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# SPECIAL LAGRANGIAN MANIFOLDS OBTAINED FROM COMPLEX GRASSMANNIANS

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**Abstract:** This paper gives an example of special Lagrangian manifold obtained from a hypersurface of a complex Grassmannian with vanishing first Chern class. The obtained manifold is a 1-torus bundle over the two dimensional real projective space. Such manifolds are interesting for mirror symmetry theory. Other examples of the same type are provided at the end of this article.

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## 1. Introduction

In 1982, when F. Harvey and H. Lawson introduced special Lagrangian manifolds in [7], their main interest was calibration problems. Now, these manifolds give another approach to mirror symmetry and string theory and become crucial in these fields. A conjecture due to Strominger, Yau and Zaslow in [9] explains mirror symmetry in a fairly mathematical way. An important class of examples of special Lagrangian manifolds $L$ may be found among the submanifolds of a $n$

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complex dimensional manifold $M$, which is compact Kählerian with vanishing first Chern class. Then, the definition is given by the following points:

— the symplectic form $\omega$ associated to the kählerian structure restricted to the $n$ real dimensional submanifold $L$ must identically vanish, i.e. $i^*\omega = 0$, where $i : L \rightarrow M$ is the canonical injection. So the maximal isotropic submanifold $L$, is endowed by a Lagrangian structure.

— There exists a $(n, 0)$-holomorphic volume form $\Omega$ such that

$$\Omega \wedge \overline{\Omega} = \frac{2^n(-i)^{n^2}}{n!}\omega^n.$$

This last one is given by the solution of Calabi conjecture (given by Aubin in [2] and Yau in [10]).

— Finally, $L$ is a special Lagrangian manifold if we have $i^*\Omega = dV_L$ (in the general case we have $i^*\Omega = \lambda dV_L$ where $\lambda \in S^1$).

For a good initiation to the problem of Calabi conjecture and its solution, one can refer to the book [1]. In our case and in order to give examples of special Lagrangian manifolds, we use, as it was done by R.L. Bryant [4] and D. Joyce [8], a real structure $c$ on $M$, i.e. an anti-holomorphic involution $c : M \rightarrow M$ such that $c^*\omega = -\omega$ and $c^*\Omega = \overline{\Omega}$. The set $L$ of fixed points of this involution is a special Lagrangian manifold. $L$ can be viewed as the real locus of $M$.

This paper gives an example of a 3 dimensional complex hypersurface of the Grassmannian $G_{2,4}\mathbb{C}$ with vanishing $C_1$. It is well known that its real locus is a special Lagrangian manifold. Actually, it is a complete intersection of a quadric and a quartic in $\mathbb{P}_5\mathbb{C}$. The first interpretation (hypersurface of $G_{2,4}\mathbb{C}$) does not require particular skills in Algebraic Geometry. Therefore, both approaches will be developed in this article. We also give an interpretation of the obtained 3-dimensional real locus as a 1-torus bundle over $\mathbb{P}_2\mathbb{R}$. Using the second point of view (complete intersection in $\mathbb{P}_5\mathbb{C}$), we show that this is nothing but $S^3/\mathbb{Z}_4$. The first section of this paper is devoted to the description of the manifold $X$ (definition and calculus of its first Chern class). In the second part, we stand and prove the main result (Theorem 3), giving both interpretations of the real locus $L$ of $X$. Finally, we choose to describe two other examples among a large class of manifolds which are similar to $X$.

## 2. Definition and Description of the Manifold $X$

Let us consider the following equation:

$$F(u, u') = (z_0 z_1' - z_1 z_0')^4 + (z_0 z_2' - z_2 z_0')^4 + (z_0 z_3' - z_3 z_0')^4$$

$$- \quad (z_1z_2' - z_2z_1')^4 - (z_1z_3' - z_3z_1')^4 - (z_2z_3' - z_3z_2')^4 = 0, \qquad (1)$$

where $u = (z_0, z_1, z_2, z_3)$ and $u' = (z_0', z_1', z_2', z_3')$ are two independent vectors of $\mathbb{C}^4$. It is easy to see that this equation depends only on the 2-plane of $\mathbb{C}^4$ given by $u$ and $u'$. Consequently, (1) defines a subset of the Grassmannian $G_{2,4}\mathbb{C}$, set of complex two dimensional linear spaces of $\mathbb{C}^4$.

**Lemma 1.** *The equation (1) defines a holomorphic hypersurface of $G_{2,4}\mathbb{C}$ which may be identified with a complete intersection of a quadric and a quartic in $\mathbb{P}_5\mathbb{C}$.*

*Proof.* In order to find the rank of the linear tangent map of $F$ in a $G_{2,4}\mathbb{C}$ classical coordinates system $(\zeta_i)_{i\in\{1,..,4\}}$, we consider:

$$\left\{\begin{array}{r} \zeta_1^3 - \zeta_4\left(\zeta_1\zeta_4 - \zeta_2\zeta_3\right)^3 = 0\,, \\ \zeta_2^3 + \zeta_3\left(\zeta_1\zeta_4 - \zeta_2\zeta_3\right)^3 = 0\,, \\ -\zeta_3^3 + \zeta_2\left(\zeta_1\zeta_4 - \zeta_2\zeta_3\right)^3 = 0\,, \\ -\zeta_4^3 - \zeta_1\left(\zeta_1\zeta_4 - \zeta_2\zeta_3\right)^3 = 0\,. \end{array}\right. \qquad (2)$$

Using equation (1), we firstly prove that the solutions of this system are necessarily of norm one. Taking into account all different cases, explicit computations lead to a contradiction with the system (2).

Let us give another description of $X$. To this end, we use the classical identification of $G_{2,4}\mathbb{C}$ with the quadric in $\mathbb{P}_5\mathbb{C}$ given by:

$$\eta_0\eta_5 - \eta_1\eta_4 + \eta_2\eta_3 = 0,$$

via the map which associates to every point

$$\begin{pmatrix} z_0 & z_0' \\ z_1 & z_1' \\ z_2 & z_2' \\ z_3 & z_3' \end{pmatrix} \in G_{2,4}\mathbb{C}$$

described in the above coordinates, the point of $\mathbb{P}_5\mathbb{C}$:

$$[\eta_0 = z_0z_1' - z_1z_0', \eta_1 = z_0z_2' - z_2z_0', \eta_2 = z_0z_3' - z_3z_0',$$
$$\eta_3 = z_1z_2' - z_2z_1', \eta_4 = z_1z_3' - z_3z_1', \eta_5 = z_2z_3' - z_3z_2'].$$

So, $X$ appears as a complete intersection of a quadric and a quartic in $\mathbb{P}_5\mathbb{C}$, given by:

$$\left\{\begin{array}{r} \eta_0\eta_5 - \eta_1\eta_4 + \eta_2\eta_3 = 0\,, \\ \eta_0^4 + \eta_1^4 + \eta_2^4 - \eta_3^4 - \eta_4^4 - \eta_5^4 = 0\,. \end{array}\right. \qquad (3)$$

$\square$

**Lemma 2.** *$X$ is a manifold with vanishing first Chern class.*

*Proof.* In a first time, we shall give a "self-contained" proof, using the description of $X$, given by equation (1). Then, in a second time, taking into account equation (3), we give a shorter proof which uses some concepts of algebraic geometry.

1) Let us prove that the determinant bundle $\Lambda^3 T^* X$ is trivializable by giving a $(3,0)$-holomorphic volume form $\Omega$ from a $(4,0)$-meromorphic form on $G_{2,4}\mathbb{C}$, using Poincaré residue. It corresponds to the generalization of the classical Cauchy residue at a point of a domain of $\mathbb{C}$ to the concept of residue in a hypersurface of a $n$ dimensional complex manifold. If $\eta$ is a $n$-meromorphic form of such a manifold which has first order poles on the hypersurface $X$ locally defined by the equation $g = 0$, then $\eta$ may be locally written as:

$$\eta = \frac{\gamma \wedge dg}{g} + \delta,$$

where $\gamma$ and $\delta$ are respectively $(n-1)$ and $n$ holomorphic forms. Then, the restriction of $\gamma$ to $X$ is well defined as a $(n-1)$ holomorphic form on $X$. We say that $\gamma$ is the Poincaré residue of $\eta$.

In our case, let us consider the open chart set $U_{01}$ of $G_{2,4}\mathbb{C}$ where the $\{U_{ij}\}_{0\leq i<j\leq 3}$ are given by:

$$U_{ij} = \{(u,u') \in \mathbb{C}^4 \times \mathbb{C}^4 \\ : u = (z_0, z_1, z_2, z_3), u' = (z'_0, z'_1, z'_2, z'_3), z_i z'_j - z_j z'_i \neq 0\}.$$

The chart maps $\varphi_{ij} : U_{ij} \to \mathbb{C}^4 \sim M_2(\mathbb{C})$, are defined, similarly to $\varphi_{01}$ in the following manner:

$$\varphi_{01}(u,u') = \begin{pmatrix} z_2 & z'_2 \\ z_3 & z'_3 \end{pmatrix} \times \begin{pmatrix} z_0 & z'_0 \\ z_1 & z'_1 \end{pmatrix}^{-1} = \begin{pmatrix} \zeta_3 & \zeta_1 \\ \zeta_4 & \zeta_2 \end{pmatrix}.$$

The expression of $F$ in $U_{01}$ is given by

$$f_{01}(\zeta_1,\zeta_2,\zeta_3,\zeta_4) = 1 + \zeta_1^4 + \zeta_2^4 - \zeta_3^4 - \zeta_4^4 - (\zeta_1\zeta_4 - \zeta_2\zeta_3)^4,$$

In $U_{01}$, let us consider :

$$\eta = \frac{d\zeta_1 \wedge d\zeta_2 \wedge d\zeta_3 \wedge d\zeta_4}{1 + \zeta_1^4 + \zeta_2^4 - \zeta_3^4 - \zeta_4^4 - (\zeta_1\zeta_4 - \zeta_2\zeta_3)^4}.$$

$\eta$ is the local expression in the open chart set $U_{01}$ of a $n$ meromorphic form globally defined on $G_{2,4}\mathbb{C}$ whose poles are along $X$. Indeed, the power 4 is the correct one. This can easily be seen in proceeding to the change of charts. There are two types of change of charts in $G_{2,4}$. The first ones (or the easiest) are of the form:

$$(\zeta_1, \zeta_2, \zeta_3, \zeta_4) \longrightarrow (-\frac{\zeta_2}{\zeta_1}, \frac{1}{\zeta_1}, -\frac{D}{\zeta_1}, \frac{\zeta_3}{\zeta_1}),$$

where $D = (\zeta_1\zeta_4 - \zeta_2\zeta_3)$. The determinant of such a change of charts is equal to $1/\zeta_1^4$, and so $\eta$ may be extended to the new chart. The second change of charts is:

$$(\zeta_1, \zeta_2, \zeta_3, \zeta_4) \longrightarrow (-\frac{\zeta_4}{D}, \frac{\zeta_2}{D}, \frac{\zeta_3}{D}, -\frac{\zeta_1}{D}).$$

Its determinant (much more difficult to compute) is equal to $1/D^4$, so, we may extend the expression to the new chart. This result has been established in the general case of $G_{p,p+q}\mathbb{C}$ par J. Grivaux in [6]. Using a skill calculation, he has found the Einstein constant used in the Kähler potential of Grassmannians. The Poincaré residue of

$$\eta = \frac{d\zeta_1 \wedge d\zeta_2 \wedge d\zeta_3 \wedge d\zeta_4}{1 + \zeta_1^4 + \zeta_2^4 - \zeta_3^4 - \zeta_4^4 - (\zeta_1\zeta_4 - \zeta_2\zeta_3)^4}$$

is given, in a submersion chart $X$ obtained as a sub-chart of $(U_{01}, \varphi_{01})$ considering the condition $\partial f_{01}/\partial\zeta_1 \neq 0$, by the expression:

$$\gamma = \frac{-d\zeta_2 \wedge d\zeta_3 \wedge d\zeta_4}{\partial f_{01}/\partial\zeta_1},$$

which defines a holomorphic 3 volume form on $X$. This proves that $C_1(X) = 0$.

2) Using the description (3) of $X$, we can directly establish this last result. Actually, a similar proof to the preceding shows that a hypersurface of degree $m+1$ of $\mathbb{P}_m\mathbb{C}$ has necessarily a vanishing $C_1$. Taking into account that the sum of the degrees of a quadric and a quartic is equal to 6, and because our intersection is complete in $\mathbb{P}_5\mathbb{C}$, we obtain the result thanks to the adjunction formula. □

## 3. Lagrangian Submanifold

**Theorem 3.** *The real locus $L$ of $X$ is a special Lagrangian submanifold endowed with a circle bundle structure over $\mathbb{P}_2\mathbb{R}$ which may be identified with $S^3/\mathbb{Z}_4$.*

*Proof.* In order to show that $L$ is a special Lagrangian submanifold, we use the result given by R.L. Bryant [4]: the real locus of a trivial first Chern class manifold (that is to say the fixed points of an anti-holomorphic involution), when it is non-empty, is a special Lagrangian manifold which is, according to Lemma 1 and Lemma 2, the case of the manifold $X$ (the anti-holomorphic involution we use here is the classical conjugation). $L = L'/\sim$, where $L'$ is the set of the 2-planes of $\mathbb{R}^4$ seen as pairs of independent vectors $(u, u') \in \mathbb{R}^4 \times \mathbb{R}^4$, $u = (x_0, x_1, x_2, x_3), u' = (x'_0, x'_1, x'_2, x'_3)$ (defining a point of $G_{2,4}\mathbb{R}$) such that:

$$(x_0x'_1 - x_1x'_0)^4 + (x_0x'_2 - x_2x'_0)^4 + (x_0x'_3 - x_3x'_0)^4 = 1\,, \qquad (E_1)$$
$$(x_1x'_2 - x_2x'_1)^4 + (x_1x'_3 - x_3x'_1)^4 + (x_2x'_3 - x_3x'_2)^4 = 1\,, \qquad (E_2)$$

and $\sim$ is the equivalence relation defined by: $(u, u') \sim (v, v') \in \mathbb{R}^4 \times \mathbb{R}^4$, if and only if $v = au + cu'$ and $v' = bu + du'$ with $ad - bc = \pm 1$.

$(E_1)$ and $(E_2)$ come from the equation (1) and a given normalization. So $L'$ is a submanifold of $\mathbb{R}^8$, diffeomorphic to the pull-back of $(1, 1) \in \mathbb{R}^2$ by the submersion

$$\psi : (\mathbb{R} \times \mathbb{R}^3) \times (\mathbb{R} \times \mathbb{R}^3) \to \mathbb{R}^2$$

defined by

$$\psi((\alpha, u_0), (\alpha', u_0')) = (\|u_0 \wedge u_0'\|^2, \|\alpha' u_0 - \alpha u_0'\|^2).$$

In the former description $u = (\alpha, u_0) \in \mathbb{R}^4$ where $\alpha = x_0$ and $u_0 = (x_1, x_2, x_3) \in \mathbb{R}^3$ is the projection of $u = (x_0, x_1, x_2, x_3)$ on $\mathbb{R}^3 = \{(0, x, y, z) \in \mathbb{R}^4\}$ (it is the same for the prime items). So the expression $u_0 \wedge u_0'$ is intrinsic. The result is more difficult to obtain for $\|\alpha' u_0 - \alpha u_0'\|^2$, after quotienting by $\sim$. Indeed, if

$$v = au + cu' = (\beta, v_0) \text{ and } v' = bu + du' = (\beta', v_0') \text{ with } ad - bc = \pm 1,$$

we have

$$\beta = a\alpha + c\alpha', \quad \beta' = b\alpha + d\alpha', \;\; v_0 = au_0 + cu_0' \text{ and } v_0' = bu_0 + du_0'.$$

So

$$\begin{aligned} \beta' v_0 - \beta v_0' &= (b\alpha + d\alpha')(au_0 + cu_0') - (a\alpha + c\alpha')(bu_0 + du_0') \\ &= (ad - bc)(\alpha' u_0 - \alpha u_0') = \pm(\alpha' u_0 - \alpha u_0')\,. \end{aligned}$$

The interpretation of $L$ as a circle bundle over $\mathbb{P}_2\mathbb{R}$ may be realized as follows:

— $(E_2)$ determines the basis of this bundle and corresponds to the fact that the vectorial product of the vectors projections $u = (x_0, x_1, x_2, x_3)$ and $u' = (x_0', x_1', x_2', x_3')$ on $\mathbb{R}^3 = \{(0, x, y, z) \in \mathbb{R}^4\}$ is unitary for a certain norm. According to $(E_2)$ the vectors $u_0 = (0, x_1, x_2, x_3)$ and $u_0{}' = (0, x_1', x_2', x_3')$ are linearly independent. Then they define a point of the Grassmannian $G_{2,3}\mathbb{R}$, that is to say a point of $\mathbb{P}_2\mathbb{R}$ (given by their vectorial product).

— $(E_1)$ describes the fibres above $\mathbb{P}_2\mathbb{R}$. To any pair $(w, w')$ of independent vectors of $\mathbb{R}^3$, corresponds the circle in the basis $\{w, w'\}$ given by the equation $\|\alpha' w - \alpha w'\|^2 = 1$.

Using the second interpretation of $X$ (intersection of a quadric and a quartic), and the notations used in the equations (3), $L$ may be identified with $S^3/\mathbb{Z}_4$ in the following way: It is the set of the vectors pairs of $\mathbb{R}^3$ $(u, v)$ where $u = (\eta_0, \eta_1, \eta_2)$ and $v = (\eta_5, -\eta_4, \eta_3)$ with vanishing scalar product fulfilling the normalization condition

$$\eta_0^4 + \eta_1^4 + \eta_2^4 = \eta_5^4 + (-\eta_4)^4 + \eta_3^4 = 1,$$

which is topologically equivalent to $\|u\| = \|v\|$. If one considers the unit vectors $u' = u/\|u\|$ and $v' = v/\|v\|$, the triple $(u', v', u' \wedge v')$ is a direct orthonormal basis of $\mathbb{R}^3$ which can be viewed as a matrix in $SO(3)$. The one to one correspondence

$$[\eta_0, .., \eta_5] \longrightarrow \{(u', v'), (-u', -v')\},$$

defines a map from $L$ in $SO(3)/\mathbb{Z}_2$, where $\mathbb{Z}_2$ is the subgroup $\{Id, \sigma\}$ of $SO(3)$ generated by the matrix $\sigma = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, which corresponds to the map

$$(u', v', u' \wedge v') \longrightarrow (-u', -v', u' \wedge v').$$

This map being a diffeomorphism, $L$ is diffeomorphic to $SO(3)/\mathbb{Z}_2$ or equivalently $SU(2)/\mathbb{Z}_4$ that is $S^3/\mathbb{Z}_4$, using the double cover $SU(2) \longrightarrow SO(3)$. We can recover the description of $L$ (given in 1)) as a circle bundle over $\mathbb{P}_2\mathbb{R}$ by projecting

$$\begin{array}{ccc} SO(3)/\mathbb{Z}_2 & \longrightarrow & \mathbb{P}_2\mathbb{R}\,, \\ \{(u', v', u' \wedge v'), (-u', -v', u' \wedge v')\} & \longrightarrow & \{u', -u'\}. \end{array}$$

□

## 4. Other Examples

To get other natural examples of special Lagrangian submanifolds similar to $X$ it suffices to multiply, for example, some factors by constants in equation (1), keeping at least one minus sign opposite to the others (in order to get a non empty real locus). The nature of the obtained special Lagrangian submanifolds may be quite different of the submanifold $L$.

1) Let us consider the following example whose interpretation is of similar interest to the above one:

$$(z_0z_1' - z_1z_0')^4 - (z_0z_2' - z_2z_0')^4 - (z_0z_3' - z_3z_0')^4 - (z_1z_2' - z_2z_1')^4 \\ -(z_1z_3' - z_3z_1')^4 - 2(z_2z_3' - z_3z_2')^4 = 0\,, \qquad (4)$$

where $u = (z_0, z_1, z_2, z_3)$ and $u' = (z_0', z_1', z_2', z_3')$ are two independent vectors of $\mathbb{C}^4$, defining a 2-plane of $\mathbb{C}^4$, that is to say a point of $G_{2,4}\mathbb{C}$. As it was realized above for $X$, we prove that this hypersurface is a trivial first Chern class submanifold. Its real locus which is a special Lagrangian submanifold is given (using a good normalization) by the equations

$$\hat{E}_1 : \; (x_0x_1' - x_1x_0')^4 = 1$$

and

$$\hat{E}_2 : \; (x_1x_2' - x_2x_1')^4 + (x_1x_3' - x_3x_1')^4 + 2(x_2x_3' - x_3x_2')^4 \\ +(x_0x_2' - x_2x_0')^4 + (x_0x_3' - x_3x_0')^4 = 1,$$

quotiented by the action $\sim$ described above for $X$.

— $\hat{E}_1$ indicates that $L'$ is a subset of the open chart set $(x_0x_1' - x_1x_0') \neq 0$ of $G_{2,4}\mathbb{R}$.

— $\hat{E}_2$ may be written as:

$$\hat{E}_2 \; : \; (x_1x_2' - x_2x_1')^4 + (x_1x_3' - x_3x_1')^4 + (x_2x_3' - x_3x_2')^4 \\ (x_2x_3' - x_3x_2')^4 + (x_0x_2' - x_2x_0')^4 + (x_0x_3' - x_3x_0')^4 = 1,$$

which is a set diffeomorphic to

$$\|u_0 \wedge u_0'\|^2 + \|u_1 \wedge u_1'\|^2 = 1,$$

where $u_0$ and $u_1$ are the projections of $u = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4$ respectively on $\{x_0 = 0\}$ and $\{x_1 = 0\}$ seen as $\mathbb{R}^3$ (we have the same situation for the prime items).

2) A second example is given for any $u = (z_0, z_1, z_2, z_3)$ and $u' = (z'_0, z'_1, z'_2, z'_3)$, independent vectors of $\mathbb{C}^4$ (seen as a point of $G_{2,4}\mathbb{C}$), by:

$$(z_0 z'_1 - z_1 z'_0)^4 + (z_0 z'_2 - z_2 z'_0)^4 - (z_0 z'_3 - z_3 z'_0)^4 - 2(z_1 z'_2 - z_2 z'_1)^4 \\ -2(z_1 z'_3 - z_3 z'_1)^4 - 2(z_2 z'_3 - z_3 z'_2)^4 = 0. \qquad (5)$$

and defining a hypersurface of $G_{2,4}\mathbb{C}$ with trivial first Chern class. Its real locus $R$, which is a special Lagrangian manifold is given, after the normalization used in both former examples, by the equations:

$$(x_0 x'_1 - x_1 x'_0)^4 + (x_0 x'_2 - x_2 x'_0)^4 = 1\,, \qquad (\widetilde{E}_1)$$

$$(x_0 x'_3 - x_3 x'_0)^4 + 2(x_1 x'_2 - x_2 x'_1)^4 + 2(x_1 x'_3 - x_3 x'_1)^4 \\ + 2(x_2 x'_3 - x_3 x'_2)^4 = 1\,, \qquad (\widetilde{E}_2)$$

again quotiented by $\sim$ (as described above). Using a normalization, this system is equivalent to the following one:

$$(x_0 x'_1 - x_1 x'_0)^4 + (x_0 x'_2 - x_2 x'_0)^4 + (x_0 x'_3 - x_3 x'_0)^4 = 1\,, \qquad (\widetilde{E'}_1)$$

$$2(x_0 x'_3 - x_3 x'_0)^4 + 2(x_1 x'_2 - x_2 x'_1)^4 + 2(x_1 x'_3 - x_3 x'_1)^4 \\ + 2(x_2 x'_3 - x_3 x'_2)^4 = 1\,. \qquad (\widetilde{E'}_2)$$

As it was noticed in the case of $X$, $(\widetilde{E'}_1)$ describes the circle $\|x_0 u'_0 - x'_0 u_0\|^2 = 1$. Recall that $u_0$ is the projection of $u = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4$ on $\{x_0 = 0\}$ (idem for the prime items). The interpretation of $(\tilde{E}'_2)$ using classical geometric objects seems to be a little more difficult. However on can affirm that the set described by $(\tilde{E}'_2)$ contains the ball $\|u_0 \wedge u'_0\|^2 < 1/2$.

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