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GENERAL EQUATION FOR CHINESE CHECKER CONICS AND FOCUS-DIRECTRIX CHINESE CHECKER CONICS

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Abstract: In [6] and [7] the two foci CC-conics have been studied. In this work a general equation for CC-conics is given and the focus directrix CC-conics are studied. The foci and focus-directrix definitions give different figures.

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Key Words: Chinese checker plane, Chinese checker conics

1. Introduction

In [4] E.F. Krause, made a suggestion for the idea of Chinese checker geometry (throughout this study we write CC instead of *Chinese checker* for the sake of short) and in [1] G. Chen, ıntroduced it by defining the metric

$$d_c(X,Y) = d_L(X,Y) + \left(\sqrt{2}-1\right) d_S(X,Y)\ ,$$

where

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$$d_L(X,Y) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$$

and

$$d_s(X,Y) = \min\{|x_1 - x_2|, |y_1 - y_2|\}$$

for any points $X = (x_1, y_1)$ and $Y = (x_2, y_2)$ in the analytical plane. In [4] A.Ç. Uymaz, studied the CC-circle and in [6], [7] M. Turan, M. Özcan, . studied the two-foci CC-conics. In [2] general equation for taxicab conics are given and they were classificated for all possible cases. The aim of this study is to give a general equation for all CC-conics. The focus directrix CC-conics are examined and compared with that in [6] and [7]. Throughout the study, we denote foci by $F = F_1 = (x_1, y_1)$ and $F_2 = (x_2, y_2)$ and directrix $l : ax + by + c = 0$ with slope m, instead of $\sqrt{2} - 1$ we write q.

2. A General Equation For CC-Conics

In Euclidean geometry conic is defined as follows.

Definition 2.1. A conic is any curve which is the locus of a point which moves so that the ratio of its distance from a fixed point to its distance from a fixed line is constant. The ratio is the eccentricity of a curve, the fixed point the focus, and the fixed line the directrix. The eccentricity is always denoted by e. When $e = 1, e < 1, e > 1$ the conic is a parabola, an ellipse, or a hyperbola, respectively.

The Euclidean conics defined as focus-directrix or two-foci coincide in the same position. Therefore one have to calculate the CC-distance of a point to a line in the CC-plane.

Lemma 2.2. *CC-distance of point P= $(x_{0,}y_0)$ to the line $l : ax+by+c = 0$ in the CC-plane is*

$$d_c(P,l) = \begin{cases} \left|\dfrac{ax_0 + by_0 + c}{b}\right|, & \text{if } |m| \leq q\,, \\ \dfrac{\sqrt{2}}{|b|}\left|\dfrac{ax_0 + by_0 + c}{1+m}\right|, & \text{if } q \leq |m| \leq q+2\,, \\ \left|\dfrac{ax_0 + by_0 + c}{a}\right|, & \text{if } |m| \geq q+2\,, \end{cases}$$

where $m = \frac{a}{b}$ is the slope of l.

Proof. For a detailed proof see [5]. □

Equation of a CC-conic with the focus $F = F_1 = (x_1, y_1)$ and directrix $l : ax + by + c = 0$ has the form

$$\frac{d_c(P,F)}{d_c(P,l)} = e\,, \tag{1}$$

where e is the eccentricity of the CC-conic.

A CC-conic given in equation (1) is called a focus-directrix CC-ellipse, parabola or hyperbola if $0 < e < 1$, $e = 1$ or $e > 1$ respectively. Equation (1) is equivalent to

$$d_c(P,F) - ed_c(P,l) = 0\,,$$

or

$$\max\{|x-x_1|\,,|y-y_1|\} + q\min\{|x-x_1|\,,|y-y_1|\} - ed_c(P,l) = 0\,.$$

Now a general equation for all CC-conics can be given. This equation contains all the two-foci CC-conics given in [6] and [7] . In addition it contains focus-directrix CC-conics examined in this study.

Theorem 2.3. *Equation of a CC-conic, with the focus $F = F_1 = (x_1, y_1)$ and $F_2 = (x_2, y_2)$ or the focus $F_1(x_1, y_1)$ and directrix $l : ax + by + c = 0$ has the folowing form*

$$\begin{aligned}\max\{|x-x_1|-|y-y_1|\} &+ q\min\{|x-x_1|-|y-y_1|\}\\ &+\lambda(\max\{|x-x_2|-|y-y_2|\} + q\min\{|x-x_2|-|y-y_2|\}\\ &\qquad+\mu\,|ax+by+c| \pm \lambda t = 0\,,\end{aligned} \tag{2}$$

where $\lambda \in \{-1, 0, 1\}$, $t \le 0$, e is the eccenricity of related conic and

$$\mu = \begin{cases} \dfrac{e(\lambda^2-1)}{|b|}\,, & \text{if } \ |m| \le q\,,\\ \dfrac{\sqrt{2}e(\lambda^2-1)}{|b|\,(1+|m|)}\,, & \text{if } \ q \le |m| \le q+2\,,\\ \dfrac{e(\lambda^2-1)}{|a|}\,, & \text{if } \ |m| \ge q+2\,. \end{cases}$$

Proof. Equation of a CC-conic with the foci (x_1, y_1) and (x_2, y_2) has the form

$$\begin{aligned}\max\{|x-x_1|-|y-y_1|\} &+ q\text{min}\ \{|x-x_1|-|y-y_1|\}\\ &+t_1(\max\{|x-x_2|-|y-y_2|\} + q\min\{|x-x_2|-|y-y_2|\} = \pm t\,,\end{aligned} \tag{3}$$

where $t_1 \epsilon \{-1,1\}$, $t \geq 0$,

Similarly, equation of CC-conic with the focus (x_1, y_1) and directrix l : $ax+by+c=0$ has the form:

$$\max\{|x-x_1|-|y-y_1|\}+q\min\{|x-x_1|-|y-y_1|\} + t_2\,|ax+by+c|=0\,, \quad (4)$$

where $t_2<0$. Hence the linear combination of equations (3) and (4)

$$s_1\max\{|x-x_1|\,,|y-y_1|\}+q\min\{|x-x_1|\,,|y-y_1|\} + s_2(\max\{|x-x_2|\,,|y-y_2|\}+q\min\{|x-x_2|\,,|y-y_2|\} + s_3\,|ax+by+c|\pm s_4=0\,, \quad (5)$$

represents all of the CC-conics, where $s_1\neq 0$, say $s_1=1$. Equation (5) contains equation (3) iff $s_2\,\{-1,1\}$, $s_3=0$, $s_4\leq 0$.

Thus $s_3=(s_2^2-1)s$, $s\in R$. Equation (5) contains equation (4) iff $s_2=0$, $s_3<0$ and $s_4=0$.

Consequently, $s_4=s_2t$ and $s>0$, $t\leq 0$. Now, let $s_2=\lambda$. Thus equation (5) becomes

$$\max\{|x-x_1|-|y-y_1|\}+q\min\{|x-x_1|-|y-y_1|\} \lambda(\max\{|x-x_2|-|y-y_2|\}+q\min\{|x-x_2|-|y-y_2|\} +(\lambda^2-1)\,|ax+by+c|\pm\lambda t=0\,. \quad (6)$$

with $\lambda\in\{-1,0,1\}$, $s>0$ and $t\leq 0$. In the case $\lambda=0$ we get

$$\frac{\max\{|x-x_1|-|y-y_1|\}+q\min\{|x-x_1|-|y-y_1|\}}{|ax+by+c|}=s\,,$$

form the above equation (6). Using Lemma 2.2, we obtain

$$e=\frac{\max\{|x-x_1|-|y-y_1|\}+q\min\{|x-x_1|-|y-y_1|\}}{|ax+by+c|}=ks\,,$$

$$k=\begin{cases} |b| & \text{if} \quad |m|\leq q\,,\\ |b|\ (\ 1+|m|\ ) & \text{if} \quad q\leq|m|\leq q+2\,,\\ |a| & \text{if} \quad |m|\geq q+2\,.\end{cases}$$

Clearly e represents the eccentricity of the conic. That is,

$$s=ek^{-1}\,,\quad \text{and} \quad (\lambda^2-1)s=e,(\lambda^2-1)k^{-1}\,(\mu)$$

which comletes the proof. □

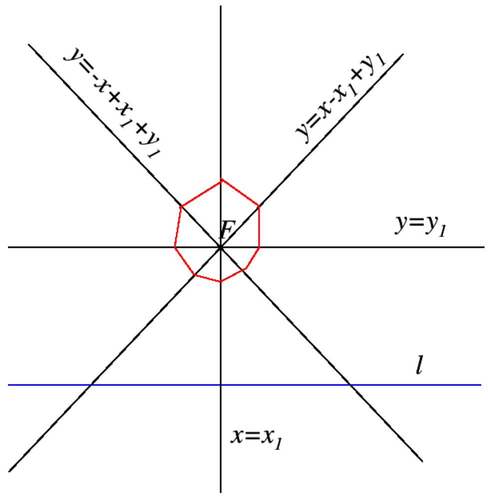
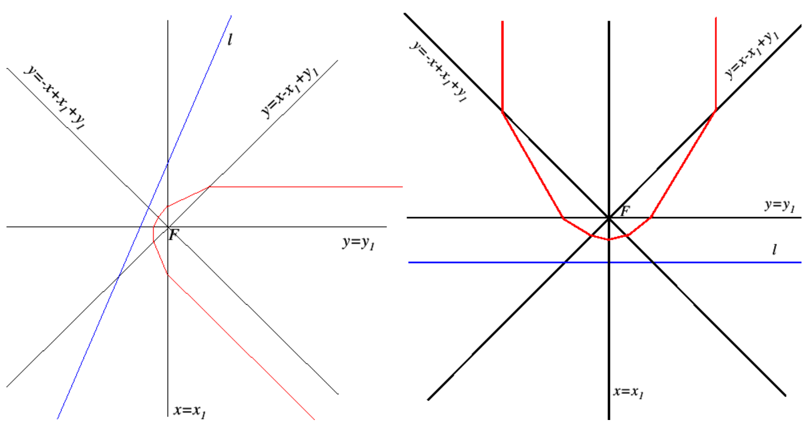


Figure 1: Focus-directrix CC-ellipses

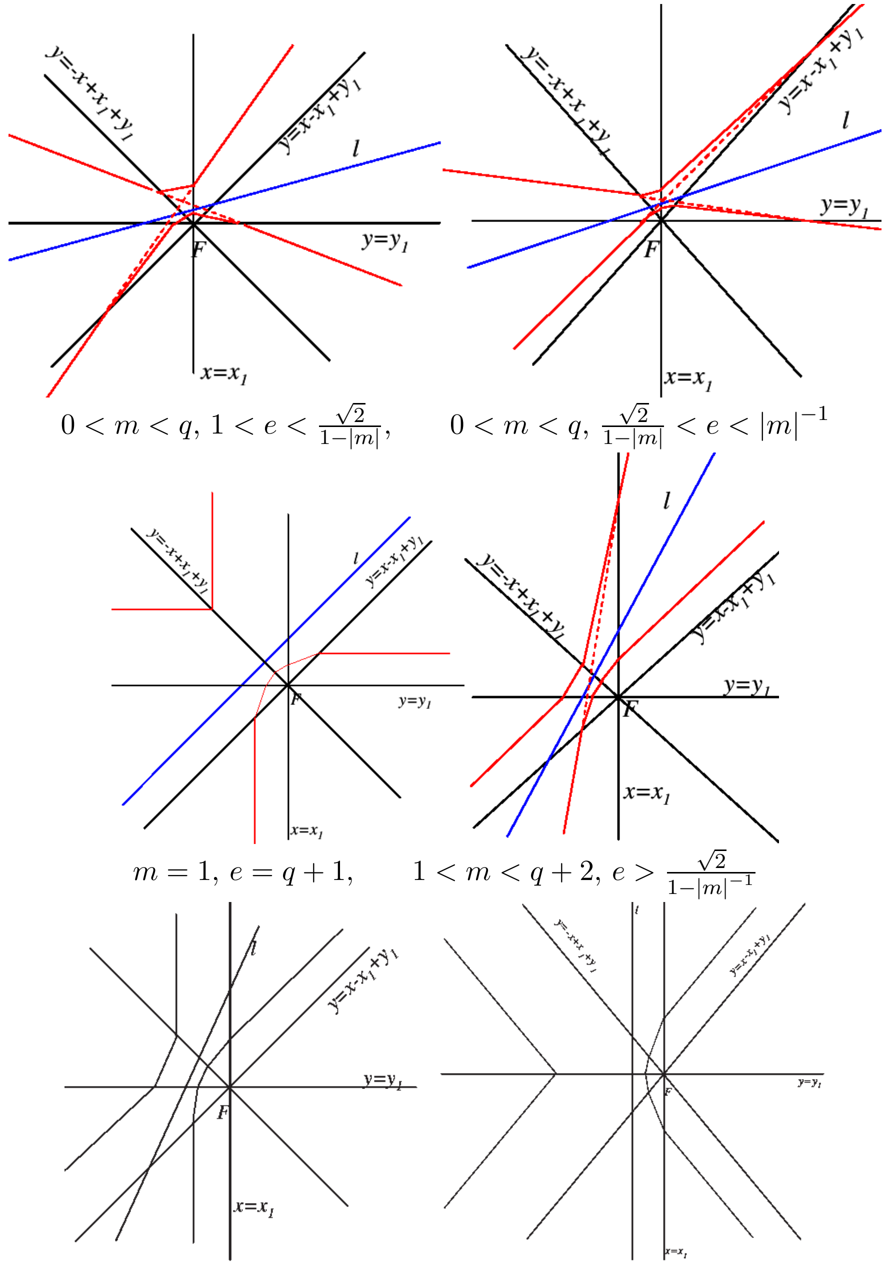


$m = q$ $\quad$ $0 < m < q$

Figure 2: Focus-directrix CC-parabolas

3. Classification of the CC-Conics

All of the CC-conics represented by equation (1) are seperated into two classes by using the cofficient λ. A CC-conic represented by equation (1) is called a *focus-directrix CC-conic if* $\lambda = 0$ and a *two-foci CC-conic* if $\lambda = \pm 1$. A focus-directrix conic obtained while $\lambda = 0$ is called a focus-directrix ellipse, parabola or hyperbola if $0 < e < 1$, $e = 1$, or $e > 1$, respectively. A two-foci CC-conic is

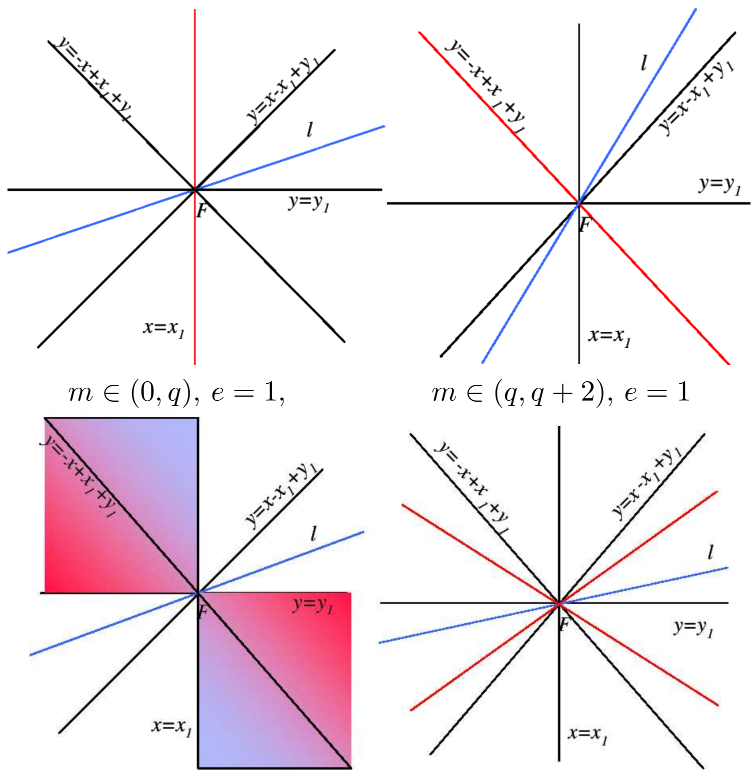


$0 < m < q,\ 1 < e < \frac{\sqrt{2}}{1-|m|}$, $0 < m < q,\ \frac{\sqrt{2}}{1-|m|} < e < |m|^{-1}$

$m = 1,\ e = q+1$, $1 < m < q+2,\ e > \frac{\sqrt{2}}{1-|m|^{-1}}$

$m = q+2,\ e = q+2$, $m = \infty,\ e \leq q+1$

Figure 3: Focus-directrix CC-parabolas



$m \in (0, q)$, $e = 1$, $m \in (q, q+2)$, $e = 1$

$m = q$, $e = 1$, $m = \infty$, $e = q + 1$,

Figure 4: Focus-directrix CC-parabolas

called a two-foci CC-ellipse or CC-hyperbola if $\lambda = 1$ or $\lambda = -1$ respectively.

Two-foci CC-conics were examined and classified for all possible cases in [7], [8]. Therefore we examine types of all focus-directrix CC-conics, in this section of the paper. We classify them as focus-directrix CC-conics and degenerate focus-directrix CC-conics.

Theorem 3.1. *Focus-directrix CC-conics represented by equation* (2) *have types as in the following Table 1. They are uniquely determinedby the eccentricity e, the position (slope m) of the directrix $l : ax + by + c = 0$ and the focus $F = F_1 = (x_1, y_1)$.*

Corollary 3.2. *Tte set of the CC-ellipses and hyperbolas obtained in* [7], [8] *and this study are distinct from eaach other. That is, two-foci and focus-directrix CC-conic definitions give different figures.*

e	m	**Geometric Locus**
$0 < e < 1$	$\forall m$	A non-regular octagon.
$e = 1$	$m \in \{q, q+2\}$	A true CC-parabola, consisting of five line segments and two rays.Where the slope of a ray is -1, the other is horizantal or vertical, according as $m = q$ or $m = q+2$.
	$m \notin \{q, q+2\}$	A true CC-parabola, consisting of six line segments and two rays. Those rays are horizontal or the slope of one -1 and the other 1or vertical, according as $m \in [0, q)$ or $m \in (q, q+2)$ or $m \in (q+2, \infty]$.
$e > 1$	$m \in \{q, q+2\}$, $e = q+2$	A CC-hyperbola with two branches. One of the branches consists of three line segments and two rays one being vertical. The second branch consists of a line segment and two rays one being vertical. The two non-vertical rays are on a line.
	$m \in \{0, \infty\}$, $e = q+1$	A CC-hyperbola with two branches. One of the branches consists of four line segments and two rays. The second branches consists of two rays. Where the slope of the rays are -1 and 1.
	$m \in (0, q]$, $e > \|m\|^{-1}$, $m \in (q, 1)$, $e \geq \dfrac{\sqrt{2}}{1 - \|m\|}$	A CC-hyperbola with two branches. Each branch consists of three line segments and two rays. Where all rays on two lines with intersection point on the directrix.
	$m \in (1, q+2)$, $e > \dfrac{\sqrt{2}}{1 - \|m\|^{-1}}$	A CC-hyperbola with two branches. One of the branches consists of three line segments and two rays. The second branch consists of two line segments and two rays. Where the slope of the rays one being 1. The other rays are on a line.
	$m \in (q+2, \infty)$, $e \geq \|m\|$	A CC-hyperbola with two branches. One of the branches consists of three line segments and two rays one being vertical. The second branch consists of two line segments and two rays one being vertical. The two non-vertical rays are on a line.
	$m \in (0, q]$, $e < \frac{\sqrt{2}}{1-\|m\|}$; $m \in (q, 1)$; $e \leq \|m\|^{-1}$; $m \in (1, q+2]$, $e < \|m\|$; $m \in (q+2, \infty)$, $e < \frac{\sqrt{2}}{1-\|m\|^{-1}}$	A CC-hyperbola with two branches. One of the branches consists of five line segments and two rays.The second branch consists of a line segment and two rays having opposite directions with the first rays. The four rays are on two lines. Each of these two lines, the directrix and two line containg one of the line segments are concurrent.
	$m \in [0, q)$, $\frac{\sqrt{2}}{1-\|m\|} \leq e \leq \|m\|^{-1}$; $m \in (q, 1]$, $\|m\|^{-1} < e < \frac{\sqrt{2}}{1-\|m\|}$; $m \in [1, q+2)$, $\|m\| < e < \frac{\sqrt{2}}{1-\|m\|^{-1}}$; $m \in (q+2, \infty)$, $\frac{\sqrt{2}}{1-\|m\|^{-1}} < e < \|m\|$	A CC-hyperbola with two branches. One of the branches consists of four line segments and two rays. The second branch consists of two line segment and two rays. Where all rays are on two lines with intersection point on the directrix.

Table 1: Focus-directrix CC-Conics ($\lambda = 0$)

The focus F is not on the directrix l for the CC-conics given in Table 1. The focus F is not on the directrix l for the CC-conics given Table 1. On the otherhand, a focus-directrix CC-conics represented by. Equation (1) is called *degenerate* if the focus is on the directrix.

The degenerate CC-conics must be examined in three cases and their possible subcases (see [6]) in Figures 1-3.

Figure 4 presents examples of the degenerate focus-directris CC-conics.

4. Graphs of The CC-Conics

Graph of CC-conic can be drawn if it is represented by equation of the form given in Theorem 1. Graphs of some of the focus-directrix CC-conics and degenarete focus-directrix CC-conics are given on Figures 1-4.

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