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ON ITERATIVE APPROXIMATIONS OF SOLUTIONS FOR A SYSTEM GENERALIZED NONLINEAR VARIATIONAL-LIKE INCLUSIONS

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Abstract: In this paper we introduce a new class of system of generalized nonlinear variational-like inclusions in Hilbert spaces. By using fixed point method and resolvent operator technique, we suggest iterative algorithms for solving this class of system of generalized nonlinear variational-like inclusions. The results presented in this paper improve and unify many known results in the literature.

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1. Introduction

It is well known that variational inequality theory has wide applications to a lot of fields including, for example, mechanics, physics, optimization and control, nonlinear programming, economics, and engineering sciences (see, [1-4, 6-11]). Useful and important generalizations of variational inequalities are variational inclusions.

Recently, Chidume, Kazmi and Zegeye [2] introduced various types of η-accretive mappings in real Banach spaces, established some properties of η-proximal point mappings for η-m-accretive mappings, and developed an iterative algorithm for a class of general variational-like inclusions involving η-accretive mappings in Banach spaces. Cai, Liu, Shim and Kang [1], Kazmi and Bhat [4], Nie, Liu, Kim and Kang [6] and Verma [7] introduced a few of systems of nonlinear variational inequalities and variational-like inclusions in Hilbert spaces, proved the existence of solutions and suggested several iteration algorithms for these systems of variational inequalities and variational-like inclusions and discussed convergence criteria for the iteration algorithms.

Inspired and motivated by recent research works [1-4, 6-11], in this paper, we introduce and study a new system of generalized nonlinear variational-like inclusions in Hilbert spaces. By using the resolvent operator technique for maximal η-monotone mapping, we establish the equivalence among the system of generalized nonlinear variational-like inclusions, resolvent equations and fixed point problems. Employing this equivalence we construct some iterative algorithms and prove the convergence of the iterative sequences generated by these algorithms. The results presented here also include the corresponding results in [1-4, 6-11] as special cases.

2. Preliminaries

Throughout this paper, we assume that H is a real Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Recall the following definitions and lemmas, which are used in the sequel.

Definition 2.1. Let $N : H \times H \to H$ and $g : H \to H$ be two mappings.

(1) g is said to be a-Lipschitz continuous if there exists a constant $a > 0$ such that

$$\|g(u) - g(v)\| \le a\|u - v\|, \quad \forall u, v \in H;$$

(2) g is said to be b-strongly monotone if there exists a constant $b > 0$ such that

$$\langle g(u) - g(v), u - v\rangle \ge b\|u - v\|^2, \quad \forall u, v \in H;$$

(3) g is said to be relaxed γ-r-cocoercive if there exist constants $\gamma, r > 0$ such that

$$\langle g(u) - g(v), u - v\rangle \ge (-\gamma)\|g(u) - g(v)\|^2 + r\|u - v\|^2, \quad \forall u, v \in H;$$

(4) N is said to be δ-Lipschitz continuous in the first argument if there exists a constant $\delta > 0$ such that

$$\|N(u, x) - N(v, x)\| \le \delta\|u - v\|, \quad \forall u, v, x \in H;$$

(5) N is said to be τ-strongly monotone in the first argument if there exists a constant $\tau > 0$ such that

$$\langle N(u, x) - N(v, x), u - v\rangle \ge \tau\|u - v\|^2, \quad \forall u, v, x \in H;$$

(6) N is said to be relaxed γ-r-cocoercive in the first argument if there exist constants $\gamma, r > 0$ such that

$$\langle N(u, x) - N(v, x), u - v\rangle \ge (-\gamma)\|N(u, x) - N(v, x)\|^2 + r\|u - v\|^2, \quad \forall u, v, x \in H.$$

Similarly, we can define the Lipschitz continuity, strong monotonicity and relaxed cocoercivity of the mapping N in the second argument.

Definition 2.2. A mapping $\eta : H \times H \to H$ is said to be:
(1) δ-strongly monotone if there exists a constant $\delta > 0$ such that

$$\langle u - v, \eta(u, v)\rangle \ge \delta\|u - v\|^2, \quad \forall u, v \in H;$$

(2) τ-Lipschitz continuous if there exists a constant $\tau > 0$ such that

$$\|\eta(u, v)\| \le \tau\|u - v\|, \quad \forall u, v \in H.$$

Definition 2.3. Let $\eta, N : H \times H \to H$ and $g : H \to H$ be single-valued mappings, $M : H \to 2^H$, where 2^H is the power set of H, be a multi-valued mapping.

(1) M is said to be η-monotone if

$$\langle x - y, \eta(u, v)\rangle \geq 0, \quad \forall u, v \in H, x \in Mu, y \in Mv;$$

(2) M is said to be (δ, η)-strongly monotone if there exists a constant $\delta > 0$ such that

$$\langle x - y, \eta(u, v)\rangle \geq \delta\|u - v\|^2, \quad \forall u, v \in H, x \in Mu, y \in Mv;$$

(3) M is said to be maximal η-monotone if M is η-monotone and $(I + \rho M)(H) = H$, for any $\rho > 0$, where I stands for an identity operator.

(4) g is said to be (δ, η)-strongly monotone if there exists a constant $\delta > 0$ such that

$$\langle g(u) - g(v), \eta(u, v)\rangle \geq \delta\|u - v\|^2, \quad \forall u, v \in H;$$

(5) N is said to be (δ, η)-strongly monotone in the first argument if there exists a constant $\delta > 0$ such that

$$\langle N(u, x) - N(v, x), \eta(u, v)\rangle \geq \delta\|u - v\|^2, \quad \forall u, v, x \in H.$$

Similarly, we can define N is (δ, η)-strongly monotone in the second argument.

Lemma 2.1. (see [2, 3]) *Let $\eta : H \times H \to H$ be δ-strongly monotone and τ-Lipschitz continuous mapping and let $M : H \to 2^H$ be maximal η-monotone. Then the following conclusions hold:*

(1) $\langle x - y, \eta(u, v)\rangle \geq 0, \forall (y, v) \in \text{Graph}\,(M)$ *implies* $(x, u) \in \text{Graph}\,(M)$, *where* $\text{Graph}\,(M) = \{(x, u) \in H \times H : x \in Mu\}$;

(2) *the resolvent operator* $J_\rho^{\partial\varphi} = (I + \rho M)^{-1}$ *of M is single-valued for any* $\rho > 0$.

Lemma 2.2. (see [2]) *Let $\eta : H \times H \to H$ be δ-strongly monotone and τ-Lipschitz continuous and let $M : H \to 2^H$ be maximal η-monotone. Then the resolvent operator J_ρ^M is $\frac{\tau}{\delta}$-Lipschitz continuous, i.e.,*

$$\|J_\rho^M(u) - J_\rho^M(v)\| \leq \frac{\tau}{\delta}\|u - v\|, \quad \forall u, v \in H, \tag{2.3}$$

where $\rho > 0$ is a constant.

Lemma 2.3. (see [5]) *Let* $\{a_n\}_{n\geq 0}, \{b_n\}_{n\geq 0}, \{c_n\}_{n\geq 0}$ *and* $\{t_n\}_{n\geq 0}$ *be four sequences of nonnegative numbers satisfying*

$$a_{n+1} \leq (1-t_n)a_n + t_n b_n + c_n, \quad n \geq 0,$$

where $\{t_n\}_{n\geq 0} \subseteq [0,1], \sum_{n=0}^{\infty} t_n = +\infty, \lim_{n\to\infty} b_n = 0$ *and* $\sum_{n=0}^{\infty} c_n < \infty$. *Then* $\lim_{n\to\infty} a_n = 0$.

Let $N, \eta : H \times H \to H, \quad T, f, g, h : H \to H$ be single-valued mappings and let $M : H \to 2^H$ be a maximal η-monotone mapping. We now introduce the following system of generalized nonlinear variational-like inclusions (SGNVLI):

Find $u, v \in H$ such that

$$\begin{cases} 0 \in f(u) - g(v) + \rho(N(u, g(v)) + M(h(u))), & \rho > 0, \\ 0 \in g(v) - f(u) + \gamma(T(u) + M(g(v))), & \gamma > 0. \end{cases} \tag{2.4}$$

Remark 2.1. It is clear that SGNVLI (2.4) includes various classes of systems of variational inequalities and variational-like inclusions in [1-4, 6-11] as special cases.

3. Main Results

Lemma 3.1. *Let* $N, \eta : H \times H \to H, T, f, g, h : H \to H$ *be single-valued mappings and let* $M : H \to 2^H$ *be a maximal* η*-monotone mapping. Then the following statements are equivalent:*

(a) *SGNVLI (2.4) has a solution* $(u, v) \in H \times H$.

(b) *There exists* $(u, v) \in H \times H$ *satisfying*

$$\begin{aligned} h(u) &= J_{\rho}^{M}[h(u) + g(v) - f(u) - \rho N(u, g(v))], \\ g(v) &= J_{\gamma}^{M}[f(u) - \gamma T(u)], \end{aligned} \tag{3.1}$$

where $\rho, \gamma > 0$ *are two constants.*

(c) *The mapping* $G : H \to H$ *defined by*

$$\begin{aligned} G(x) = x - h(x) + J_{\rho}^{M}[h(x) - f(x) + J_{\gamma}^{M}(f(x) - \gamma T(x)) \\ - \rho N(x, J_{\gamma}^{M}(f(x) - \gamma T(x)))], \quad \forall x \in H, \end{aligned} \tag{3.2}$$

has a fixed point $u \in H$ *and* $g(v) = J_{\gamma}^{M}(f(u) - \gamma T(u))$.

Exploiting Lemma 3.1, we now suggest the following iterative algorithms for solving SGNVLI (2.4).

Algorithm 3.1. For given $u_0 \in H$, compute $\{u_n\}_{n\geq 0}$ and $\{g(v_n)\}_{n\geq 0}$ from the iterative schemes

$$\begin{aligned} &t_n = (1-b_n)u_n + b_n G(u_n) + q_n, \quad u_{n+1} = (1-a_n)u_n + a_n G(t_n) + l_n, \\ &g(v_n) = J_\gamma^M(f(u_n) - \gamma T(u_n)) + p_n, \end{aligned} \tag{3.3}$$

for all $n \geq 0$, where G is defined by (3.2), $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ are any sequences in $[0,1]$ and $\{q_n\}_{n\geq 0}, \{l_n\}_{n\geq 0}$ and $\{r_n\}_{n\geq 0}$ are arbitrary sequences in H satisfying

$$\sum_{n=0}^{\infty} a_n = +\infty, \quad \sum_{n=0}^{\infty} \|l_n\| < +\infty, \qquad \lim_{n\to\infty} \|q_n\| = \lim_{n\to\infty} \|p_n\| = 0. \tag{3.4}$$

Algorithm 3.2. Let $M : H \to 2^H, N, \eta : H \times H \to H, T, f, g, h : H \to H$ with $h(H) \supseteq \mathrm{dom}\,(M)$, $g(H) \supseteq \mathrm{dom}\,(M)$. For given $u_0 \in H$, compute $\{h(u_n)\}_{n\geq 0}$ and $\{g(v_n)\}_{n\geq 0}$ from the iterative schemes

$$\begin{aligned} h(u_{n+1}) &= J_\rho^M[h(u_n) + g(v_n) - f(u_n) - \rho N(u_n, g(v_n))], \\ g(v_n) &= J_\gamma^M[f(u_n) - \gamma T(u_n)], \end{aligned} \tag{3.5}$$

for all $n \geq 0$ and $\rho, \gamma > 0$ are two constants.

Now we prove the existence of solutions for SGNVLI (2.4) and the convergence of the iterative sequences generated by Algorithm 3.1 and Algorithm 3.2, respectively.

Theorem 3.1. *Let $\eta : H \times H \to H$ be δ-strongly monotone and τ-Lipschitz continuous mapping, $M : H \to 2^H$ be maximal η-monotone. Let $h : H \to H$ be a-Lipschitz continuous and b-strongly monotone, $f : H \to H$ be c-Lipschitz continuous and d-strongly monotone, $g : H \to H$ be ε-strongly monotone and $g(H) \supseteq \mathrm{dom}\,(M)$, $N : H \times H \to H$ be σ_1-Lipschitz continuous in the first argument and σ_2-Lipschitz continuous in the second argument, respectively, and be relaxed μ_2-r_2-cocoercivity in the second argument, T be relaxed s-t-cocoercivity and λ-Lipschitz continuous. Let $k_1 = \sqrt{1-2d+c^2} < 1, k_2 = \sqrt{1-2b+a^2}$ and $e = 1 - [(1+\frac{\tau}{\delta})k_2 + \frac{\tau}{\delta}k_1] \in (0,1)$. If there exist constants $\gamma > 0$ and $\rho > 0$ satisfying*

$$\begin{aligned} &\left|\gamma - \frac{t - s\lambda^2}{\lambda^2}\right| < \frac{\sqrt{\lambda^2[(1-k_1)^2 - 1] + (t - s\lambda^2)^2}}{\lambda^2}, \\ &t > s\lambda^2, \quad \lambda^2[(1-k_1)^2 - 1] + (t - s\lambda^2)^2 > 0, \end{aligned} \tag{3.6}$$

$$\rho < \frac{\delta e}{\tau \sigma_1}, \tag{3.7}$$

and either

$$
\begin{aligned}
&\sigma_2 > \frac{\delta}{\tau}\sigma_1, \quad (r_2 - \mu_2\sigma_2^2) > \frac{\delta^3}{\tau^3}e\sigma_1,\\
&\left|\rho - \frac{(r_2 - \mu_2\sigma_2^2) - \frac{\delta^3}{\tau^3}e\sigma_1}{\sigma_2^2 - \frac{\delta^2}{\tau^2}\sigma_1^2}\right|\\
&\qquad < \frac{\sqrt{(\frac{\delta^4}{\tau^4}e^2 - 1)(\sigma_2^2 - \frac{\delta^2}{\tau^2}\sigma_1^2) + [(r_2 - \mu_2\sigma_2^2) - \frac{\delta^3}{\tau^3}e\sigma_1]^2}}{\sigma_2^2 - \frac{\delta^2}{\tau^2}\sigma_1^2}, \qquad (3.8)\\
&\left(\frac{\delta^4}{\tau^4}e^2 - 1\right)\left(\sigma_2^2 - \frac{\delta^2}{\tau^2}\sigma_1^2\right) + \left[(r_2 - \mu_2\sigma_2^2) - \frac{\delta^3}{\tau^3}e\sigma_1\right]^2 > 0\,,
\end{aligned}
$$

or

$$
\begin{aligned}
&\sigma_2 < \frac{\delta}{\tau}\sigma_1, \qquad \left|\rho - \frac{(r_2 - \mu_2\sigma_2^2) - \frac{\delta^3}{\tau^3}e\sigma_1}{\sigma_2^2 - \frac{\delta^2}{\tau^2}\sigma_1^2}\right|\\
&\qquad > \frac{\sqrt{(\frac{\delta^4}{\tau^4}e^2 - 1)(\sigma_2^2 - \frac{\delta^2}{\tau^2}\sigma_1^2) + [(r_2 - \mu_2\sigma_2^2) - \frac{\delta^3}{\tau^3}e\sigma_1]^2}}{\frac{\delta^2}{\tau^2}\sigma_1^2 - \sigma_2^2}, \qquad (3.9)
\end{aligned}
$$

then SGNVLI (2.4) *has a solution* $(u, v) \in H \times H$ *and the sequences* $\{u_n\}_{n\geq 0}$ *and* $\{v_n\}_{n\geq 0}$ *generated by Algorithm 3.1 satisfy*

$$\lim_{n\to\infty} u_n = u \quad \textit{and} \quad \lim_{n\to\infty} v_n = v.$$

Proof. Now we prove that G in (3.2) is a contraction mapping. By Lemma 2.2, we know that for any $x, y \in H$,

$$
\begin{aligned}
\|Gx - Gy\| &\leq \|x - y - [h(x) - h(y)]\| + \frac{\tau}{\delta}\|x - y - [h(x) - h(y)]\|\\
&+ \frac{\tau}{\delta}\|x - y - [f(x) - f(y)]\| + \frac{\tau}{\delta}\|J_\gamma^M(f(x) - \gamma T(x)) - J_\gamma^M(f(y) - \gamma T(y))\\
&\quad - \rho[N(x, J_\gamma^M(f(x) - \gamma T(x))) - N(x, J_\gamma^M(f(y) - \gamma T(y)))]\|\\
&\quad + \frac{\tau}{\delta}\rho\|N(x, J_\gamma^M(f(y) - \gamma T(y))) - N(y, J_\gamma^M(f(y) - \gamma T(y)))\|\\
&\quad \leq \left(1 + \frac{\tau}{\delta}\right)\sqrt{1 - 2b + a^2}\|x - y\| + \frac{\tau}{\delta}\sqrt{1 - 2d + c^2}\|x - y\|\\
&\quad + \frac{\tau}{\delta}\|J_\gamma^M(f(x) - \gamma T(x)) - J_\gamma^M(f(y) - \gamma T(y))\\
&\quad - \rho[N(x, J_\gamma^M(f(x) - \gamma T(x))) - N(x, J_\gamma^M(f(y) - \gamma T(y)))]\|\\
&\quad + \frac{\tau}{\delta}\rho\sigma_1\|x - y\|. \qquad (3.10)
\end{aligned}
$$

Since N is σ_1 and σ_2-Lipschitz continuous in the first and second arguments, respectively, and relaxed μ_2-r_2-cocoercive in the second argument, T is relaxed s-t-cocoercivive and λ-Lipschitz continuous, we have

$$\begin{aligned}\|J_\gamma^M(f(x)-\gamma T(x)) - J_\gamma^M(f(y)-\gamma T(y)) - \rho[N(x, J_\gamma^M(f(x)-\gamma T(x)))\\ - N(x, J_\gamma^M(f(y)-\gamma T(y)))]\| \le \sqrt{1-2\rho(r_2-\mu_2\sigma_2^2)+\rho^2\sigma_2^2}\\ \times \|J_\gamma^M(f(x)-\gamma T(x)) - J_\gamma^M(f(y)-\gamma T(y))\|,\end{aligned} \quad (3.11)$$

and

$$\begin{aligned}&\|J_\gamma^M(f(x)-\gamma T(x)) - J_\gamma^M(f(y)-\gamma T(y))\|\\ &\le \frac{\tau}{\delta}\|x-y-[f(x)-f(y)]\| + \frac{\tau}{\delta}\|x-y-\gamma[T(x)-T(y)]\| \le \frac{\tau}{\delta}\theta_1\|x-y\|,\end{aligned}$$

where $\theta_1 = \sqrt{1-2d+c^2} + \sqrt{1-2\gamma(t-s\lambda^2)+\gamma^2\lambda^2}$. It follows from (3.6) that $\theta_1 < 1$ and

$$\|J_\gamma^M(f(x)-\gamma T(x)) - J_\gamma^M(f(y)-\gamma T(y))\| \le \frac{\tau}{\delta}\|x-y\|. \quad (3.12)$$

Using (3.11) and (3.12) in (3.10), we obtain that

$$\|Gx-Gy\| \le \theta\|x-y\|, \quad (3.13)$$

where $\theta = (1+\frac{\tau}{\delta})k_2 + \frac{\tau}{\delta}k_1 + \frac{\tau^2}{\delta^2}\sqrt{1-2\rho(r_2-\mu_2\sigma_2^2)+\rho^2\sigma_2^2} + \frac{\tau}{\delta}\rho\sigma_1$. Clearly (3.7) and one of (3.8) and (3.9) ensure that $\theta \in (0,1)$. Hence G is a contraction mapping and it has a unique fixed point $u \in H$. Because $g(H) \supseteq \mathrm{dom}\,(M)$ and $(I+\rho M)(H) = H$, we easily conclude that there exists $v \in H$ such that $g(v) = J_\gamma^M(f(u)-\gamma T(u))$. It follows from Lemma 3.1 that (u,v) is a solution of SGNVLI (2.4).

Next we prove that sequences $\{u_n\}_{n\ge 0}$ and $\{v_n\}_{n\ge 0}$ generated by Algorithm 3.1 converge strongly to u and v, respectively. Using (3.3) and (3.13) we get that

$$\|t_n-u\| \le (1-(1-\theta)b_n)\|u_n-u\| + \|q_n\| \le \|u_n-u\| + \|q_n\|$$

and

$$\begin{aligned}\|u_{n+1}-u\| &\le (1-a_n)\|u_n-u\| + a_n\theta\|t_n-u\| + \|l_n\|\\ &\le (1-(1-\theta)a_n)\|u_n-u\| + a_n\|q_n\| + \|l_n\|\end{aligned}$$

for all $n \geq 0$. It follows from Lemma 2.3 and (3.4) that $\lim_{n\to\infty} u_n = u$. By (3.3), (3.12) and strong monotonicity of g we deduce that

$$\varepsilon\|v_n - v\| \leq \|g(v_n) - g(v)\| \leq \frac{\tau}{\delta}\|u_n - u\| + \|p_n\| \to 0$$

as $n \to \infty$. Hence $\lim_{n\to\infty} v_n = v$. This completes the proof. □

Theorem 3.2. *Let η, M, f be as in Theorem 3.1. Let $h : H \to H$ be a-Lipschitz continuous, (b,η)-strongly monotone and $h(H) \supseteq \operatorname{dom}(M)$, $g : H \to H$ be ε-strongly monotone and continuous and $g(H) \supseteq \operatorname{dom}(M)$, $N : H \times H \to H$ be σ_1-Lipschitz continuous in the first argument and σ_2-Lipschitz continuous in the second argument, respectively, and be (μ_1,η)-strongly monotone in the first argument and be μ_2-strongly monotone in the second argument, respectively; T be s-strongly monotone and λ-Lipschitz continuous. Let $k_1 = \sqrt{1-2d+c^2} < \frac{\delta}{\tau}$, $e = \frac{\delta b}{\tau^2} - c \in (0,1)$ and $\theta(\rho) = \sqrt{1-2\rho\mu_2+\rho^2\sigma_2^2}$. If there exist constants $\gamma > 0$ and $\rho > 0$ satisfying*

$$\left|\gamma - \frac{s}{\lambda^2}\right| < \frac{\sqrt{\lambda^2[(\frac{\delta}{\tau}-k_1)^2-1]+s^2}}{\lambda^2}, \tag{3.14}$$

$$\lambda^2\Big[\Big(\frac{\delta}{\tau}-k_1\Big)^2-1\Big]+s^2 > 0,$$

$$\left|\rho - \frac{\mu_2}{\sigma_2^2}\right| < \frac{\sqrt{\sigma_2^2(e^2-1)+\mu_2^2}}{\sigma_2^2}, \quad \sigma_2^2(e^2-1)+\mu_2^2 > 0\,, \tag{3.15}$$

and

$$\left|\rho - \frac{\mu_1-\sigma_1\sqrt{\tau^2-2b+a^2}}{\sigma_1^2}\right| < \frac{\sqrt{\sigma_1^2[(e-\theta(\rho))^2-a^2]+(\mu_1-\sigma_1\sqrt{\tau^2-2b+a^2})^2}}{\sigma_1^2}, \tag{3.16}$$

$$\sigma_1^2[(e-\theta(\rho))^2-a^2]+(\mu_1-\sigma_1\sqrt{\tau^2-2b+a^2})^2 > 0,$$

$$\mu_1 > \sigma_1\sqrt{\tau^2-2b+a^2},$$

then SGNVLI (2.4) has a solution $(u,v) \in H \times H$ and the sequences $\{u_n\}_{n\geq 0}$ and $\{v_n\}_{n\geq 0}$ generated by Algorithm 3.2 satisfy

$$\lim_{n\to\infty} u_n = u \quad \text{and} \quad \lim_{n\to\infty} v_n = v.$$

Proof. From (3.5) and Lemma 2.2, we have

$$\begin{aligned}
&\|h(u_{n+2}) - h(u_{n+1})\| \\
&\quad \le \frac{\tau}{\delta}\|h(u_{n+1}) - h(u_n) - \rho[N(u_{n+1}, g(v_{n+1})) - N(u_n, g(v_{n+1}))]\| \\
&\qquad + \frac{\tau}{\delta}\|g(v_{n+1}) - g(v_n) - \rho[N(u_n, g(v_{n+1})) - N(u_n, g(v_n))]\| \\
&\qquad + \frac{\tau}{\delta}\|f(u_{n+1}) - f(u_n)\|. \quad (3.17)
\end{aligned}$$

Using the assumptions that N is (μ_1, η)-strongly monotone, σ_1-Lipschitz continuous in the first argument, h is a-Lipschitz continuous, (b, η)-strongly monotone and η is τ-Lipschitz continuous, we get that

$$\begin{aligned}
&\|h(u_{n+1}) - h(u_n) - \rho[N(u_{n+1}, g(v_{n+1})) - N(u_n, g(v_{n+1}))]\|^2 \\
&\le a^2\|u_{n+1} - u_n\|^2 - 2\rho\langle \eta(u_{n+1}, u_n), N(u_{n+1}, g(v_{n+1})) - N(u_n, g(v_{n+1}))\rangle \\
&+ 2\rho\langle \eta(u_{n+1}, u_n) - [h(u_{n+1}) - h(u_n)], N(u_{n+1}, g(v_{n+1})) - N(u_n, g(v_{n+1}))\rangle \\
&+ \rho^2\sigma_1^2\|u_{n+1} - u_n\|^2 \\
&\le a^2\|u_{n+1} - u_n\|^2 - 2\rho\mu_1\|u_{n+1} - u_n\|^2 + \rho^2\sigma_1^2\|u_{n+1} - u_n\|^2 \\
&+ 2\rho\|\eta(u_{n+1}, u_n) - [h(u_{n+1}) - h(u_n)]\|\|N(u_{n+1}, g(v_{n+1})) - N(u_n, g(v_{n+1}))\| \\
&\le [a^2 - 2\rho(\mu_1 - \sigma_1\sqrt{\tau^2 - 2b + a^2}) + \rho^2\sigma_1^2]\|u_{n+1} - u_n\|^2. \quad (3.18)
\end{aligned}$$

Since N is μ_2-strongly monotone and σ_2-Lipschitz continuous in the second argument, we have

$$\begin{aligned}
&\|g(v_{n+1}) - g(v_n) - \rho[N(u_n, g(v_{n+1})) - N(u_n, g(v_n))]\| \\
&\qquad \le \sqrt{1 - 2\rho\mu_2 + \rho^2\sigma_2^2}\|g(v_{n+1}) - g(v_n)\|. \quad (3.19)
\end{aligned}$$

It follows from Lipschitz continuity and strong monotonicity of f and T, Lemma 2.3 and (3.14) that

$$\begin{aligned}
&\|g(v_{n+1}) - g(v_n)\| \\
&\le \frac{\tau}{\delta}[\|u_{n+1} - u_n - (f(u_{n+1}) - f(u_n))\| + \|u_{n+1} - u_n - \gamma(T(u_{n+1}) - T(u_n))\|] \\
&\le \frac{\tau}{\delta}(\sqrt{1 - 2d + c^2} + \sqrt{1 - 2s\gamma + \lambda^2\gamma^2})\|u_{n+1} - u_n\| \\
&\le \|u_{n+1} - u_n\|. \quad (3.20)
\end{aligned}$$

Because h is (b, η)-strongly monotone and η is τ-Lipschitz continuous, we have

$$\begin{aligned} b\|u_{n+2} - u_{n+1}\|^2 &\le \langle h(u_{n+2}) - h(u_{n+1}), \eta(u_{n+2}, u_{n+1})\rangle \\ &\le \tau\|h(u_{n+2}) - h(u_{n+1})\|\|u_{n+2} - u_{n+1}\|, \end{aligned}$$

that is,

$$\|u_{n+2} - u_{n+1}\| \le \frac{\tau}{b}\|h(u_{n+2}) - h(u_{n+1})\|. \tag{3.21}$$

Using (3.17)-(3.20) in (3.21), we obtain that

$$\|u_{n+2} - u_{n+1}\| \le \vartheta\|u_{n+1} - u_n\|, \tag{3.22}$$

where $\vartheta = \frac{\tau^2}{\delta b}[\sqrt{a^2 - 2\rho(\mu_1 - \sigma_1\sqrt{\tau^2 - 2b + a^2}) + \rho^2\sigma_1^2} + \theta(\rho) + c]$. Since g is ε-strongly monotone, by (3.20), we have

$$\|v_{n+1} - v_n\| \le \frac{1}{\varepsilon}\|g(v_{n+1}) - g(v_n)\| \le \frac{1}{\varepsilon}\|u_{n+1} - u_n\|. \tag{3.23}$$

It follows from (3.15), (3.16), (3.22) and (3.23) that $\{u_n\}_{n\ge 0}$ and $\{v_n\}_{n\ge 0}$ are Cauchy sequences in H. Therefore, there exist $u, v \in H$ such that $u_n \to u$ and $v_n \to v$ as $n \to \infty$. It follows from the continuity of J_ρ^M, N, h, g, f, T and (3.5) that (3.1) holds. That is, (u, v) is a solution of SGNVLI (2.4) by Lemma 3.1. This completes the proof. □

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