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ERGODICITY CONDITIONS IN A SIMPLE BEST EFFORT QUEUEING NETWORKS

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Abstract: In this article, we are interested in a network of the type "best effort" where the band-width available is distributed in the equitable way between the users, by considering a policy known as "min" which is in fact an approximation of the traditional policy of "Equite Max-Min". A simple example of such a model is a star distribution system which we give necessary and sufficient conditions of ergodicity by using Lyapunov functions.

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1. Introduction

In this work, we are interrested on a network managing the floods of data for several users, in which no guarantee of service is offered. The most reasonable model of the service in this context is the service known as "best effort", which allocates the band-width in an equitable way between the active users (see [2]). In fact, there are several concepts of equity in this kind of problems, most traditional being equity Max-Min. Even if one starts with well understanding the way in which the band-width is distributed by the various mechanisms of control of congestion. Relatively, little works allow to account for the impact of the random nature of the traffic, in particular on the quality of service perceived. One of the studies in this direction is that of Roberts and Massoulié [7] which

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analyze a linear network topology.

Our intention is to study more complex topologies under differents angles. It is about a Markovian model of a network "best effort", it is intended to account for the impact of topology, like the parameters of traffic and the policy of division of band-width on the performance. General information on the Lyapunov functions and stability of the stochastic networks will be given in Section 2. The ergodicity conditions of the star distribution system are considered in Section 3.

2. Lyapunov Fonctions, Drift

A manner at the same time natural and effective to study the stability of Markov chains is by means of Lyapunov functions. The latter is a mapping $f\colon E \longrightarrow \mathbb{R}_+$. The drift associated with f is defined by

$$\forall i \in E, \;\; \triangle f(i) = \mathbb{E}\left(f(X(n+1)) - f(X(n)) \to X(n) = i\right) ,$$

where E is the space of states (here discrete) of the chain.

In a heuristic way, f represents a potential energy, and $\triangle f$ the tendency of Markov chain "to converge" towards an area of finished energy, or "to diverge" towards an area from infinite energy, according to the sign of the drift outside a unit finished. More formally, there are the following criteria of stability.

Theorem 2.1. (Foster Criterion) *Let χ be an irreductible Markov chain. A sufficient condition so that χ is positive, is that there exist a finite set A and a Lyapunov function f, such that $\sup_{i\in A} \triangle f(i) < \infty$ and for certain constant $\varepsilon > 0$,*

$$\forall i \notin A, \quad \triangle f(i) \leq -\varepsilon.$$

Proof. Let τ_A be the time of return towards A. We have for all $n \geq 1$,

$$\begin{aligned}\mathbb{E}(f(X_{n+1}).\mathbb{I}_{\{\tau_A>n+1\}}/X(n),\ldots,X(0))\\ &\leq \mathbb{E}(f(X(n+1)).\mathbb{I}_{\{\tau_A>n\}}/X(n),\ldots,X(0))\\ &= \mathbb{E}(f(X(n+1))/X(n)).\mathbb{I}_{\{\tau_A>n\}} \leq (f(X(n))-\varepsilon).\mathbb{I}_{\{\tau_A>n\}}.\end{aligned}$$

Let $i \in E$. For all $n \geq 1$, we have

$$0 \leq \mathbb{E}_i[f(X(n+1))\mathbb{I}_{\{\tau_A>n+1\}}] \leq \mathbb{E}_i[f(X(n).\mathbb{I}_{\{\tau_A>n\}}] - \varepsilon.\mathbb{P}_i(\tau_A > n).$$

By iteration of this inequality, one obtains

$$0 \leq \mathbb{E}_i[f(X(1)).\mathbb{I}_{\{\tau_A>1\}}] - \varepsilon. \sum_{k=1}^{n} \mathbb{P}_i(\tau_A > k).$$

While making tighten n towards the infinity, and by noticing that

$$\mathbb{E}_i[f(X(1)).\mathbb{I}_{\{\tau_A>1\}}] \leq \mathbb{E}_i[f(X(1))] = f(i) + \triangle f(i),$$

it gets $\varepsilon. \sum_{k=1}^{\infty} \mathbb{P}_i(\tau_A > k) \leq f(i) + \triangle f(i)$.

Finally,

$$\forall i \in E,\ \mathbb{E}_i(\tau_A) \leq 1 + \varepsilon^{-1}(f(i) + \triangle f(i)).$$

That is to say in particular, $\forall i \in A,\ \mathbb{E}_i(\tau_A) < \infty$. Thus, there is a positive unit finished A, which implies the positivity of the chain. □

For all functions $m : E \longrightarrow \mathbb{N}$, we define

$$\forall i \in E, \triangle^m f(i) = \mathbb{E}(f(X(n+m(i))) - f(X(n))/X(n) = i).$$

A natural generalisation of Foster's criterion is given by the following result.

Corollary 2.1. (Generalized Foster Criterion) *Let χ be an irreductible Markov chain. A sufficient condition so that χ is positive is that exist a finite set A, a Lyapunov function f, a function m defined on E and taking nonnull integer values, such that*

$$\sup_{i\in A} \triangle^m f(i) < +\infty$$

and for a certain constant $\varepsilon > 0$,

$$\forall i \notin A,\ \triangle^m f(i) \leq -\varepsilon.m(i).$$

3. Star Networks

3.1. Description of the Model

We consider a network including $\mathcal{L}$ bonds. Several sources establish connections along a roads which borrow these bonds. We are interested to the way of bandwidth dividing offered by each bond between active connections, like with the effect of allowance policy on the dynamic of the network.

In the star distribution systems, all the roads are length 2, which represents rather well a large switch with many roads. Each branch of star comprises two bonds then (entering and outgoing) and each road is isomorphous with a couple of branches.

That is to say $\mathcal{R}$ the set of network's roads. Connections are created on a road according to a process of Poisson of intensity $\lambda_r, r \in \mathcal{R}$. Each connection is maintained until to have transmitted through the network the data, whose volume follows an exponential law of average σ_r. A bond $l \in \mathcal{L}$ has a band-width $\mathcal{C}_l$ and the intensity of arrival to this bond is noted

$$\rho_l = \frac{1}{\mathcal{C}_l} \sum_{r \in l} \lambda_r \sigma_r.$$

Summation being carried out on all the roads using the bond l. Each road r has at every moment a ζ_r fraction of the band-width available.

The state of the system at the moment $t \in \mathbb{R}$ is defined by the number of active connections on each road, it is noted $X(t) = (x_r, r \in \mathbb{R})$ and the quantity $X_l(t) = \sum_{l \in r} x_r(t)$ represents the number of connections on the bond $l \in \mathcal{L}$. The vector $X(t)$ is a Markov process from which the possible transitions are:

— Arrival of a connection on a road r, with intensity λ_r.

— Departure of a connection on a road r, with intensity $\frac{\xi_r(X(t))}{\sigma_r}$.

Distribution policy of the band-width considered in our work is the policy known as of -MIN- proposed by the authors L. Massoulié and J. Roberts in [7], where a connection on a road r is seen allotting a flow

$$\xi_r^{\min}(X(t)) = \min_{l \in r} \frac{\mathcal{C}_l}{X_l(t)}.$$

3.2. Ergodicity Conditions

It proves that the stationary regime exists under the usual conditions, which ensure that flow entering in each bond l is lower than $\mathcal{C}_l$.

Theorem 3.2.1. *1. If $\max_{l \in \mathcal{L}} \rho_l < 1$, so the network is ergodic.*

2. If $\max_{l \in \mathcal{L}} \rho_l > 1$, so the network is transient.

Proof. We consider a Markov chain with discrete time $(X(n), n \in \mathbb{N})$, which describe the sequence of states visited by the process at continuous time $(X(t))_{t \in \mathbb{R}}$. On basis of a fixed state $X = (x_r, r \in \mathcal{R})$, the transitions are defined by:

$$\mathbb{P}[x_r(n+1) - x_r(n) = 1/X(n) = X] = \frac{\lambda_r}{D},$$
$$\mathbb{P}[x_r(n+1) - x_r(n) = -1/X(n) = X] = \frac{1}{D}.\frac{x_r}{\sigma_r}\min_{l\in r}\frac{\mathcal{C}_l}{X_l},$$

where

$$D = \sum_{r\in\mathcal{R}}(\lambda_r + \frac{x_r}{\sigma_r}\min_{l\in r}\frac{\mathcal{C}_l}{X_l}) \le |\mathcal{R}|.(\max_{r\in\mathcal{R}}\lambda_r + \max_{r\in\mathcal{R}}\frac{1}{\sigma_r}.\max_{l\in\mathcal{L}}\mathcal{C}_l).$$

Let us note

$$D^{'} = |\mathcal{R}|.(\max_{r\in\mathcal{R}}\lambda_r + \max_{r\in\mathcal{R}}\frac{1}{\sigma_r}.\max_{l\in\mathcal{L}}\mathcal{C}_l).$$

The ergodicity of the process $(X_t)_{t\in\mathbb{R}}$ is implied by that of $(X_n, n \in \mathbb{N})$ (since the borrowing rates of a state X are undervalued uniformly by a positive constant).

i) Let us show that if $\max_{l\in\mathcal{L}}\rho_l < 1$, then the process $(X_t)_{t\in\mathbb{R}}$ is ergodic.

We suppose that $\rho_M = \max_{l\in\mathcal{L}}\rho_l < 1$. We introduce Lyapunov function

$$f(X) = \sum_{r\in\mathcal{R}}\sum_{1\le k\le x_r}\gamma_r^k,$$

where $\gamma_r > 1$ will be clarified further.

In order to express the rates of transition according to x_r and ρ_l, we write:

$$X_l = \sum_{l\in r} x_r = \sum_{l\in r}\lambda_r\sigma_r\frac{x_r}{\lambda_r\sigma_r} \le \rho_l\mathcal{C}_l\max_{l\in r}\frac{x_r}{\lambda_r\sigma_r},$$

and we introduce quantities

$$\bar{x}_r = \frac{x_r}{\lambda_r\sigma_r}, \quad \bar{x}_M = \max_{r\in\mathcal{R}}\bar{x}_r.$$

Thus, $\forall r \in \mathcal{R}$,

$$\mathbb{P}[(x_r(n+1) - x_r(n) = -1)/X(n) = X] \ge \frac{\lambda_r}{D.\rho_M}.\frac{\bar{x}_r}{\bar{x}_M}.$$

Then, we can write

$$\mathbb{E}[(f(X(n+1)) - f(X(n)))/X(n) = X]$$

$$= \sum_{r\in\mathcal{R}} \{\gamma_r^{\bar{x}_r+1}.\mathbb{P}[(x_r(n+1) - x_r(n) = 1)/X(n) = X]$$

$$-\gamma^{\bar{x}_r}.\mathbb{P}[(x_r(n+1)-x_r(n) = -1)/X(n) = X]\} \le \sum_{r\in\mathcal{R}} \frac{\lambda_r}{\rho_M.D}.\gamma_r^{\bar{x}_r}[\rho_M.\gamma_r - \frac{\bar{x}_r}{\bar{x}_M}].$$

For a fixed θ and under condition $\rho_M < \theta < 1$, one poses $\gamma_r = \gamma^{\frac{1}{\lambda_r \sigma_r}}$, where γ is selected such as

$$\rho_M.\gamma_r = \rho_M.\gamma^{\frac{1}{\lambda_r}\sigma_r} < \theta < 1, \quad r \in \mathcal{R}.$$

It resultes that γ and θ satisfy the preceding constraints

$$\mathbb{E}[(f(X(n+1) - f(X_n))/X(n) = X] \le \sum_{r\in\mathcal{R}} \frac{\lambda_r \gamma^{\bar{x}_r}}{\rho_M.D}[\theta - \frac{\bar{x}_r}{\bar{x}_M}].$$

Let α be a real such that $\theta < \alpha < 1$. The following quantities will be evaluated separately

$$\begin{cases} \Sigma_1 &= \displaystyle\sum_{r:\bar{x}_r > \alpha.\bar{x}_M} \frac{\lambda_r \gamma^{\bar{x}_r}}{\rho_M.D}[\theta - \frac{\bar{x}_r}{\bar{x}_M}], \\ \Sigma_2 &= \displaystyle\sum_{r:\bar{x}_r \le \alpha.\bar{x}_M} \frac{\lambda_r \gamma^{\bar{x}_r}}{\rho_M.D}[\theta - \frac{\bar{x}_r}{\bar{x}_M}]. \end{cases}$$

We raise the sum Σ_1, of which all the terms are negatives, by only one term corresponding to r_0 for which $\bar{x}_{r_0} = \bar{x}_M$:

$$\Sigma_1 \le \frac{\lambda_{r_0}.\gamma^{\bar{x}_M}}{\rho_M.D}(\theta - \alpha) \le \frac{\gamma^{\bar{x}_M}}{\rho_M.D}(\theta - \alpha).\min_{r\in\mathcal{R}} \lambda_r < 0.$$

Similarly, for Σ_2, one has

$$\Sigma_2 \le \sum_{r:\bar{x}_r \le \alpha.\bar{x}_M} \frac{\lambda_r \gamma^{\alpha \bar{x}_M}}{\rho_M.D}.\theta \le \frac{\gamma^{\bar{x}_M}}{\rho_M.D}\gamma^{(\alpha-1)}.\bar{x}_M.|\mathcal{R}|.\theta \max_{r\in\mathcal{R}} \lambda_r.$$

Finally, if $C > 0$ and $\varepsilon > 0$ are selected in order to carry out the inequality

$$(\theta - \alpha)\min_{r\in\mathcal{R}} \lambda_r + \gamma^{(\alpha-1).C}|\mathcal{R}|.\theta \max_{r\in\mathcal{R}} \lambda_r \le -\varepsilon,$$

we obtain $\forall x \in \{\bar{x}_M > C\}$,

$$\mathbb{E}[(f(X(n+1)) - f(X(n))/X(n) = X] = \Sigma_1 + \Sigma_2$$

$$\leq \frac{-\varepsilon.\gamma^{\bar{x}_M}}{\rho_M.D} \leq \frac{-\varepsilon.\gamma^C}{\rho_M.D'} < 0.$$

Since the set $\{x_M \leq C\}$ is compact, one can apply the theorem of Foster, and to deduce that the chain is ergodic.

ii) Let us suppose now that $\max_{l\in\mathcal{L}} \rho_l > 1$, i.e. that there exists l_0 such that $\rho_{l_o} > 1$.

While posing

$$g(X) = \sum_{l_0\in r} \sigma_r.\bar{x}_r.$$

Thus, one has

$$\mathbb{E}[(g(X(n+1)) - g(X(n)))/X(n) = X] =$$
$$= \sum_{l_0\in r} \sigma_r\{\mathbb{P}[(\bar{x}_r(n+1) - \bar{x}_r(n) = 1)/X(n) = X]$$
$$- \mathbb{P}[(\bar{x}_r(n+1) - \bar{x}_r(n) = -1/X(n) = X]\} \geq \frac{1}{D'}[\mathcal{C}_{l_0}.\rho_{l_0} - \mathcal{C}_{l_0}] > 0.$$

According to Foster's criterion for the transience, and as the jumps are limited, the system is thus transient. □

4. Conclusion

— The configuration which was studied (star network) is a configuration that corresponds to a telecommunications network, such as the switching networks of packages or the switching networks of cells (ATM).
— The study of ergodicity aims to lay down the processing capacity of communications's machines, to know the operating cost of a node, then to result in changing the support of transmission (the band-width) and the nodes (machines of communications).
— A future work will be as to seek the conditions of ergodicity for a switching network in other types, and with another policy of the band-width allowance.

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