|||\cdot|||_s$-sense.

The action (22) by $(\mathfrak{VK}_2)(D)$-kernels from the left on $A_2(D)$ can be extended to $L_2[0,1]$ by replacing first $f(r'e^{i\theta})$ with $(re^{i\theta})^{-1}u(r'/r)$ and afterwards the variable of integration r' with rr', whence the transformation

$$\int_0^1 K(r,rr',\theta)u(r')dr'=<K(r,r\cdot,\theta)|\overline{u}> = \sum_{n=0}^{\infty}<u|k_n>(re^{i\theta})^{n}(re^{i\theta}\in D) \quad (31)$$

with operator norm (as element of $\mathcal{L}(L_2[0,2], A_{s2}(D))$) bounded by $\gamma|||K|||_s$ (9). If on the one hand the kernel $K(r,r',\theta) \in (\mathfrak{V}\mathfrak{K}_2)_c(D)$, then the inner product expression $< K(r,r\cdot,\theta)|\overline{u} >$ defines an $A_{c2}(D)$-function of variable $re^{i\theta}$, because the $L_2[0,1]$-norm of $\sqrt{r} < K(r,r\cdot,\theta) - K(r,r\cdot,\theta')|\overline{u} >$ is dominated by $|||K(\cdot,\cdot,\theta) - K(\cdot,\cdot,\theta')||| \parallel u \parallel$. Given any maximal orthonormal set Q_ν $(\nu \geq 0)$ of $L_2[0,1]$, the expressions $g_\nu(re^{i\theta}) \equiv < K(r,r\cdot,\theta)|Q_\nu >$ and $(g_\nu \oslash Q_\nu)(r,r',\theta)$ belong respectively to $A_{c2}(D)$ and $(\mathfrak{V}\mathfrak{K}_2)_c(D)$. The Bessel Inequality for $L_2[0,1]$ establishes further that $\parallel (K - \sum_{\nu=0}^{N} g_\nu \oslash Q_\nu)(r,r\cdot,\theta) \parallel^2$ is a decreasing sequence of Lebesgue-measurable functions of variable r, bounded above by $\parallel K(r,r\cdot,\theta) \parallel^2$, converging to 0 a.e. in r as $N \to \infty$. As a result of Dini's Theorem,

$$|||(K - \sum_{\nu=0}^{N} g_\nu \oslash Q_\nu)(\cdot,\cdot,\theta)||| = [\int_0^1 r \parallel (K - \sum_{\nu=0}^{N} g_\nu \oslash Q_\nu)(r,r\cdot,\theta) \parallel^2 dr]^{1/2}$$

converges uniformly to 0 in θ as $N \to \infty$, namely $\lim_{N\to\infty} |||K - \sum_{\nu=0}^{N} g_\nu \oslash Q_\nu|||_s = 0$. On the other hand, for the approximation of $K(r,r',\theta) \in (\mathfrak{V}\mathfrak{K}_2)(D)$ by means of $(A_{c2}(D) \oslash L_2[0,1])$-kernels $(g_\nu \oslash Q_\nu)(r,r',\theta)$, the best that can be said is that

$$\parallel (K - \sum_{\nu=0}^{N} g_\nu \oslash Q_\nu)(\cdot,\cdot,\theta) \parallel_{L_2(\Delta_n)} \to 0 \text{ uniformly in } \theta \qquad \text{as } N \to \infty \text{ for each } \Delta_n$$

and thus we obtain the following result.

Theorem 8. *If Q_ν $(\nu \geq 0)$ is an orthonormal basis of $L_2[0,1]$ and $K(r,r',\theta) \in (\mathfrak{V}\mathfrak{K}_2)(D)$, then for the $A_{c2}(D)$-functions*

$$g_\nu(re^{i\theta}) \equiv < K(r,r\cdot,\theta)|Q_\nu > \ (\nu \geq 0) \lim_{N\to\infty} \parallel (K - \sum_{\nu=0}^{N} g_\nu \oslash Q_\nu)(\cdot,\cdot,\theta) \parallel_{L_2(E)} = 0$$

uniformly in θ $(-\pi \leq \theta \leq \pi)$ for all Lebesgue-measurable sets E satisfying $\overline{E} \subset \Delta \backslash (\{1\} \times [0,1])$. E may also be Δ if $K(r,r',\theta) \in (\mathfrak{V}\mathfrak{K}_2)_c(D)$.

5. Conditions for Uniform Square Integrability

The kernel $K(r,r',\theta) \in (\mathfrak{V}\mathfrak{K}_2)_c(D)$ if and only if $((t,\theta) \mapsto K_t(r,r',\theta)) \in C([0,1]\times[-\pi,\pi], L_2(\Delta))$ (28). If this is the case, then $|||K-K_t|||_s \to 0$ as $t \to 1^-$ or equivalently, the validity of the approximations in Theorem 6 and Theorem 7, which may not be so easy to establish directly. Alternately, the results of G.H. Hardy, J.E. Littlwood and A. Zygmund ([3], pp.74-78), as to when the boundary values of $H(D)$-functions belong to Λ_α and Λ_* in terms of order conditions on the first and second derivatives, provide handy criteria for deciding when $K(r,r',\theta) \in (\mathfrak{V}\mathfrak{K}_2)_c(D)$. Accordingly, the Banach space Λ_α of Lipschitz-continuous functions U of order α on ∂D is normed by $\|\cdot\|_\alpha \equiv \|\cdot\|_\infty + h_\alpha(\cdot)$ with semi-norm $h_\alpha(U) \equiv \sup\limits_{t\neq s} |U(e^{is}) - U(e^{it})||t-s|^{-\alpha}$, whereas Λ_* is the Banach space all $U \in C(\partial D)$, satisfying $h_*(U) \equiv \sup\limits_{0<\epsilon<\pi, |z|=1} |U(ze^{i\epsilon}) - 2U(z) + U(z^{-i\epsilon})| < \infty$ with norm $\|\cdot\|_* \equiv \|\cdot\|_\infty + h_*(\cdot)$. One now defines for every $K(r,r',\theta) \in (\mathfrak{V}\mathfrak{K}_2)_c(D)$ with representation (19) the λ-parameter family of $(\mathfrak{V}\mathfrak{K}_2)_c(D)$-kernels ($\lambda > 0$)

$$K_*^\lambda(r,r',\theta) \equiv [\Gamma(\lambda)]^{-1} \int\limits_0^1 (1-t)^{\lambda-1} K_t(r,r',\theta)dt$$
$$= \sum_{n=0}^{\infty} n![\Gamma(n+\lambda+1)]^{-1} (\cdot^n \oslash k_n)(r,r',\theta). \quad (32)$$

That these kernels indeed belong to $(\mathfrak{V}\mathfrak{K}_2)_c(D)$ is an immediate consequence of $|||K_*^\lambda(\cdot,\cdot,\theta) - K_*^\lambda(\cdot,\cdot,\theta')||| \le [\Gamma(\lambda)]^{-1} \int\limits_0^{1-\eta} (1-t)^{\lambda-1} |||K_t(\cdot,\cdot,\theta) - K_t(\cdot,\cdot,\theta')|||dt + 2|||K|||_s [\Gamma(\lambda)]^{-1} \int\limits_{1-\eta}^1 t^{-1}(1-t)^{\lambda-1}dt$ $(0 < \eta < 1)$, Minkowski integral inequality and $|||K_t(\cdot,\cdot,\theta)||| \le t^{-1} \| K(\cdot,\cdot,\theta) \|_{L_2(t\Delta)}$. Herein the second expression tends toward 0 as $\eta \to 0^+$ and $((t,\theta) \mapsto K_t(r,r',\theta)) \in C([0,1-\eta]\times[-\pi,\pi], L_2(\Delta))$. Moreover, equation (32) for $K_*^\lambda(r,r',\theta)$ and the norm-equality of statement (19) guarantee that the associated transformations

$$(F_K^\lambda u)(e^{i\theta}) \equiv \int\limits_0^1 K_*^\lambda(1,r',\theta)u(r')e^{i\theta}dr' = < K_*^\lambda(1,\cdot,\theta)|\overline{u} > e^{i\theta} \quad (33)$$

define linear maps $F_K^\lambda : L_2[0,1] \mapsto H_2(\partial D)$ (Hilbert space of boundary values of $H_2(D)$-functions for all $\lambda \geq 2^{-1}$). For the case $\lambda > 2^{-1}$ more can be said, namely $F_K^\lambda : L_2[0,1] \mapsto C(\partial D)$ as becomes apparent from

$$|e^{-i\theta}(F_K^\lambda u)(e^{i\theta}) - e^{-i\theta'}(F_K^\lambda u)(e^{i\theta'})|$$
$$\leq [\Gamma(\lambda)]^{-1} \parallel u \parallel \int_0^{1-\eta} |||K(t,t\cdot,\theta) - K(t,t\cdot,\theta')|||(1-t)^{\lambda-1}dt$$
$$+ 2[\Gamma(\lambda)]^{-1} \parallel u \parallel |||K|||_s \left[\int_{1-\eta}^{1} t^{-1}(1-t)^{2\lambda-2}dt\right]^{1/2} \qquad (0<\eta<1)$$

(Cauchy-Schwarz inequality), wherein the second term has limit 0 as $\eta \to 0^+$ and $((r,\theta) \mapsto K(r,r\cdot,\theta)) \in C([0,1-\eta]\times[-\pi,\pi], L_2[0,1])$ is uniformly continuous (norm-inequality of statement (19)). Among the associated transformations, the case $\lambda = 1$ is of special significance, because $(F_K^1 u)'(e^{i\theta}) = < K(r,r\cdot,\theta)|\overline{u} >$ ((31)) and therefore, the results of G.H. Hardy, J.E. Littlewood and A. Zygmund (see [3], pp. 74-78) are applicable; in particular we shall prove the following theorem.

Theorem 9. *$F_K^1 \in \mathcal{L}(L_2[0,1], \Lambda_\alpha)$ if and only if*
$\parallel K(r,\cdot,\theta) \parallel = O((1-r)^{\alpha-1})$ uniformly in θ $(-\pi \leq \theta < \pi)$ as $r \to 1^-$.

Proof. The proof is that given by P.L. Duren ([3], pp. 74-75) of the Hardy-Littlewood Theorem with $f(re^{i\theta}) = (F_K^1 u)(re^{i\theta})$ and noting that its derivative $f'(re^{i\theta}) =< K(r,r\cdot,\theta)|\overline{u} >$. Utilizing $A = h_\alpha(F_K^1 u) \leq \parallel F_K^1 \parallel\parallel u \parallel$ and $A_\alpha \equiv \sup_{0\leq r<1} (1-r)^{\alpha-1})([1+r^2-2r\cos\cdot]^{-1} * |\cdot|^\alpha)(0) < \infty$ yields $|f'(re^{i\theta})| \leq A_\alpha \parallel F_K^1 \parallel (1-r)^{\alpha-1} \parallel u \parallel$, which in turn entails that $r^{-1/2} \parallel K(r,\cdot,\theta) \parallel \leq A_\alpha \parallel F_K^1 \parallel (1-r)^{\alpha-1}$.

For the sufficiency, one assumes $\parallel K(r,\cdot,\theta) \parallel \leq R(1-r)^{\alpha-1}(0 \leq r < 1)$ for some R independent of θ. After writing $(F_K^1 u)(e^{i\theta})$ as a double integral over $\Box$ ((32) and (33)), the estimate $\parallel F_K^1 u \parallel_\infty \leq \alpha^{-1}R \parallel u \parallel$ follows. Further, the path-integral used by P.L. Duren ([3], p. 75) with $|f'(re^{i\theta})| \leq R \parallel u \parallel (1-r)^{\alpha-1}$ leads to $h_\alpha(F_K^1 u) \leq (1+2\alpha^{-1})R \parallel u \parallel$, specifically $\parallel F_K^1 \parallel \leq (1+3\alpha^{-1})R$, and thereby completes the proof. $\Box$

Corollary 10. *If $F_K^1 \in \mathcal{L}(L_2[0,1], \Lambda_\alpha)$ for some $\alpha > 1/2$, then $K(r,r',\theta) \in (\mathfrak{V}\mathfrak{K}_2)_c(D)$.*

Proof. $\parallel K(r,\cdot,\theta) \parallel \leq R(1-r)^{\alpha-1}$ $(0 \leq r < 1)$, for an R independent of θ, implies

$$\| K(\cdot,\cdot,\theta) \|_{L_2(\Delta\setminus\Delta_n)} = \left[\int\limits_{1-1/n}^{1} \| K(r,\cdot,\theta) \|^2 \, dr \right]^{1/2} \leq R(2\alpha-1)^{-1/2} n^{1/2-\alpha},$$

thereby completing the proof. □

Prior to formulating the analog of Zygmund's Theorem in $(\mathfrak{V}\mathfrak{K}_2)(D)$, the derivative with respect to the dot "$\cdot$" of representation (19) of $K(r,r',\theta)$, namely $(\mathcal{D}K)\ (r,r',\theta) \equiv \sum_{n=0}^{\infty} n(\cdot^{n-1} \oslash k_n)(r,r',\theta)$, must be introduced. $(\mathcal{D}K)(r,r',\theta)$ does not necessarily belong to $(\mathfrak{V}\mathfrak{K}_2)(D)$, nonetheless it is indispensable for

Theorem 11. $F_K^1 \in \mathcal{L}(L_2[0,1],\Lambda_*)$ *if and only if* $\| (\mathcal{D}K)(r,\cdot,\theta) \| = O((1-r)^{-1})$ *uniformly in* θ $(-\pi \leq \theta \leq \pi)$ *as* $r \to 1^-$.

Proof. The proof follows the verification of the Zygmund Theorem by P.L. Duren ([3], pp. 76-78) with the f and f' as in Theorem 9 and especially $f''(re^{i\theta}) =< (\mathcal{D}K)\ (r,r\cdot,\theta)|\overline{u} >$. Although not explicitly stated by P.L. Duren, he avails himself of the estimates

$$|f_\theta(re^{i\theta})| \leq 2\pi^{-1}(1-r)^{-1} \| f(e^{i\cdot}) \|_\infty,\ |f_{\theta\theta}(re^{i\theta})| \leq \pi^{-1}(1-r)^{-1} h_*(f(e^{i\cdot})),$$

which under $f''(re^{i\theta}) = (re^{i\theta})^{-2}[if_\theta(re^{i\theta}) - f_{\theta\theta}(re^{i\theta})]$ combines to $|f''(re^{i\theta})| \leq 2\pi^{-1}(2-\sqrt{3})^{-2}(1-r)^{-1} \| f(e^{i\cdot}) \|_*$ for $2-\sqrt{3} < r < 1$. $f(e^{i\theta}) = (F_K^1 u)(e^{i\theta})$ implies $\| f(e^{i\cdot}) \|_* \leq \| F_K^1 \| \| u \|$ and thus

$$\| (\mathcal{D}K)(r,r\cdot,\theta) \| \leq 2\pi^{-1}(2-\sqrt{3})^{-2} \| F_K^1 \| (1-r)^{-1} \quad (2-\sqrt{3} < r < 1).$$

If $F^1_{\cdot^0\oslash k_0}$ stands for F_K^1, where $K = \cdot^0 \oslash k_0$, then in representation (19) $(\mathcal{D}\cdot^0 \oslash k_0)(r,r',\theta) = 0$ and $F^1_{\cdot^0\oslash k_0} \in \mathcal{L}(L_2[0,1],\Lambda_*)$ on account of $\| F^1_{\cdot^0\oslash k_0} \| \leq 1+\pi$, which readily follows from: $(F^1_{\cdot^0\oslash k_0} u)(re^{i\theta}) =< u|k_0 > e^{i\theta}$ (33), $\| F^1_{\cdot^0\oslash k_0} u \|_\infty = | < u|k_0 > |$ and $h_*(F^1_{\cdot^0\oslash k_0}) \leq \pi| < u|k_0 > |$. Without loss of generality therefore, $k_0 = 0$ in equation (19) as well as $\| (\mathcal{D}K)\ (r,r\cdot,\theta) \| \leq R(1-r)^{-1}$ (R independent of θ) may be assumed, which has the following two consequences for the sufficiency:

$$|f''(re^{i\theta})| \ \leq \ R(1-r)^{-1} \| u \|, \quad |f'(re^{i\theta})| \ \leq \ -R \| u \| \log(1-r). \tag{34}$$

Since $f'(re^{i\theta}) =< K(r,r\cdot,\theta)|\overline{u} >$, the second inequality of (34) implies that $|||K(r,r\cdot,\theta)||| \leq -R\log(1-r)$ and thus $\| F_K^1 u \|_\infty \leq R \| u \|$, after writing

$(F_K^1 u)(e^{i\theta})$ as a double integral over $\square$ (32), (33) and utilizing $\int\limits_0^1 \log(1-r)dr = -1$.

Underneath the rationale of P.L. Duren ([3], pp. 77-78) is the estimate of the semi-norm value $h_*(f(e^{i\cdot}))$ by means of

$$f(e^{i(\theta+h)}) - 2f(e^{i\theta}) + f(e^{i(\theta-h)}) = \Phi(\rho,\theta,h) - 2\rho e^{i\theta}\int\limits_0^h f'(\rho e^{i(\theta+t)})\sin t dt$$

$$- \rho e^2 e^{i2\theta}\int\limits_0^h e^{-it}\int\limits_{-t}^t f''(\rho e^{i(\theta+\tau)})e^{i\tau}d\tau dt \quad (h>0), \quad (35)$$

where

$$\Phi(\rho,\theta,h) = (1-\rho)e^{i\theta}[(e^{ih}-1)f'(\rho e^{i(\theta+h)}) - (e^{-ih}-1)f'(\rho e^{i\theta})]$$

$$+ i\rho(1-\rho)e^{i2\theta}\int\limits_0^h [f''(\rho e^{i(\theta+t)})e^{it} - e^{-i2h}f''(\rho e^{i(\theta-t)})e^{-it}]dt$$

$$+ e^{i2\theta}\int\limits_\rho^1 (1-r)[e^{i2h}f''(re^{i(\theta+h)}) - 2f''(re^{i\theta}) + e^{-i2h}f''(re^{i(\theta-h)})]dr.$$

The majorization of $|\Phi(\rho,\theta,h)|$ via inequalities (34) for $\rho = 1-h$ yields a constant C_1 independent of (h,θ,u) such that $|\Phi(1-h,\theta,h)| \leq C_1 R \parallel u \parallel h$. Proceeding in like manner ($\rho = 1-h$), the integrals of relation (35) have moduli bounded by $-R \parallel u \parallel h^2 \log h$ and $R \parallel u \parallel h$ respectively. Thus $C_2 \equiv \sup\limits_{0<h\leq 1} -h\log h$ leads to $h_*(F_K^1 u) \leq (C_1 + 2C_2 + 1)R \parallel u \parallel$, that is $\parallel F_K^1 \parallel \leq (C_1 + 2C_2 + 1)R$, and thereby completes the proof. $\square$

Corollary 12. *If $F_K^1 \in \mathcal{L}(L_2[0,1], \Lambda_*)$, then $K(r,r',\theta) \in (\mathfrak{V}\mathfrak{K}_2)_c(D)$.*

Proof. The second inequality of statement (34) entails

$$\parallel K(\cdot,\cdot,\theta) \parallel_{L_2(\Delta\backslash\Delta_n)} \leq R[\int\limits_{1-1/n}^1 (\log(1-r))^2 dr]^{1/2}$$

$$= Rn^{-1/2}[(\log n)^2 + 2\log n + 2]^{1/2}$$

and thereby completes the proof. $\square$

6. Counterexample

The structured behavior of $H(D)$-functions w in a deleted neighborhood of a pole lying on ∂D entails $\| w \|_s = \infty$ and thus $(w \oslash k)$ (r, r', θ) cannot be a candidate for a $K(r, r', \theta) \in (\mathfrak{V}\mathfrak{K}_2)(D) \setminus (\mathfrak{V}\mathfrak{K}_2)_c(D)$. However, the haphazardness of a holomorphic function in the vicinity of an isolated essential singularity (Cassorati-Weierstraß Theorem) makes $w(z) \equiv \exp((1-z)^{-1})(1-z)^{-1}$ a suitable choice, whence the kernel

$$(w \oslash k)(r, r', \theta) \equiv \exp((1 - re^{i\theta})^{-1})(1 - re^{i\theta})^{-1}\overline{k}(r'/r)\chi_{[0,r]}(r'). \tag{36}$$

It possesses the obvious properties of $((w \oslash k)f)(z) = zw(z) < f(z\cdot)|k > \in H(D)$ for all $f \in A_2(D)$, $\| w \oslash k \|_s = \| k \| \sup\limits_{-\pi \le \theta \le \pi} \| \cdot^{1/2} w(\cdot e^{i\theta}) \|$ and $\| w(\cdot e^{i\theta}) \|$ is continuous on $[-\pi, \pi]$ with the exception of $\theta = 0$, at which $\| w(\cdot e^{i0}) \| = 2^{-1/2}e^{-1}$. Hence $(w \oslash k)(r, r', \theta) \in (\mathfrak{V}\mathfrak{K}_2)(D)$ hinges on the following theorem.

Theorem 13. $\limsup\limits_{\theta \to 0^+} \| w(\cdot e^{i\theta}) \| \le [1 + (6e^2)^{-1} + (2e)^{-1}]^{1/2}$.

Proof. The substitution of $r = t \sin\theta + \cos\theta$ and $x = \text{ctg}\ \theta$ into

$$\| w(\cdot e^{i\theta}) \|^2 = \int_0^1 \exp(2(r\cos\theta - 1)[1 + r^2 - 2r\cos\theta]^{-1})[1 + r^2 - 2r\cos\theta]^{-1} dr,$$

under which in this integral the limit $\theta \to 0^+$ changes to $x \to \infty$, results in

$$\| w(\cdot e^{i\theta}) \|^2 = \sqrt{x^2+1}\Big(\int_0^1 + \int_1^2 + \int_2^x\Big) \exp(-2(xt+1)[1+t^2]^{-1})[1+t^2]^{-1} dt$$
$$+ \sqrt{x^2+1} \int_0^{\sqrt{x^2+1}-x} \exp(2(xt-1)[1+t^2]^{-1})[1+t^2]^{-1} dt$$
$$= (I_1 + I_2 + I_3 + I_4)(x). \tag{37}$$

The function $-2t[1+t^2]^{-1}$ decreases strictly from 0 to -1 on $[0,1]$ and increases strictly from -1 to 0 on $[1, \infty]$. If $0 \le t \le 1$, then $-2xt[1+t^2]^{-1} \le -tx$ $(x \ge 0)$ implies $I_1(x) \le \sqrt{x^2+1} \int_0^1 e^{-xt} dt$ – i.e. $\limsup\limits_{x \to \infty} I_1(x) \le 1$. Given $1 \le t \le 2$, $-2xt[1+t^2]^{-1} \le 0{,}8x$ $(x \ge 0)$ leads to $I_2(x) \le (2e^{0{,}4})^{-1}e^{-0{,}8x}$ - i.e. $\limsup\limits_{x \to \infty} I_2(x) = 0$. For $t \ge 2$ one notes that the function $x[1+t^2]^{-1} = \{-2x(1-t^2)[1+t^2]^{-2}\}\{2^{-1} + [1+t^2]^{-1}\}$ of variable t is the product of two positive terms, the first being the derivative of $-2xt[1+t^2]^{-1}$ with respect to t, whereas the second is bounded by 5/6. Factoring x^{-1} from the last term of the integrand in $I_3(x)$ yields $I_3(x) \leq x^{-1}\sqrt{x^2+1}\int_2^x \{\exp(-2xt[1+t^2]^{-1})\}_t(5/6)dt = 5(6x)^{-1}\sqrt{x^2+1}\{\exp(-2x^2[1+x^2]^{-1}) - e^{-0,8x}\}(x \geq 2)$ - i.e. $\limsup_{x\to\infty} I_3(x) \leq 5(6e^2)^{-1}$. The Intermediate Value Theorem for integrals guarantees the existence of a $\xi = \xi(x)$ satisfying: $0 < \xi < [x+\sqrt{x^2+1}]^{-1}$, whereby the integral $I_4(x) = \sqrt{x^2+1}[x+\sqrt{x^2+1}]^{-1}\exp(2(x\xi-1)[1+\xi^2]^{-1})[1+\xi^2]^{-1}$. Out of $2x\xi[1+\xi^2]^{-1} \leq 2x[x+\sqrt{x^2+1}]^{-1}(x>0)$ the conclusion $\limsup_{x\to\infty} I_4(x) \leq (2e)^{-1}$ immediately follows and consequently completes the proof. □

Theorem 14. $\sup_{-\pi\leq\theta\leq\pi} \| w(\cdot e^{i\theta}) \|_{L_2[r,1]} \geq \sqrt{2e}\; 6^{-1}\sqrt{e-1}\ \ (0 \leq r \leq 1)$.

Proof. The square of the $L_2[r,1]$-norm of $w(\cdot e^{i\theta})$ exceeds the modulus of the path-integral of $\exp(2(1-z)^{-1})(1-z)^{-2}$ from $re^{i\theta}$ to $e^{i\theta}$. The modulus of this path-integral is $2^{-1}|e^{-1} - \exp(2(r\cos\theta - 1)[1+r^2-2r\cos\theta]^{-1})|$, which equals $2^{-1}(e^{-1}-e^{-2})$ whenever $\cos\theta = r$ and thereby completes the proof. □

One now turns to the kernel $(w \oslash k)(r, r', \theta)$. The $L_2(\Delta \setminus \Delta_n)$-norm of $(w\oslash k)(\cdot,\cdot,\theta)$ coincides with the product of the $L_2[0,1]$- and $L_2[1-1/n,1]$-norms of k and $\cdot^{1/2}w(\cdot e^{i\theta})$ respectively; however, the $L_2[1-1/n,1]$-norm of $\cdot^{1/2}w(\cdot e^{i\theta})$ exceeds the $L_2[1-1/n,1]$-norm of $w(\cdot e^{i\theta})$ multiplied by $\sqrt{1-1/n}$. Inconsequence of Theorem 14, the supremum of the $L_2(\Delta\setminus\Delta_n)$-norm of the functions $(w\oslash k)(\cdot,\cdot,\theta)$ $(-\pi \leq \theta \leq \pi)$ is greater than $\| k \| (\sqrt{2}e)^{-1}\sqrt{e-1}\sqrt{1-1/n}$ and thus $(w \oslash k)(r,r',\theta) \notin (\mathfrak{V}\mathfrak{K}_2)_c(D)$ $(\| k \| > 0)$.

Theorem 15. *$(\mathfrak{V}\mathfrak{K}_2)_c(D)$ is a proper closed left, but not a right ideal of $(\mathfrak{V}\mathfrak{K}_2)(D)$.*

Proof. Since $|(LK)(r,r',\theta)|$ does not exceed the product of the $L_2[r',r]$-norms of $L(r,\cdot,\theta)$ and $K(\cdot,r',\theta)$, it follows that the $L_2(\Delta\setminus\Delta_n)$-norm of function $(LK)(\cdot,\cdot,\theta)$ is bounded by the $L_2(\Delta\setminus\Delta_n)$-norm $L(\cdot,\cdot,\theta)$ multiplied by $|||K|||_s$. Also, $(LK)(r,r',\theta) \in (\mathfrak{V}\mathfrak{K}_2)_c(D)$ by Theorem 3 and Theorem 4, provided $L(r,r',\theta) \in (\mathfrak{V}\mathfrak{K}_2)_c(D)$. Furthermore, $(w \oslash k)(r,r',\theta) \in (\mathfrak{V}\mathfrak{K}_2)(D) \setminus (\mathfrak{V}\mathfrak{K}_2)_c(D)$ $(\| k \| > 0)$ implies $(\mathfrak{V}\mathfrak{K}_2)_c(D)$ is proper closed left ideal of $(\mathfrak{V}\mathfrak{K}_2)(D)$.

That $(\mathfrak{V}\mathfrak{K}_2)_c(D)$ fails to be a right ideal of $(\mathfrak{V}\mathfrak{K}_2)(D)$ is demonstrated as follows. For the choices $L(r,r',\theta) = (w\oslash k)(r,r',\theta)((36))$ and $K(r,r',\theta) = (\cdot^m \oslash l)(r,r',\theta) \in (\mathfrak{V}\mathfrak{K}_2)_c(D)$ (Theorem 7) $(LK)(r,r',\theta) = (w_m \oslash l_m)(r,r',\theta)$ with $w_m(z) = z^{m+1}w(z)$ and $l_m(t) =< \cdot^{m+1}\chi_{[t,1]}\overline{k}|l >$, for which $\| l_m \| \leq (2m+$

$4)^{-1/2} \parallel k \parallel\parallel l \parallel$ on account of $|l_m(t)| \leq\parallel k \parallel\parallel \cdot^m l(t/\cdot)\chi_{[t,1]} \parallel$. There always exist $L_2[0,1]$-functions k and l satisfying $\parallel l_m \parallel > 0$, because $< \cdot^N|K > \neq 0$ for some $N \geq m$ (Müntz-Szász Theorem) and for $l \equiv \cdot^{N-m}$ one has that $t^{N-m}l_m(t) =< \cdot^{N+1}\chi_{[t,1]}|k >$ is non-trivial $C[0,1]$-function, since $r''^{N+1}\overline{k}(r'')dr''$ determines an absolutely continuous Borel-measure with respect to m_1. Modifying the argument in the proof of Theorem 14 for the function $\cdot^{m+3/2}w(\cdot e^{i\theta})$ results in the supremum of the $L_2(\Delta\setminus\Delta_n)$-norm of $(w_m \oslash l_m)(r,r',\theta)$ $(-\pi \leq \theta \leq \pi)$ being bounded below by the positive quantity $\parallel l_m \parallel (\sqrt{2}e)^{-1}\sqrt{e-1}(1-1/n)^{m+3/2}$ - i.e. $(LK)(r,r',\theta) \notin (\mathfrak{V}\mathfrak{K}_2)_c(D)$ - and thereby completes the proof. □

In terms of Theorem 1 and Theorem 7, the representation (19) of $(\mathfrak{V}\mathfrak{K}_2)(D)$-kernel $(w \oslash k)(r,r',\theta)$ (36) cannot be $|||\cdot|||_s$-convergent, although it is uniformly $\parallel\ \parallel_{L_2(\Delta_n)}$-convergent in θ for each $n \geq 1$. Furthermore, the $A_{s2}(D)$-functions $w_m(z) = z^m w(z)$ also serve as counter-examples verifying that $A_{c2}(D)$ is a closed proper subspace of $A_{s2}(D)$.

7. Applications

Given $(\mathfrak{V}\mathfrak{K}_2)(D)$-kernel $K(r,r',\theta)$, the $H(D\times\mathbb{C})$-function $F = F(z,w)$ is said to satisfy a "$A_{s2}(D)(A_2(D))$-*radial Lipschitz condition with respect to* $K(r,r',\theta)$" if F satisfies: $F(\cdot,0) \in A_{s2}(D)(A_2(D))$, $|F(z,w_2) - F(z,w_1)| \leq c(r,\theta)|w_2 - w_1|$ $(z = re^{i\theta})$ and $\gamma_K(c) \equiv \sup\limits_{-\pi\leq\theta\leq\pi} [\int\limits_0^1 \parallel K(r,\cdot,\theta)c(\cdot,\theta) \parallel^2 dr]^{1/2} < \infty$. Thus for a given $f \in A_{s2}(D)(A_2(D))$, the radial Hammerstein integral equation for unknown $y \in A_{s2}(D)(A_2(D))$ is

$$y(re^{i\theta}) = f(re^{i\theta}) + \lambda \int\limits_0^r K(r,r',\theta)F(r'e^{i\theta},y(r'e^{i\theta}))e^{i\theta}dr' \quad (\lambda \in \mathbb{C}), \qquad (38)$$

wherein the appendage "a.e. in r for each θ" is omitted because of equation (27). To expedite the solution for y by means of the Weisinger Fixed Point Theorem (see [4], Seiten 138-139), the entire function $G(z) \equiv \sum\limits_{n=0}^{\infty}(n!)^{-1/2}z^n$ is introduced and the right hand expression of equation (38) is designated $(F_K y)(re^{i\theta})$ or $(F_K y)(r,\theta)$ according as $F(r'e^{i\theta},y(r'e^{i\theta}))$ or $F(r'e^{i\theta},y(r'))$ appears under the integral sign. For two arbitrary $y_\kappa \in L_2[0,1]$ $(\kappa = 1,2)$

$$|({F_K}^{n+1}y_2 - {F_K}^{n+1}y_1)(r,\theta)| \leq |\lambda|^{n+1} \parallel K(r,\cdot,\theta)c(\cdot,\theta) \parallel \qquad (39)$$

$$\times\Big[(n!)^{-1}\Big(\int_0^r \| K(t,\cdot,\theta)c(\cdot,\theta) \|^2 \, dt\Big)^n\Big]^{1/2} \| y_2 - y_1 \|_{L_2[0,r]} \quad (n \geq 0)$$

as consequence of: induction, radial Lipschitz condition, Cauchy-Schwarz Inequality and the integral expression being an absolutely continuous θ-parameter family of functions with a.e.-derivative $\| K(r,\cdot,\theta)c(\cdot,\theta) \|^2$ for each θ. Replacing in the estimates (39) the $L_2[0,1]$-functions y_κ $(\kappa = 1,2)$ with $A_{s2}(D)(A_2(D))$-functions $y_\kappa = y_\kappa(z)$ $(\kappa = 1,2)$ yields:

$$|({F_K}^{n+1}y_2 - {F_K}^{n+1}y_1)(re^{i\theta})| \leq |\lambda|^{n+1} \| K(r,\cdot,\theta)c(\cdot,\theta) \|$$

$$\times\Big[(n!)^{-1}\Big(\int_0^r \| K(t,\cdot,\theta)c(\cdot,\theta) \|^2 \, dt\Big)^n\Big]^{1/2} \| (y_2 - y_1)(\cdot e^{i\theta}) \|_{L_2[0,r]} \quad (n \geq 0),$$

from which follows

$$\| {F_K}^{n+1}y_2 - {F_K}^{n+1}y_1 \|_\xi \leq [(n+1)!]^{-1/2}[\gamma_K(c)\|\lambda|]^{n+1} \| y_2 - y_1 \|_\xi$$

with $\xi = s$ or 0 according as $y_\kappa \in A_{s2}(D)$ or $A_2(D)$ $(\kappa = 1,2)$. Since, $\sum_{n=0}^{\infty}(n!)^{-1/2}[\gamma_K(c)|\lambda|]^n = G(\gamma_K(c)|\lambda|)$, the Weisinger Fixed Point Theorem guarantees for the operator F_K a unique fixed point y $(y = F_K y)$ attainable by iteration - i.e. $y = \lim_{n\to\infty} {F_K}^{n+1}h$ with any $h \in A_{s2}(D)$ or $A_2(D)$ according as $\xi = s$ or 0 in the appropriate norms $\| \cdot \|_\xi$. This y is the unique solution of the radial Hammerstein integral equation (38) and setting $h = f$ out of sheer convenience gives error estimates

$$\| y - {F_K}^{n+1}f \|_\xi \leq [(n+1)!]^{-1/2}[\gamma_K(c)\|\lambda|]^{n+1}G(\gamma_K(c)\|\lambda|) \| F_K f - f \|_\xi$$

with $\xi = s$ or 0 according as $F(\cdot,0)$ and $f \in A_{s2}(D)$ or $A_2(D)$.

For $H(D\times\mathbb{C})$-function $F(z,w) \equiv w$ one has: $c(r,\theta) = 1$, $\| K(r,\cdot,\theta)c(\cdot,\theta) \| = \| K(r,\cdot,\theta) \|$, $\gamma_K(c) = |||K|||_s$, equation (38) becomes the radial Volterra integral equation

$$y(re^{i\theta}) = f(re^{i\theta}) + \lambda\int_0^r K(r,r',\theta)y(r'e^{i\theta})e^{i\theta}dr' \quad (\lambda \in \mathbb{C}), \tag{40}$$

and setting $\lambda = 1, y_1 = 0$ and $y_2 = K(\cdot,r',\theta)$ in relation (39) yields

$$|K^{n+2}(r,r',\theta)| \leq \| K(r,\cdot,\theta) \|$$

$$\times\Big[(n!)^{-1}\Big(\int_0^r \| K(t,\cdot,\theta) \|^2 \, dt\Big)^n\Big]^{1/2} \| K(\cdot,r',\theta) \| \quad (n \geq 0). \tag{41}$$

$\lim_{n\to\infty}[|||K^n|||_s]^{1/n} = 0$, because $|||K^{n+2}|||_s \leq [(n+1)!]^{-1/2}[|||K|||_s]^{n+1}$ (41), and therefore $(\mathfrak{V}\mathfrak{K}_2)(D)$-kernels are quasi-nilpotent ([5], p. 49) in the Banach algebra $(\mathfrak{V}\mathfrak{K}_2)(D)$. Quasi-nilpotency means that

$$H_\lambda(r,r',\theta) \equiv H_\lambda(K)(r,r',\theta) = \sum_{n=0}^{\infty} \lambda^n K^{n+1}(r,r',\theta) \text{ is a } (\mathfrak{V}\mathfrak{K}_2)(D)\text{-valued}$$

entire function of λ, which fact is valid for all $K(r,r',\theta) \in (\mathfrak{V}\mathfrak{K}_2)(D)$. Moreover, the validity of the Fredholm resolvent equations

$$\lambda(KH_\lambda)(r,r',\theta) = \lambda(H_\lambda K)(r,r',\theta) = H_\lambda(r,r',\theta) - K(r,r',\theta)$$

in the Banach algebra $(\mathfrak{V}\mathfrak{K}_2)(D)$ without identity implies $(I-\lambda K)^{-1} = I + \lambda H_\lambda$ for all $\lambda \in \mathbb{C}$ in the Banach algebras $\mathcal{L}(A_{s2}(D))$ or $\mathcal{L}(A_2(D))$ with identity. Thus the unique $A_{s2}(D)$- or $A_2(D)$-solution of the radial Volterra integral equation (40) is

$$y(re^{i\theta}) = f(re^{i\theta}) + \lambda \int_0^r H_\lambda(r,r',\theta) f(r'e^{i\theta}) e^{i\theta} dr' = \sum_{n=0}^{\infty} \lambda^n (K^n f)(re^{i\theta}), \quad (42)$$

for which the error estimates

$$\| y - \sum_{n=0}^{N} \lambda^n K^n f \|_\xi \leq (n!)^{-1/2}[|||K|||_s|\lambda|]^{n+1} G(|||K|||_s|\lambda|) \| f \|_\xi$$

hold with $\xi = s$ or 0 according as $f \in A_{s2}(D)$ or $A_2(D)$.

Returning to the $H(D\times\mathbb{C})$-function F and the radial Hammerstein integral equation (7.1) for unknown $y \in A_2(D)$, F is said to satisfy an "*angular Lipschitz condition with respect to* $K(r,r',\theta)$" provided there exists a L_2-Voltera kernel $K(r,r')$ such that

$$\| K(r,r',\cdot)[F(r'e^{i\cdot}, y_2(r'e^{i\cdot})) - F(r'e^{i\cdot}, y_1(r'e^{i\cdot}))] \| \leq K(r,r') \| (y_2 - y_1)(r'e^{i\cdot}) \|$$

for all y_1 and $y_2 \in A_2(D)$. The analogs of relations (39) for F_K are

$$\| ({F_K}^{n+1} y_2 - {F_K}^{n+1} y_1)(re^{i\theta}) \| \leq |\lambda|^{n+1} \| K(r,\cdot) \| \times \left[(n!)^{-1}\left(\int_0^r \| K(t,\cdot) \|^2 \, dt\right)^n\right]^{1/2} \| y_2 - y_1 \|_0 \quad (n>0), \quad (43)$$

which estimates are proved via the Minkowski integral inequality and the arguments invoked for inequalities (39). Taking the $L_2[0,1]$-norms of both sides of relation (43) yield the norm-inequality $\| {F_K}^{n+1} y_2 - {F_K}^{n+1} y_1 \|_0 \leq [(n+1)!]^{-1/2}(|||\lambda K|||)^{n+1} \| y_2 - y_1 \|_0$ $(n>0)$. Further, by the Weisinger Fixed Point

Theorem as before with $|||\lambda K|||$ replacing $\gamma_K(c)|\lambda|$, the unique $A_2(D)$-solution of equation (38) is $y = \lim_{n\to\infty} F_K{}^{n+1} f$ with error estimates $\| y - F_K{}^{n+1} f \|_0 \leq [(n+1)!]^{-1/2}(|||\lambda K|||)^{n+1} G(|||\lambda K|||) \| F_K f - f \|_0$ $(n \geq 0)$.

To apply $(\mathfrak{V}\mathfrak{K}_2)(D)$-kernels to an n-th order linear differential operator in monic form with coefficients that are restrictions of $H(\overline{D})$-functions to the segment $0 \leq r \leq 1$ of the real axis, one must recall the structure of the Green's function. Given the linear differential operator $L(D) = \sum_{\nu=0}^{n} a_\nu(x) D^{n-\nu}$ $(D = d/dx$, all a_ν's continuous on $a \leq x \leq b$, and $a_0(x) \neq 0)$ and n linearly independent linear functionals $U_\mu(y) = \sum_{\nu=0}^{n-1} [\alpha_{\mu\nu} y^{(\nu)}(a) + \beta_{\mu\nu} y^{(\nu)}(b)]$ on $C^{(n-1)}[a,b]$, the Green's function is constructed as follows ([6], Abschnitte 122., 123. und 124.). Let $\Phi(x)$ be the $n \times n$ fundamental matrix, whose first row $\Phi_1(x)$ consists of the integral basis $\phi_\nu(x)$ of $N(L(D))$ and the derivatives $\Phi_1{}^{(\mu)}(x)$ as subsequent rows. By means of $\Phi(x)$'s first $n-1$ rows $\Phi_{n-1}(x)$ and the n linear functionals U_μ one defines the fundamental solution $\gamma(x,\xi)$

$$\gamma(x,\xi) \equiv \frac{\operatorname{sgn}(x-\xi)}{2a_0(\xi)(\det\circ\Phi)(\xi)} \det \begin{pmatrix} \Phi_{n-1}(\xi) \\ \Phi_1(x) \end{pmatrix} \tag{44}$$

and the auxiliary quantity $\det(U\Phi_1)$, where $U\Phi_1$ is the $n \times n$ matrix with rows

$$U_\mu(\Phi_1) \equiv (U_\mu(\phi_1), U_\mu(\phi_2), \cdots, U_\mu(\phi_n)) \quad (1 \leq \mu \leq n).$$

From these, the Green's function $\Gamma(x,\xi)$ of the boundary value problem $L(D)y = 0$ with $U_\mu(y) = 0$ $(1 \leq \mu \leq n)$ is constructed a follows:

$$\Gamma(x,\xi) = \gamma(x,\xi) + \frac{Z(x,\xi)}{\det(U\Phi_1)}, \quad Z(x,\xi) \equiv \det \begin{pmatrix} 0 & \Phi_1(x) \\ (U\gamma)(\cdot,\xi) & U\Phi_1 \end{pmatrix}, \tag{45}$$

where the $(\mu+1)$-th row of the expression $Z(x,\xi)$ is

$$((U_\mu\gamma)(\cdot,\xi), U_\mu(\Phi_1)) = ((U_\mu\gamma)(\cdot,\xi), U_\mu(\phi_1), U_\mu(\phi_2), \cdots, U_\mu(\phi_n)) \quad (1 \leq \mu \leq n)$$

and $(\det\circ\Phi)(\xi)$ stands for the Wronskian $W(\Phi_1)(\xi) = W(\phi_1, \phi_2, \cdots, \phi_n)(\xi)$.

However, for the initial value problem $L(D)y = 0$ with $y^{(\mu-1)}(a) = c_\mu$ $(1 \leq \mu \leq n)$, viewed as the boundary value problem $L(D)y = 0$ with $U_\mu(y) = y^{(\mu-1)}(a)$ $(\alpha_{\mu\nu} = \delta_{(\mu-1)\nu}, \beta_{\mu\nu} = 0,\ 1 \leq \mu \leq n)$, the Green's function $\Gamma(x,\xi)$ reduces to the Volterra kernel

$$K(x,\xi) = \frac{1}{a_0(\xi) W(\Phi_1)(\xi)} \det \begin{pmatrix} \Phi_{n-1}(\xi) \\ \Phi_1(x) \end{pmatrix} \quad (a \leq \xi \leq x \leq b). \tag{46}$$

Hence, the Green's function of the general boundary value problem may be written as the sum of the Volterra Kernel $K(x,\xi)$ and a kernel $(\sum_{\nu=1}^{n} \phi_\nu \otimes \psi_\nu)(x,\xi)$ of finite rank

$$\Gamma(x,\xi) = K(x,\xi) + \sum_{\nu=1}^{n} \phi_\nu(x)\overline{\psi_\nu(\xi)},\ \mathrm{Rang}(\sum_{\nu=1}^{n} \phi_\nu \otimes \psi_\nu) \leq n \text{ and} \tag{47}$$

$$\sum_{\nu=1}^{n} \phi_\nu(x)\overline{\psi_\nu(\xi)} = -\frac{1}{2a_0(\xi)(\det\circ\Phi)(\xi)} \det\begin{pmatrix}\Phi_{n-1}(\xi)\\ \Phi_1(x)\end{pmatrix} + \frac{Z(x,\xi)}{\det(U\Phi_1)}$$
$$(a \leq x, \xi \leq b),$$

where expansion of the last two determinants along the respective rows $\Phi_1(x) = (\phi_1(x), \phi_2(x), \cdots, \phi_n(x))$ yields

$$\overline{\psi_\nu(\xi)} = \frac{(-1)^{n+\nu+1} W(\cdots, \phi_{\nu-1}, \phi_{\nu+1}, \cdots)(\xi)}{2a_0(\xi)W(\Phi_1)(\xi)} \ + \tag{48}$$

$$\frac{(-1)^\nu}{\det(U\Phi_1)} \det\begin{pmatrix}(U_1\gamma)(\cdot,\xi) & \cdots & U_1(\phi_{\nu-1}) & U_1(\phi_{\nu+1}) & \cdots\\ (U_2\gamma)(\cdot,\xi) & \cdots & U_2(\phi_{\nu-1}) & U_2(\phi_{\nu+1}) & \cdots\\ \vdots & \cdots & \vdots & \vdots & \cdots\\ (U_n\gamma)(\cdot,\xi) & \cdots & U_n(\phi_{\nu-1}) & U_n(\phi_{\nu+1}) & \cdots\end{pmatrix} \ (1 \leq \nu \leq n).$$

Moreover, by means of utilization of equation (44) one sees clearly that

$$(U_\mu\gamma)(\cdot,\xi) = \sum_{\nu=0}^{n-1} [\alpha_{\mu\nu}({D_1}^\nu\gamma)(a,\xi) + \beta_{\mu\nu}({D_1}^\nu\gamma)(b,\xi)]$$
$$= \frac{1}{2a_0(\xi)(\det\circ\Phi)(\xi)} \sum_{\nu=0}^{n-1}\left[-\alpha_{\mu\nu}\det\begin{pmatrix}\Phi_{n-1}(\xi)\\ {\Phi_1}^{(\nu)}(a)\end{pmatrix} + \beta_{\mu\nu} det\begin{pmatrix}\Phi_{n-1}(\xi)\\ {\Phi_1}^{(\nu)}(b)\end{pmatrix}\right]$$
$$(1 \leq \mu \leq n). \tag{49}$$

Consequently, the solution of the initial value problem $L(\mathcal{D})y = f(r)$ ($\mathcal{D} = d/dr$) with $y^{(\mu-1)}(0) = c_\mu$ $(1 \leq \mu \leq n)$ in terms of $\Phi(r)$, $\Phi_{n-1}(r')$, $\Phi_1(r)$ and $K(r,r')$ (obtained from setting $x = r$, $\xi = r'$, $[a,b] = [0,1]$, $a_0(r) \equiv 1$ and $\mathbf{c} = (c_1, c_2, \cdots, c_n)^T$ in expressions (44) and (46)) is

$$y(r) = -[(\det\circ\Phi)(0)]^{-1}\det\begin{pmatrix}0 & \Phi_1(r)\\ \mathbf{c} & \Phi(0)\end{pmatrix} + \int_0^r K(r,r')f(r')dr' \quad (0 \leq r \leq 1)$$

$$\text{with } K(r,r') = \frac{1}{(\det\circ\Phi)(r')}\det\begin{pmatrix}\Phi_{n-1}(r')\\ \Phi_1(r)\end{pmatrix} \quad (0 \le r' \le r \le 1). \quad (50)$$

Assuming $f(r)$ and $\Phi_1(r)$ to be the restrictions of $H(\overline{D})$-functions $f(z)$ and $\Phi_1(z)$ to the segment $0 \le r \le 1$ of the real axis, the question arises as to what the analytic extension of the solution $y(r)$ to all of $\overline{D}$ looks like. Since $a_0(r') \equiv 1$, $(\det\circ\Phi)(r') = W(\Phi_1)(r')$, $\Phi_{n-1}(r')$ are also restrictions of $H(\overline{D})$-functions to $[0,1]$, there exists a $(\mathfrak{V}\mathfrak{K}_2)(D)$-kernel $K(r,r',\theta)$ such that $K(r,r',0) = K(r,r')$. Theorem 1 and equation (25) for $\theta = 0$ imply $< K(r,r\cdot)|\overline{u} >=< K(r,r\cdot,0)|\overline{u} >= \sum_{n=0}^{\infty} r^n < u|k_n >$ for $0 \le r \le 1$. The unique determination of $H(\overline{D})$-functions by their values on $[0,1]$ gives $< K(re^{i\theta},\cdot re^{i\theta})|\overline{u} >=< K(r,r\cdot,\theta)|\overline{u} >$ for all $u \in L_2[0,1]$. Therefore, the holomorphic extension of y to all of $\overline{D}$ with the suitable $(\mathfrak{V}\mathfrak{K}_2)(D)$-kernel $K(r,r',\theta)$ is

$$y(re^{i\theta}) = -[(\det\circ\Phi)(0)]^{-1}\det\begin{pmatrix}0 & \Phi_1(re^{i\theta})\\ \mathbf{c} & \Phi(0)\end{pmatrix} + \int_0^r K(r,r',\theta)f(r'e^{i\theta})e^{i\theta}dr', \quad (51)$$

$$K(r,r',\theta) = [(\det\circ\Phi)(r'e^{i\theta})]^{-1}\det\begin{pmatrix}\Phi_{n-1}(r'e^{i\theta})\\ \Phi_1(re^{i\theta})\end{pmatrix} \quad ((r,r') \in \Box; -\pi \le \theta \le \pi).$$

Solving the λ-parameter family of initial value problems $L(\mathcal{D})y - \lambda y = f(r)$ with $y^{(\nu-1)}(0) = c_\nu$ $(1 \le \nu \le n)$, under the assumptions for $L(\mathcal{D})$ and f as before, is equivalent to solving the λ-parameter family of Volterra integral equations $y(r) - \lambda\int_0^r K(r,r',0)y(r')dr' = g(r)$, where $g(r)$ is again the restriction of the $H(\overline{D})$-function $g(re^{i\theta}) = \int_0^r K(r,r',\theta)f(r'e^{i\theta})e^{i\theta}dr'$ to $[0,1]$. This amounts, in operator notation, to solving $(I - \lambda K)y = g$ in $L_2[0,1]$ with $g = Kf$. Its unique $L_2[0,1]$-solution is $y = (I + \lambda H_\lambda)g$ with L_2-kernel $H_\lambda(r,r') = H_\lambda(K)(r,r',0)$, where H_λ satisfies the Fredholm resolvent equation $\lambda K H_\lambda = \lambda H_\lambda K = H_\lambda - K$ in the Banach algebra of $L_2[0,1]$-Volterra kernels on $\Box(\triangle)$([12], Chapter 2). Hence, $y(re^{i\theta})$ given by equation (42) is the holomorphic extension of the solution $y(r)$ to all of $\overline{D}$, namely

$$y(re^{i\theta}) = g(re^{i\theta}) + \lambda \int_0^r H_\lambda(r,r',\theta) g(r'e^{i\theta}) e^{i\theta} dr'$$
$$= \int_0^r H_\lambda(r,r',\theta) f(r'e^{i\theta}) e^{i\theta} dr' = \sum_{n=0}^{\infty} \lambda^n (K^{n+1} f)(re^{i\theta})$$
$$(\lambda \in \mathbb{C}, re^{i\theta} \in \overline{D}), \quad (52)$$

on account of: $(I + \lambda H_\lambda)g = (I + \lambda H_\lambda)Kf = (K + \lambda H_\lambda K)f = H_\lambda f$, the quasi-nilpotency K in the Banach algebra $(\mathfrak{V}\mathfrak{K}_2)(D)$ and equation (42).

If however, in the immediately preceding λ-parameter family of initial value problems the λy is replaced by $\lambda F(r,y)$, F satisfying a radial or angular Lipschitz condition, then the unique solutions of the λ-parameter family of non-linear initial value problems $L(\mathcal{D})y - \lambda F(r,y) = g(r)$ and $y^{(\nu-1)}(0) = c_\nu$ $(1 \leq \nu \leq n)$ are the unique solutions of the λ-parameter family of Hammerstein integral equations $y(r) = f(r) + \lambda \int\limits_0^r K(r,r',0) F(r', y(r')) dr'$, where $f(r) = \lambda \int\limits_0^r K(r,r',0) g(r') dr'$. Their holomorphic extensions $y(re^{i\theta})$ to all of $\overline{D}$ are the unique solutions of the λ-parameter family of radial Hammerstein integral equations (38), with f as already given and the $(\mathfrak{V}\mathfrak{K}_2)(D)$-kernel $K(r,r',\theta)$ appearing in equations (51).

Let the differential operator $L(\mathcal{D})$ and $f(r)$ be as before. Finally, the solution of the boundary value problem $L(\mathcal{D})y = f(r)$ with linearly independent linear functionals $U_\mu(y) = \sum\limits_{\nu=0}^{n-1} [\alpha_{\mu\nu} y^{(\nu)}(0) + \beta_{\mu\nu} y^{(\nu)}(1)] = 0$ $(1 \leq \mu \leq n)$ has only trivially solvable associated homogeneous boundary problem ([6], Abschnitt 124.), and the unique solution is $y(r) = \int\limits_0^1 \Gamma(r,r') f(r') dr'$, where the Green's function $\Gamma(r,r')$ is given by equation (45) under the substitution: $x = r$, $\xi = r'$ and $[a,b] = [0,1]$. By utilizing the representation (47) for the Green's function

$$\Gamma(r,r') = K(r,r') + \sum_{\nu=1}^{n} \phi_\nu(r) \overline{\psi_\nu(r')} = \big(K + \sum_{\nu=1}^{n} \phi_\nu \otimes \psi_\nu\big)(r,r')$$

in terms of ψ_ν given by expression (48) for $\xi = r'$, the solution of the boundary value problem becomes the restriction of the $H(\overline{D})$-function

$$y(re^{i\theta}) = \int_0^r K(r,r',\theta) f(r'e^{i\theta}) e^{i\theta} dr' + \sum_{\nu=1}^{n} < f|\psi_\nu > \phi_\nu(re^{i\theta}),$$

$$\text{where } < f|\psi_\nu >= \int_0^1 f(r')\overline{\psi_\nu(r')}dr' \ (1 \le \nu \le n), \quad (53)$$

to the interval $0 \le r \le 1$ of the real axis. In particular in terms of the notation in ([12], Chapter III)

$$y(r) = \int_0^r K(r, r', 0)f(r')dr' + \sum_{\nu=1}^n < f|\psi_\nu > \phi_\nu(r)$$

$$= \big([K + \sum_{\nu=1}^n \phi_\nu \otimes \psi_\nu] f\big)(r) \text{ with } K(r, r', 0) = K(r, r') \text{ on } \Box. \quad (54)$$

One has furthermore concomitant to this, the λ-parameter family of boundary value problems $L(\mathcal{D})y - \lambda y = f(r)$ with linearly independent linear functionals U_μ and $f \in H(\overline{D})$ as in the preceding paragraph. "Operating on the left" by the operator determined by the Green's function $\Gamma(r, r')$ on the differential equation $L(\mathcal{D})y - \lambda y = f$ leads to the equation $(I - \lambda K)y - \lambda(\sum_{\nu=1}^n \phi_\nu \otimes \psi_\nu)y = Kf + \sum_{\nu=1}^n < f|\psi_\nu > \phi_\nu \in L_2[0, 1]$. Thereafter, operating by $I + \lambda H_\lambda \in \mathcal{L}(L_2[0, 1])$ on the left of this expression produces

$$y - \lambda \sum_{\nu=1}^n < y|\psi_\nu > (I + \lambda H_\lambda)\phi_\nu$$

$$= (I + \lambda H_\lambda)Kf + \sum_{\nu=1}^n < f|\psi_\nu > (I + \lambda H_\lambda)\phi_\nu,$$

which written in the tensor product notation in $\mathcal{L}(L_2[0, 1])$ means

$$[I - \lambda \sum_{\nu=1}^n (I + \lambda H_\lambda)\phi_\nu \otimes \psi_\nu]y = [I + \lambda H_\lambda + \sum_{\nu=1}^n (I + \lambda H_\lambda)\phi_\nu \otimes \psi_\nu]f, \quad (55)$$

where $f \in L_2[0, 1]$. By letting $k(\lambda) = (k_{\mu\nu}(\lambda))$ be the $n \times n$ matrix whose entries are the entire functions $k_{\mu\nu}(\lambda) =< (I + \lambda H_\lambda)\phi_\nu|\psi_\mu >$ of variable λ, one defines $\Omega(\lambda) = (\Omega_{\mu\nu}(\lambda))$ to be the classical adjoint $\text{Adj}(I - \lambda k(\lambda))$ of matrix $I - \lambda k(\lambda)$ and $d(\lambda) \equiv \det(I - \lambda k(\lambda))$. $\text{Adj}(I - \lambda k(\lambda))$ and $d(\lambda)$ are $\mathbb{C}^{(n \times n)}$-valued and $\mathbb{C}$-valued entire functions of λ respectively; in particular, $d(\lambda) \neq 0$ except for a countable subset of $\mathbb{C}$ not accumulating in the finite plane. The well-known matrix relations $\lambda k(\lambda)\Omega(\lambda) = \lambda\Omega(\lambda)k(\lambda) = \Omega(\lambda) - d(\lambda)I$ implies for all λ satisfying $d(\lambda) \neq 0$ that

$$[I - \lambda \sum_{\nu=1}^{n} (I + \lambda H_\lambda)\phi_\nu \otimes \psi_\nu]^{-1}$$
$$= I + \frac{1}{d(\lambda)} \sum_{\mu,\nu=1}^{n} \Omega_{\mu\nu}(\lambda)(I + \lambda H_\lambda)\phi_\mu \otimes \psi_\nu. \quad (56)$$

Operating by this expression on the left of the equation (55), under the condition $d(\lambda) \neq 0$, where $H_\lambda{}^*$ stands for the adjoint of the integral operator H_λ in the Hilbert space $L_2[0,1]$, results in

$$y(r) = \big(\big[H_\lambda + \frac{1}{d(\lambda)} \sum_{\mu,\nu=1}^{n} \Omega_{\mu\nu}(\lambda)(I + \lambda H_\lambda)\phi_\mu \otimes (I + \overline{\lambda} H_\lambda{}^*)\psi_\nu\big] f\big)(r)$$
$$= \int_0^r H_\lambda(r, r', 0) f(r') dr' + \frac{1}{d(\lambda)} \sum_{\mu,\nu=1}^{n} \Omega_{\mu\nu}(\lambda)$$
$$\times < (I + \lambda H_\lambda) f | \psi_\nu > (I + \lambda H_\lambda)\phi_\mu(r) \quad (57)$$

for $0 \leq r \leq 1$ with $H_\lambda(r, r', \theta) = \sum_{n=0}^{\infty} \lambda^n K^{n+1}(r, r', \theta)$ given immediately after equation (41). The holomorphic extension of $y(r)$ to all of $\overline{D}$ turns out to be

$$y(re^{i\theta}) = \int_0^r H_\lambda(r, r', \theta) f(r'e^{i\theta}) e^{i\theta} dr' + \frac{1}{d(\lambda)} \sum_{\mu,\nu=1}^{n} \Omega_{\mu\nu}(\lambda)$$
$$\times < (I + \lambda H_\lambda) f | \psi_\nu > \left[\phi_\mu(re^{i\theta}) + \lambda \int_0^r H_\lambda(r, r', \theta)\phi_\mu(r'e^{i\theta}) e^{i\theta} dr'\right]$$
$$\text{for } d(\lambda) \neq 0, \quad (58)$$

$$< (I + \lambda H_\lambda) f | \psi_\nu > = \int_0^1 \left[f(r) + \lambda \int_0^r H_\lambda(r, r', 0) f(r') dr'\right] \overline{\psi_\nu(r)} dr$$
$$(1 \leq \nu \leq n).$$

In equation (58) the functional expression $< (I + \lambda H_\lambda) f | \psi_\nu >$ in terms of $(I + \lambda H_\lambda) \in \mathcal{L}(L_2[0,1])$ must be strictly confined to the Hilbert space $L_2[0,1]$, where $H_\lambda \in \mathcal{L}(L_2[0,1])$ has the L_2-kernel $H_\lambda(r, r', 0)$, and not $H_\lambda(r, r', \theta)$ with arbitrary θ. Therefore, in the expressions $< (I + \lambda H_\lambda) f | \psi_\nu >$ the H_λ is cannot be extended to the other kernels $H_\lambda(r, r', \theta)$ $(-\pi \leq \theta \leq \pi)$, because substituting for $(I + \lambda H_\lambda) f(r)$ and $\overline{\psi_\nu}(r)$ the expressions $(I + \lambda H_\lambda) f(re^{i\theta})$ and $\overline{\psi_\nu}(re^{i\theta})$ respectively would lead to

$$\int_0^1 \left[f(re^{i\theta}) + \lambda \int_0^r H_\lambda(r, r', \theta) f(r'e^{i\theta}) e^{i\theta} dr'\right] \overline{\psi_\nu}(re^{i\theta}) e^{i\theta} dr$$
$$= \{1 \text{ or } e^{i\theta}\} \times \int_0^1 \left[f(r) + \lambda \int_0^r H_\lambda(r, r', 0) f(r') dr'\right] \overline{\psi_\nu}(r) dr$$
$$\text{for all} \quad -\pi \le \theta \le \pi,$$

which is not possible unless $\theta = 0$.

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