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A NOTE ON GENERALIZED (σ, τ)-DERIVATIONS ASSOCIATE WITH HOCHSCHILD 2-(σ, τ) COCYCLES OF RINGS

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Abstract: Let R be a ring, M an R-bimodule, σ and τ endomorphisms of R, and $\alpha : R \times R \to M$ a biadditive map. In present paper we introduce a new type of generalized derivations associate with Hochschild 2-(σ, τ) cocycles α and prove that generalized (σ, τ)-Jordan derivations of this type are also generalized (σ, τ)-derivations under certain conditions.

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1. Introduction

Let R be an associative ring. M is called an R-bimodule is M is a left and right R-module such that $x(my) = (xm)y$ for all $x, y \in R$ and $m \in M$. An additive map $f : R \to M$ is called a generalized derivation if there exists a derivation $d : R \to M$ such that

$$f(xy) = f(x)y + xd(y) \quad \text{for all } x, y \in R, \tag{1}$$

and f is called a generalized Jordan derivation if there exists a Jordan derivation $d : R \to M$ such that

$$f(x^2) = f(x)x + xd(x) \quad \text{for all } x \in R. \tag{2}$$

We denote above generalized derivations by $(f,\ d)$. Note that if R is a prime

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ring (resp. semiprime) and the additive map $f : R \to M$ satisfies (1), then the map $d : R \to M$ must be a derivation.

In [3], [8], [11] and [9], the properties of these type of generalized derivations were studied in different aspects. In [12], Nakajima introduced an another type of generalized derivations was defined by the author as follows. An additive map $f : R \to M$ is called a generalized derivation if there exists an element $w \in M$ such that

$$f(xy) = f(x)y + xf(y) + xwy \quad \text{for all } x,\, y \in R, \tag{3}$$

and f is said to be a generalized Jordan derivation of R if there exists an element $w \in R$ such that

$$f(x^2) = f(x)x + xf(x) + xwx \quad \text{for all } x \in R. \tag{4}$$

We denote these generalized derivations by (f, w). In [10], [14] and [12] the authors studied the properties of these type of generalized derivations. It is easily seen that if R has an identity element 1, then the two notions of the above generalized derivations (1) and (3) coincide.

Let σ and τ be endomorphisms of R. An additive map $f : R \to M$ is called a generalized (σ, τ)-derivation if there exists a (σ, τ)-derivation $d : R \to M$ such that

$$f(xy) = f(x)\tau(y) + \sigma(x)d(y) \quad \text{for all } x,\, y \in R, \tag{5}$$

$f : R \to M$ is called a generalized (σ, τ)-Jordan derivation if there exists a (σ, τ)-Jordan derivation $d : R \to M$ such that

$$f(x^2) = f(x)\tau(x) + \sigma(x)d(x) \quad \text{for all } x \in R. \tag{6}$$

We denote them by (f, d). If $f = d$, then f is called (σ, τ)-derivation. In [6], the authors generalized Herstein's result [7] to (σ, τ)-derivations. Then in [1], we introduced a new type of generalized (σ, τ)-derivations in the sense of Nakajima [12] as follows:

An additive map $f : R \to M$ is called a generalized (σ, τ)-derivation if there exists an element $w \in M$ such that

$$f(xy) = f(x)\tau(y) + \sigma(x)f(y) + \sigma(x)w\tau(y) \quad \text{for all } x, y \in R\,, \tag{7}$$

and f is called a generalized (σ, τ)-Jordan derivation if there exists an element $w \in R$ such that

$$f(x^2) = f(x)\tau(x) + \sigma(x)f(x) + \sigma(x)w\tau(x) \quad \text{for all } x \in R. \tag{8}$$

We denote these generalized derivations by (f, w). Moreover we show that every generalized (σ, τ)-Jordan derivation of a prime ring R is a generalized (σ, τ)-derivation of R under some conditions [[1], Theorem 5.1].

It was known that a Jordan derivation of a 2-torsion free prime ring is a derivation and this result was extended to a 2-torsion free semiprime ring. In [14], the author proved that a generalized Jordan derivation of the type (4) is also a generalized derivation and in [2], they proved Brešar's type of generalized Jordan derivation $(f,\ d)$ is a generalized derivation under a commutator condition.

In this study, we introduce a new type of generalized (σ, τ)-derivations as the sense of [13] and extend the above results of Jordan derivations to this type of generalized derivations.

Throughout the following, R is a ring and all maps are additive, M is R-bimodule and σ and τ are endomorphisms of R unless otherwise stated.

2. Preliminaries

Throughout the present paper we shall give some definitions and lemmas for the sake of understanding this paper.

In [13] it is defined a new type of generalized derivation. Let R be a ring. Let $\alpha : R \times R \to M$ be a biadditive map, that is, an additive map on each components. α is called a *Hochschild* 2-*cocycle* if

$$x\alpha(y,\ z) - \alpha(xy,\ z) + \alpha(x,\ yz) - \alpha(x,\ y)z = 0 \quad \text{for all } x, y, z \in R. \tag{9}$$

If $\alpha(x,\ y) = \alpha(y,\ x)$ for all $x, y \in R$, then α is called a *symmetric* 2-cocycle. A map $f : R \to M$ is called a *generalized derivation* if there exists a Hochschild 2-cocycle α such that

$$f(xy) = f(x)y + xf(y) + \alpha(x,\ y) \quad \text{for all } x, y \in R. \tag{10}$$

f is called a *generalized Jordan derivation* if there exists a Hochschild 2-cocycle α such that

$$f(x^2) = f(x)x + xf(x) + \alpha(x,\ x) \quad \text{for all } x \in R. \tag{11}$$

We denote it by $(f,\ \alpha)$. If $\alpha = 0$, then they are usual derivations and Jordan derivations.

Now let R be a ring, M an R-bimodule and $\sigma, \tau : R \to R$ endomorphisms. Then we introduce the following notions as the sense of (10) and (11);

A biadditive map $\alpha : R\times R \to M$ is said to be a Hochschild 2-(σ,τ) cocycle if it satisfies

$$\sigma(x)\alpha(y,\ z) - \alpha(xy,\ z) + \alpha(x,\ yz) - \alpha(x,\ y)\tau(z) = 0 \quad \text{for all } x, y, z \in R. \qquad (12)$$

A map $f : R \to R$ is called a *generalized* (σ,τ)*-derivation* if there exists a Hochschild 2-(σ,τ) cocycle α such that

$$f(xy) = f(x)\tau(y) + \sigma(x)f(y) + \alpha(x,\ y) \quad \text{for all } x, y \in R. \qquad (13)$$

f is called a *generalized* (σ,τ)*-Jordan derivation* if there exists a Hochschild 2-(σ,τ) cocycle α such that

$$f(x^2) = f(x)\tau(x) + \sigma(x)f(x) + \alpha(x,\ x) \quad \text{for all } x \in R. \qquad (14)$$

We denote it by (f,α). It is clear that if $\alpha = 0$, then they are usual (σ,τ)-derivations and (σ,τ)-Jordan derivations, respectively.

We give some examples of the sense our a Hochschild 2-(σ,τ) cocycle α and generalized derivations.

Examples. (1) Let $M = R$ be an R-bimodule. A biadditive mapping $\alpha : R\times R \to R$ defined by $\alpha(x,y) = \sigma(x)\tau(y)$ for all $x,y \in R$ is a Hochschild 2-(σ,τ) cocycle.

(2) Let $a \in M$. A biadditive mapping $\alpha : R\times R \to M$ defined by $\alpha(x,y) = \sigma(x)a\tau(y)$ for all $x,y \in R$ is a Hochschild 2-(σ,τ) cocycle.

The above examle shows that every generalized (σ,τ)-derivation (7) is a generalized (σ,τ)-derivation (13).

(3) Let $f : R \to M$ be a left τ-multiplier defined $f(xy) = f(x)\tau(y)$ for all $x,y \in R$. Then we have a 2-(σ,τ) cocycle $\alpha : R\times R \to M$ defined by $\alpha(x,\ y) = \sigma(x)(-f)(y)$ since $f(xy) = f(x)\tau(y) + \sigma(x)f(y) + \sigma(x)(-f)(y)$ for all $x,y \in R$. Thus a left τ- multiplier is also a generalized (σ,τ)-derivation.

(4) Let f be an ordinary derivation, that is; an additive map $f : R \to M$ satisfying $f(xy) = f(x)y + xf(y)$. Then the map $\alpha : R\times R \to M$ defined by $\alpha(x,y) = f(x)(y-\tau(y)) + (x-\sigma(x))f(y)$ is biadditive map satisfying 2-(σ,τ) cocycle condition. Since by the last relation $f(xy) = f(x)\tau(y) + \sigma(x)f(y) + \alpha(x,\ y)$ for all $x,y \in R$, ordinary derivation is a generalized $(\sigma,\ \tau)$-derivation (13).

(5) If $(f,\ d)$ and $(f,\ \omega)$ are generalized $(\sigma,\ \tau)$-derivations of types (5) and (7), respectively, then the maps $\alpha : R\times R \to M$ defined by $\alpha(x,\ y) = \sigma(x)(d-f)(y)$ and $\alpha : R\times R \to M$ defined by $\alpha(x,\ y) = \sigma(x)\omega\tau(y)$ are biadditive

maps satisfying the 2-$(\sigma,\ \tau)$ cocycle condition (12). So the generalized $(\sigma,\ \tau)$-derivations are generalized $(\sigma,\ \tau)$-derivations in our sense.

(6) Let $f : R \to M$ be an additive map and let $\alpha : R \times R \to M$ be a biadditive map. For all $x,\ y \in R$, if $f(xy) = f(x)\sigma(y) + \sigma(x)f(y) + \alpha(x,\ y)$ holds, then by the relation $f((xy)z) = f(x(yz))$, α satisfies the 2-(σ, τ) cocycle condition. Thus $(f,\ \alpha)$ is a generalized $(\sigma,\ \tau)$-derivation in our sense.

3. The Results

In the following, our main goal is to prove that a generalized (σ, τ)-Jordan derivation $(f,\ \alpha)$ associated with Hochschild 2-(σ, τ)-cocycle α is a generalized (σ, τ)-derivation under certain conditions. This gives an extension of the results of I. N. Herstein [7], Theorem 3.3, and M. Ashraf and N.-U. Rehman [2] for arbitrary derivation to this new type generalized (σ, τ)-derivation. In all that follows, we assume that α is Hochschild 2-(σ, τ)-cocycle, M is an R-bimodule, σ and τ are endomorphisms of R. Before beginning the proof of the main result, we need some lemmas.

The following lemma is useful in the calculations of 2-torsion free semiprime rings which can be found in [5], Lemma 1.1 and Lemma 1.2.

Lemma 1. (see [13], Lemma 3) (1) *Let R be a 2-torsion free semiprime ring and $a,\ b \in R$. If $axb + bxa = 0$ for all $x \in R$, then $axb = bxa = 0$ for all $x \in R$. Especially, $ab = ba = 0$.*

(2) *Let $G_1,\ G_2, ..., G_n$ be additive groups and R a semiprime ring. Suppose that mappings $S : G_1 \times G_2 \times \cdots \times G_n \to R$ and $T : G_1 \times G_2 \times \cdots \times G_n \to R$ are additive in each argument. If $S(a_1, a_2, ..., a_n)xT(a_1, a_2, ..., a_n) = 0$ for all $x \in R,\ a_i \in G_i,\ i = 1, 2, \cdots, n$, then $S(a_1, a_2, ..., a_n)xT(b_1, b_2, ..., b_n) = 0$ for all $x \in R,\ a_i,\ b_i \in G_i,\ i = 1, 2, ..., n$.*

Lemma 2. (see [13], Lemma 5) *Let R be a 2-torsion free ring and $G_1,\ G_2$ additive groups. Let $S, T : G_1 \times G_2 \to R$ be biadditive maps. Assume that $S(x_1,\ x_2)T(x_1,\ x_2) = 0$ for all $x_i \in G_i,\ i = 1, 2$. If there exists a non-zero divisor $T(a_1,\ a_2)$ for some $a_i \in G_i,\ i = 1, 2$, then $S(x_1,\ x_2) = 0$ for all $x_i \in G_i$, $i = 1, 2$.*

Now the following lemma is elementary and can be found in [4], for ordinary derivation and in [1], Lemma 5.1, for generalized derivations in rings.

Lemma 3. *Let M a 2-torsion free module and $(f,\ d) : R \to M$ a generalized (σ, τ)-Jordan derivation associated with a (σ, τ)-Jordan derivation d in the sense of (6). Then the following hold:*

(i) $f(xy + yx) = f(x)\tau(y) + \sigma(x)d(y) + f(y)\tau(x) + \sigma(y)d(x)$,

(*ii*) $f(xyx) = f(x)\tau(yx) + \sigma(x)d(y)\tau(x) + \sigma(xy)d(x)$,

(*iii*) $f(xyz+zyx) = f(x)\tau(yz)+\sigma(x)d(y)\tau(z)+\sigma\tau(xy)d(z)+f(z)\tau(yx)+\sigma(z)d(y)\tau(x) + \sigma(zy)d(x)$.

In the following, we obtain a generalization of the lemma above with respect to (σ, τ)-derivations as follows.

Lemma 4. *Let M be a 2-torsion free module and $(f,\ \alpha) : R \to M$ a generalized (σ, τ)-Jordan derivation associated with Hochschild 2-(σ, τ)-cocycle α. Then f meets the following relations.*

(*i*) $f(xy + yx) = f(x)\tau(y) + \sigma(x)f(y) + \alpha(x,\ y) + f(y)\tau(x) + \sigma(y)f(x) + \alpha(y,\ x)$.

(*ii*) $f(xyx) = f(x)\tau(yx) + \sigma(x)f(y)\tau(x) + \sigma(xy)f(x) + \sigma(x)\alpha(y, x) + \alpha(x, yx)$.

(*iii*) $f(xyz+zyx) = f(x)\tau(yz)+\sigma(x)f(y)\tau(z)+\sigma(xy)f(z)+\sigma(x)\alpha(y, z)+\alpha(x, yz) + f(z)\tau(yx) + \sigma(z)f(y)\tau(x) + \sigma(zy)f(x)\sigma(z)\alpha(y, x) + \alpha(z, yx)$.

Proof. (*i*) Since $f(x^2) = f(x)\tau(x) + \sigma(x)f(x) + \alpha(x, x)$, calculating $f(x + y)^2 = f(x^2 + y^2 + xy + yx)$ for all $x, y \in R$ in two ways, we obtain (*i*).

(*ii*) Substituting y for $xy + yx$ in (*i*) and using 2-(σ, τ) cocycles condition (12), we have

$$\begin{aligned}0 &= f(x^2y + yx^2 + 2xyx) - f(x(xy + yx) + (xy + yx)x) = f(x)\tau(xy)\\&+ \sigma(x)f(x)\tau(y) + \alpha(x,\ x)\tau(y) + f(y)\tau(x^2) + \sigma(x^2)f(y) + \sigma(y)f(x)\tau(x)\\&+ \sigma(yx)f(x) + \alpha(x^2,\ y) + \alpha(y,\ x^2) - f(x)\tau(xy) - f(x)\tau(yx) - f(x)\tau(yx)\\&- f(y)\tau(x^2) - \sigma(x)f(y)\tau(x) - \sigma(y)f(x)\tau(x) - \alpha(y,\ x)\tau(x) - \sigma(x)f(x)\tau(y)\\&+ \sigma(y)\alpha(x,\ x) - \sigma(x)f(y)\tau(x) - \sigma(x^2)f(y) - \sigma(xy)f(x) - \sigma(x)\alpha(x,\ y)\\&- \sigma(x)\alpha(y,\ x) - \sigma(xy)f(x) - \sigma(yx)f(x) - \alpha(x,\ xy) - \alpha(x,\ yx) - \alpha(xy,\ x)\\&- \alpha(yx,\ x) = 2f(xyx) + \alpha(x,\ x)\tau(y) + \alpha(x^2,\ y) + \alpha(y,\ x^2)\\&+\sigma(y)\alpha(x,\ x)-\alpha(x,\ xy)-\sigma(x)\alpha(x,\ y)-\alpha(yx,\ x)-\alpha(y,\ x)\tau(x)-2f(x)\tau(yx)\\&- 2\sigma(x)f(y)\tau(x) - 2\sigma(xy)f(x) - \alpha(x,\ yx) - \alpha(xy,\ x) - \sigma(x)\alpha(y,\ x)\\&- \alpha(x,\ y)\tau(x).\end{aligned}$$

Since $-\alpha(xy,\ x) - \alpha(x,\ y)\tau(x) = -\sigma(x)\alpha(y,\ x) - \alpha(x,\ yx)$, we obtain

$$\begin{aligned}0 &= 2f(xyx) - 2f(x)\tau(yx) - 2\sigma(x)f(y)\tau(x) - 2\sigma(xy)f(x)\\&- (\sigma(x)\alpha(y,\ x) + \alpha(x,\ yx)) - \alpha(x,\ yx) - \sigma(x)\alpha(y,\ x).\end{aligned}$$

for all $x, y \in R$. Since M is 2-torsion free, we get (*ii*) *(iii)* Replacing x by $x+z$ in *(ii)* and using *(ii)*, we easily see that *(iii)*. In the following, M is an R-module σ and τ are endomorphisms of R, α is 2-(σ,τ) cocycle condition (12) and for any $x,y,z\in R$, we set

$$F(x,\ y)=f(xy)-f(x)\tau(y)-\sigma(x)f(y) \quad \text{and} \quad \delta(x,\ y)=F(x,\ y)-\alpha(x,\ y),$$

where $f:R\to M$ an additive map. Then it is easily seen that the maps F and δ: $R\times R\to M$ are biadditive. □

Using *(ii)* and *(iii)* of Lemma 3, we have the following result.

Lemma 5. *Let M be a 2-torsion R-module, $(f,\ \alpha):R\to M$ be a generalized (σ,τ)-Jordan derivation. Then for all $x,y,z\in R$, there hold:*

(i) $\delta(x,\ y)\tau(z)[\tau(x),\ \tau(y)]+[\sigma(x),\ \sigma(y)]\sigma(z)\delta(x,\ y)=0$,

(ii) $\delta(x,\ y)[\tau(x),\tau(y)]=0$ *and* $[\sigma(x),\ \sigma(y)]\delta(x,\ y)=0$,

where $[x,\ y]=xy-yx$.

Proof. *(i)* Take xy instead of x and yx instead of y in Lemma 4 *(iii)* and using the relations *(ii)* and *(iii)* of Lemma 4, by the definitions of $F(x,y)$ and $\delta(x,\ y)$ we have

$$\begin{aligned}0 &= f((xy)z(yx)+(yx)z(xy))-f(x(yzy)x+y(xzx)y)\\ &= F(x,\ y)\tau(zyx)+\sigma(xyz)F(y,\ x)+F(y,\ x)\tau(zxy)+\sigma(yxz)F(x,\ y)\\ &+\sigma(xy)[\alpha(z,\ yx)-\alpha(z,\ y)\tau(x)]+\sigma(yx)[\alpha(z,\ xy)-\alpha(z,\ x)\tau(y)]\\ &-\sigma(x)\alpha(y,\ zy)\tau(x)-\sigma(x)\alpha(yzy,\ x)-\sigma(y)\alpha(x,\ zx)\tau(y)\\ &-\sigma(y)\alpha(xzx,\ y)+\alpha(yx,\ zxy)-\alpha(y,\ xzxy)-\alpha(x,\ yzyx)\\ &+\alpha(xy,\ zyx).\end{aligned} \tag{15}$$

Since α is a 2-(σ,τ) cocycle, we have the following relations

$$\sigma(xy)[\alpha(z,\ yx)-\alpha(z,\ y)\tau(x)]=\sigma(xy)[\alpha(zy,\ x)-\sigma(z)\alpha(y,\ x)]\,, \tag{16}$$

$$\sigma(yx)[\alpha(z,\ xy)-\alpha(z,\ x)\tau(y)]=\sigma(yx)[\alpha(zx,\ y)-\sigma(z)\alpha(x,\ y)]\,, \tag{17}$$

$$\alpha(xy,\ zyx)-\alpha(x,\ yzyx)=\sigma(x)\alpha(y,\ zyx)-\alpha(x,\ y)\tau(zyx)\,, \tag{18}$$

$$\alpha(yx,\ zxy)-\alpha(y,\ xzxy)=\sigma(y)\alpha(x,\ zxy)-\alpha(y,\ x)\tau(zxy)\,. \tag{19}$$

Substituting (16), (17), (18) and (19) in the relation (15), we obtain

$$\begin{aligned}0 &= \delta(x,y)\tau(zyx)+\delta(y,\ x)\tau(zxy)+\sigma(xyz)\delta(y,\ x)+\sigma(yxz)\delta(x,\ y)\\ &+\sigma(x)[\sigma(y)\alpha(zy,\ x)-\alpha(y(zy),\ x)+\alpha(y,\ (zy)x)-\alpha(y,\ zy)\tau(x)]\\ &+\sigma(y)[\sigma(x)\alpha(zx,\ y)-\alpha(x(zx),\ y)+\alpha(x,\ zxy)-\alpha(x,\ zx)\tau(y)]\\ &= \delta(x,\ y)\tau(zyx)+\delta(y,\ x)\tau(zxy)+\sigma(xyz)\delta(y,\ x)+\sigma(yxz)\delta(x,\ y).\end{aligned}$$

Since $\delta(x,\ y) = -\delta(y,\ x)$ for all $x, y \in R$ by Lemma 3 (i), we arrive at

$$\delta(x,\ y)\tau(z)[\tau(x),\ \tau(y)] + [\sigma(x),\ \sigma(y)]\sigma(z)\delta(x,\ y) = 0$$

for all $x, y, z \in R$, as desired.

(ii) Since $0 = f((xy)^2 + xy^2x) - f(xy(xy) + (xy)yx)$ for all $x, y \in R$, using Lemma 3 (ii) and (iii) we have

$$\begin{aligned} 0 = {} & f(xy)\tau(xy) + \sigma(xy)f(xy) + \alpha(xy,\ xy) + f(x)\tau(y^2x) + \sigma(x)(f(y)\tau(y) \\ & + \sigma(y)f(y) + \alpha(y,\ y))\tau(x) + \sigma(xy^2)f(x) + \sigma(x)\alpha(y^2,\ x) + \alpha(x,\ y^2x) \\ & - f(x)\tau(yxy) - \sigma(x)f(y)\tau(xy) - \sigma(xy)f(xy) - \sigma(x)\alpha(y,\ xy) - \alpha(x,\ yxy) \\ & - f(xy)\tau(yx) - \sigma(xy)f(y)\tau(x) - \sigma(xy)f(y)\tau(x) - \sigma(xy^2)f(x) - \sigma(xy)\alpha(y,\ x) \\ & - \alpha(xy,\ yx) = F(x,\ y)[\tau(x),\ \tau(y)] + \alpha(xy,\ xy) + \sigma(x)\alpha(y^2,\ x) + \alpha(x,\ y^2x) \\ & + \sigma(x)\alpha(y,\ y)\tau(x) - \sigma(x)\alpha(y,\ xy) - \alpha(x,\ yxy) - \sigma(xy)\alpha(y,\ x) \\ & - \alpha(xy,\ yx)\,, \end{aligned} \tag{20}$$

for all $x, y \in R$. Since α is a 2-(σ, τ) cocycle, we have the following relations

$$\begin{aligned} \alpha(x,\ y)\tau(xy) &= \sigma(x)\alpha(y,\ xy) - \alpha(xy,\ xy) + \alpha(x,\ yxy)\,, \\ \sigma(x)\alpha(y,\ yx) &= \sigma(x)(\alpha(y^2,\ x) - \sigma(y)\alpha(y,\ x) + \alpha(y,\ y)\tau(x))\,, \\ \alpha(x,\ y)\tau(yx) &= \sigma(x)\alpha(y,\ yx) - \alpha(xy,\ yx) + \alpha(x,\ y^2x). \end{aligned}$$

Using above relations in (20), we obtain

$$\delta(x,\ y)[\tau(x),\ \tau(y)] = 0 \quad \text{for all } x,\ y \in R.$$

Similarly since $0 = f((xy)^2 + yx^2y) - f((xy)xy + yx(xy))$ for all $x, y \in R$, using Lemma 3 (ii), (iii) and using the relation $\sigma(x)\alpha(x,\ y) = \alpha(x^2,\ y) + \alpha(x,\ x)\tau(y) - \alpha(x,\ xy)$ we obtain $[\sigma(x),\ \sigma(y)]\delta(x,\ y)$ for all $x, y \in R$. □

Now we are ready for proving the following result.

Theorem 2. *Let R be a 2-torsion free ring, τ onto and $(f,\ \alpha) : R \to R$ a generalized (σ, τ)-Jordan derivation on R associated with 2-(σ, τ) cocycle mapping α. If the following condition hold on R, then $(f,\ \alpha)$ is a generalized (σ, τ)-derivation.*

(i) *R is a noncommutative prime ring.*

(ii) *There exist $a, b \in R$ such that $[\tau(a), \tau(b)]$ is a non-zero divisor.*

(iii) *R is commutative, α is symmetric and $[x,\ y]_{\sigma,\tau} = 0$ for all $x, y \in R$, where $[x,\ y]_{\sigma,\tau} = x\tau(y) - \sigma(y)x$.*

Proof. (i) Multiplying (i) of Lemma 5 by $[\tau(x),\ \tau(y)]$ from the right and then using (ii) of Lemma 5, we get $\delta(x,\ y)\tau(z)[\tau(x),\ \tau(y)]^2 = 0$ for all $x, y, z \in R$. Suppose that f is not a generalized (σ,τ)-derivation. Since R is prime and τ is onto, from Lemma of [15], p. 428 and from [6], Lemma 4, we conclude that R is a commutative ring, a contradiction. Then we obtain $\delta(x,\ y) = 0$ for any $x, y \in R$, as desired.

(ii) Suppose that $[\tau(a),\ \tau(b)]$ is a nonzero divisor and so using Lemma 5 (ii), $\delta(a,\ b) = 0$ for some $a, b \in R$. Since $\delta(x,\ y)$ and $[\tau(x),\ \tau(y)] = \tau([x,\ y])$ are biadditive maps, then by Lemma 2, we have $\delta(x,\ y) = 0$ for all $x, y \in R$.

(iii) Since $[x,\ y]_{\sigma,\tau} = 0$ for all $x, y \in R$ and α is symmetric, then by Lemma 2 (i), we see that $(f,\ \alpha)$ is a generalized (σ,τ)-derivation. □

Now let μ be a left Jordan τ-multiplier on R, that is, an additive map $\mu : R \to R$ satisfying $\mu(x^2) = \mu(x)\tau(x)$ for all $x \in R$. Then we can easily obtain the following relations.

$$\mu(xy + yx) = \mu(x)\tau(y) + \mu(y)\tau(x),$$
$$\mu(xyx) = \mu(x)\tau(yx) \quad \text{and} \quad \mu(xyz + zyx) = \mu(x)\tau(yz) + \mu(z)\tau(yx).$$

Then by $\mu(xy(xy) + (xy)yx) = \mu((xy)^2 + xy^2x)$, we get

$$(\mu(xy) - \mu(x)\tau(y))[\tau(x),\ \tau(y)] = 0.$$

Thus if R is 2-torsion free ring and has the elements a and b such that $[\tau(a),\ \tau(b)]$ is a non-zero divisor, then above relations we concluded that $\mu(xy) = \mu(x)\tau(y)$ for all $x, y \in R$ by Lemma 2. Moreover by the above relations we get $(\mu(xy) - \mu(x)\tau(y))\tau(z)[\tau(x),\ \tau(y)] = 0$. So if R is a noncommutative prime ring, then using Lemma 1, we get $\mu(xy) = \mu(x)\tau(y)$ for all $x, y \in R$. Therefore by Examples in Section 2, we have the following results.

Corollary 1. *If R is a noncommutative 2-torsion free prime ring and $[\tau(a),\ \tau(b)]$ is a non-zero divisor for some a and $b \in R$, then generalized (σ,τ)-Jordan derivation $(f,\ d) : R \to R$ is a generalized (σ,τ)-derivation.*

Corollary 2. *If R is a noncommutative 2-torsion free prime ring, then generalized (σ,τ)-Jordan derivations $(f,\ d)$ and $(f,\ \omega) : R \to R$ are generalized (σ,τ)-derivations.*

Finally, we remark the following. Let x, y be arbitrary elements of R. Consider the definition of a Lie (σ,τ)-derivation, a map $d : R \to R$ satisfying the identity $d([x, y]) = [d(x), y]_{\sigma,\tau} - [d(y), x]_{\sigma,\tau}$ introduced in [1], relation (2.5). Suppose that $(f,\ \alpha)$ is a generalized (σ,τ)-derivation on R. Then $\alpha : R \times R \to R$ is symmetric if and only if f is Lie (σ,τ)-derivation. And then α is skew symmetric if and only if f satisfies the relation $d(xy + yx) =$

$d(x)\tau(y) + \sigma(x)d(y) + d(y)\tau(x) + \sigma(y)d(x)$. In case when R is 2-torsion free ring, this implies that α is skew symmetric if and only if f is (σ, τ)-Jordan derivation.

Example 1. Let R be a ring $char R \neq 2$, and for all $x \in R$ we have $x^2 = 0$. Let R contain a nonzero element $0 \neq q$ such that $xqx = 0$ for all $x \in R$ but $xqy \neq 0$ for some $0 \neq x, 0 \neq y \in R$; moreover $xq \neq 0$ for some $x \in R$. Consider the matrix ring

$$A = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in R \right\}.$$

Let σ and τ be the endomorphisms of A defined by $\sigma\left(\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}$ and $\tau\left(\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}\right) = \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}$, and α is a Hochschild 2-(σ, τ) cocycle satisfying the relation (12) defined by $\alpha\left(\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}, \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & zc \end{bmatrix}$ Then the map $f : A \to A$ defined by $f\left(\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}\right) = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}\begin{bmatrix} q & q \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} xq & xq \\ 0 & 0 \end{bmatrix}$ gives rise to generalized (σ, τ)-Jordan derivation but not to a generalized (σ, τ)-derivation on A.

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