International Journal of Pure and Applied Mathematics
Volume 30 No. 4 2006, 527-536

ON THE OPENNESS OF SURJECTIVE MAPPINGS: REPELLING POINTS

Larry Downey
School of Science
Penn State University at Erie
Station Road, Erie, PA 16563, USA
e-mail: lmd108@psu.edu

Abstract: In this note, we introduce the study of special sets in the range of certain surjective mappings. The points in these sets have the property that "nearby" points have preimages that are forced to be large in norm. For multilinear operators, these points illustrate how it is possible to not be open at the origin, despite being surjective. We also give a type of Open Mapping Theorem for surjective multilinear operators whose range has small dimension, and a corollary concerning right inverses.

AMS Subject Classification: 46T99
Key Words: open, bilinear, surjective, onto, repelling

1. Introduction

The original question of the existence of an Open Mapping Theorem for bilinear operators, posed by Walter Rudin in [5], was answered several years later in the papers [1] and [4]. In particular, Horowitz, in [4], gives an example of a bilinear operator $H : \mathbb{C}^3 \times \mathbb{C}^3 \longrightarrow \mathbb{C}^4$ which is onto but which is not open at the origin. The key to this example is that as points approach the point $(0, 0, 0, 1)$ in the range along the path $(t, t, t, 1),\ t \to 0$, the preimages of these points are forced to be large in norm. In this note, we study the phenomenon in the Horowitz example more generally, focusing on the points in the range for which points nearby have preimages which are "repelled" from the origin, i.e. have norms which tend to infinity.

The purpose of this note is to introduce these points as a topic of study, with the goal of better understanding the associated mappings, especially in the

Received: June 28, 2006

context of openness. We study general mappings where possible, although multilinear mappings are of special interest. To be precise, we make the following definition, noting here that in this paper, we study only surjective operators, and all spaces will be assumed to be Banach spaces.

Definition 1. For a surjective operator $f : \mathbb{X} \longrightarrow \mathbb{Y}$, we call a point $z \in \mathbb{Y}$ *repelling* for f if no neighborhood of z is the image of a bounded set in $\mathbb{X}$.

Note. We note here that the above definition is equivalent to: there is a sequence $(z_n) \to z$ such that $f(x_n) = z_n \implies ||x_n|| > n$.

We make the designation $\mathcal{R}_f$ for the set of all repelling points of f. It is this set that we study here. One immediate observation is that $\mathcal{R}_f$ *is closed*, for if v is any vector in the complement of $\mathcal{R}_f$, then some open neighborhood W of v is the image of a bounded subset of $\mathbb{X}$, and hence every vector in W is in the complement of $\mathcal{R}_f$. Another observation is that for each $z \in \mathcal{R}_f$, the set $f^{-1}(z)$ consists of points in the domain at which f is not open.

Among the fundamental topics which can be investigated about the set $\mathcal{R}_f$ are its algebraic structure and it is topological size.

2. The Topological Size of $\mathcal{R}_f$

If no assumptions are made about the mapping, then the set of repelling points can be very large, in fact the whole range space, as the next simple example shows.

Example 1. We define a mapping $f : \mathbb{R} \longrightarrow \mathbb{R}$ as follows: Let g be any one to one correspondence between $\mathbb{N}$ and $\mathbb{Q}$, and h be any one to one correspondence between the complement of $\mathbb{N}$ in $\mathbb{R}$ and the irrationals. Then

$$f(x) = \begin{cases} g(x) & \text{if } x \in \mathbb{N}\,, \\ h(x) & \text{else}\,. \end{cases}$$

Now, given any interval I, I contains an infinite set of rationals, which, clearly cannot be the image of a bounded set. Since no interval has a bounded preimage, every point in the range is a repelling point. □

Of course, such a function as in Example 1 is not continuous, a fact which will be easily seen as a consequence of Theorem 5 below.

We now explore conditions which guarantee that $\mathcal{R}_f$ is small topologically.

Proposition 2. *Let $f : \mathbb{X} \longrightarrow \mathbb{Y}$ be a surjective mapping between Banach spaces. If there exists a sequence (D_n) of bounded subsets of $\mathbb{X}$ so that $f(D_n)$ is closed for each n, and so that $\mathbb{Y} = \bigcup f(D_n)$, then $\mathcal{R}_f$ is nowhere dense in $\mathbb{Y}$.*

Proof. Let $z \in \mathbb{Y}$ and $\varepsilon > 0$. By the hypothesis, $\overline{B(z,\varepsilon)} = \bigcup \Big(f(D_n) \cap \overline{B(z,\varepsilon)} \Big)$. Since $\overline{B(z,\varepsilon)}$ is a complete metric space, the Baire Category Theorem gives that for some j, $f(D_j) \cap \overline{B(z,\varepsilon)}$ contains some ball $B(x,\delta)$. In particular, $||z - x|| < \varepsilon$, and since $B(x,\delta) \subset f(D_j)$ and D_j is bounded, $B(x,\delta)$ is contained in $\mathbb{Y} \backslash \mathcal{R}_f$. □

There are several corollaries to the above proposition, but we first define the following useful function.

Definition 3. For a surjective mapping $f : \mathbb{X} \longrightarrow \mathbb{Y}$, define $m_f : \mathbb{Y} \longrightarrow [0,\infty)$ by $m_f(y) = \inf\{||x|| : f(x) = y\}$.

We note here that if $z \in \mathcal{R}_f$, then there is a sequence $(z_k) \to z$ so that $m_f(z_k) \to \infty$, so that in particular, m_f is not upper semi-continuous at z.

Corollary 4. *Let $f : \mathbb{X} \longrightarrow \mathbb{Y}$ be a surjective mapping between Banach spaces. If m_f is lower semi-continuous on Y, then $\mathcal{R}_f$ is nowhere dense in Y.*

Proof. Let D_n be the closed ball of radius n in $\mathbb{X}$, and let $z_k \to z$, where each z_k is in $f(D_n)$. Because m_f is lower semi-continuous at z, and since $m_f(z_k) \leq n$ for all k, $m_f(z) \leq n$, and $z \in f(D_n)$. This shows that $f(D_n)$ is closed, and clearly $Y = \bigcup f(D_n)$. Thus, Proposition 2 gives that $\mathcal{R}_f$ is nowhere dense in Y. □

The next result is perhaps the most useful. Note that the hypothesis includes the case of continuous functions between finite dimensional spaces.

Theorem 5. *Let $f : \mathbb{X} \longrightarrow \mathbb{Y}$ be a surjective mapping between Banach spaces, where $\mathbb{X}$ is reflexive and f is weakly sequentially continuous. Then $\mathcal{R}_f$ is nowhere dense in $\mathbb{Y}$.*

Proof. We show that m_f is lower semi-continuous. Suppose the contrary, so that for some z and some $\varepsilon > 0$, there is a sequence $(z_k) \to z$ with $m_f(z_k) < m_f(z) - \varepsilon$ for all k. Let $R = m_f(z) - \varepsilon$, and for each k, let $f(x_k) = z_k$, where $||x_k|| \leq R$. Since $\overline{B(0,R)}$ is weakly compact in $\mathbb{X}$, $(x_k) \xrightarrow{w} x$ for some $x \in \overline{B(0,R)}$. By the assumption on $f, (z_k) \xrightarrow{w} f(x)$. Since the weak topology is Hausdorff, $f(x) = z$, a contradiction since $R < m_f(z)$. □

It is an open question as to whether this can be extended to infinite dimensions with no assumptions on the spaces and/or mere continuity of the mapping. Before moving to the next section, we give an example of how repelling points can be used to give a very simple proof about extensions of certain mappings. The Proposition below generalizes Example 1.

Proposition 6. *Let A and B be subsets of finite dimensional spaces $\mathbb{X}$ and $\mathbb{Y}$ respectively, where B is dense in $\mathbb{Y}$, and where A is an unbounded set with the property that each bounded subset of A is finite. If $g : A \longrightarrow B$ is any bijection, then g cannot be extended to a continuous bijection between $\mathbb{X}$ and $\mathbb{Y}$.*

Proof. Paralleling the argument in Example 1, one obtains that any continuous bijective extension f of g would be such that $\mathcal{R}_f = Y$, but this would contradict Theorem 5. $\square$

The investigation in this paper is rooted in the question concerning an Open Mapping Theorem for multilinear operators. We focus on functions including these now, for which there is at least some indication of an algebraic structure for the set of repelling points.

3. Algebraic Structure; n-Homogeneous Mappings

In the following, an n-homogeneous mapping f is one so that $f(\alpha x) = \alpha^n f(x)$ for all scalars α.

Proposition 7. (One Dimensional Subspaces) *Let $f : \mathbb{X} \longrightarrow \mathbb{Y}$ be a surjective n-homogeneous mapping between Banach spaces. If $y \in \mathcal{R}_f$, then $\alpha y \in \mathcal{R}_f$ for all $\alpha \in \mathbb{K}$.*

Proof. Let $y \in \mathcal{R}_f$, and let us first assume that $\alpha \neq 0$. Suppose that $\alpha y \notin \mathcal{R}_f$. Then there is a bounded set W in $\mathbb{X}$ so that, for some $\varepsilon > 0$, $B_\varepsilon(\alpha y) \subset f(W)$. Now, note that $B_\varepsilon(\alpha y) = \alpha B_{\frac{\varepsilon}{|\alpha|}}(y)$, and let z be any point in $B_{\frac{\varepsilon}{|\alpha|}}(y)$. Then $\alpha z \in B_\varepsilon(\alpha y)$. Hence, there is some $w \in W$ with $f(w) = \alpha z$. Thus, $f(\frac{1}{\alpha^{\frac{1}{n}}} w) = z$. We have shown that $B_{\frac{\varepsilon}{|\alpha|}}(y) \subset f(\frac{1}{\alpha^{\frac{1}{n}}} W)$, contradicting that y is repelling for f. Hence, αy is repelling. Finally, since $\mathcal{R}_f$ is closed, $0 \in \mathcal{R}_f$ as well. $\square$

Proposition 7 shows that if an n-homogeneous operator has any repelling point, then necessarily 0 is a repelling point. The next proposition shows that if the range space is finite dimensional, then 0 cannot be the only repelling point.

Proposition 8. *Let $f : \mathbb{X} \longrightarrow \mathbb{Y}$ be a surjective n-homogeneous mapping between Banach spaces, where $\mathbb{Y}$ is finite dimensional. Then either $\mathcal{R}_f$ is empty or it contains a one dimensional subspace.*

Proof. Assume $\mathcal{R}_f$ is not empty. By Proposition 7, we need only show that 0 cannot be the only repulsive point. Let $B_{\mathbb{X}}$ be the unit ball in $\mathbb{X}$ and and $S_{\mathbb{Y}}$ the unit sphere in Y. If for some $k \in \mathbb{N}$, $S_{\mathbb{Y}} \subset f(kB_{\mathbb{X}})$, then by the n-homogeneity of f, $\overline{B_{\mathbb{Y}}} \subset f(kB_{\mathbb{X}})$. This cannot be, however, since 0 is assumed to be repelling. Hence, for each $k \in \mathbb{N}$, there is a $y_k \in S_{\mathbb{Y}}$ so that $m_f(y_k) > k$. Now, as $\mathbb{Y}$ is finite dimensional, $S_{\mathbb{Y}}$ is compact. Thus, the sequence (y_k) has a subsequence which converges, say to y. Then y is a norm one vector which is repelling. □

We now give an exact description of the set of repelling points for the Horowitz example, which turns out to be a two dimensional subspace of $\mathbb{C}^4$, leaving open the possibility that $\mathcal{R}_f$ is a subspace for every surjective multilinear map f for which $\mathcal{R}_f$ is nonempty. We note here that in the following, we use the max norm on $\mathbb{C}^n$, i.e. $||(z_1, \dots z_n)|| = \text{Max}\{|z_i| : i = 1, \dots, n\}$.

Proposition 9. *Let* $H = \mathbb{C}^3 \times \mathbb{C}^3 \longrightarrow \mathbb{C}^4$ *by* $H(x, y) = (x_1y_1, x_1y_2, x_1y_3 + x_3y_1 + x_2y_2, x_3y_2 + x_2y_1)$. *Then* $\mathcal{R}_H = \{(a, b, c, d) \in \mathbb{C}^4 : a = b = 0\}$.

Proof. Let $v = (a_0, b_0, c_0, d_0) \in \mathbb{C}^4$, and suppose $a_0 \neq 0$. Let (a, b, c, d) be any point in $B_\delta(v)$, where $\delta = \frac{|a_0|}{2}$, and note then that $|a| > \delta$. We also note that there is a constant K so that for every $(z_1, z_2, z_3, z_4) \in B_\delta(v)$, $|z_i| < K$ for each $i = 1, \dots 4$. Now, letting $x = (1, \frac{d}{a}, 0)$ and $y = (a, b, c - \frac{db}{a})$, we have that $A(x, y) = (a, b, c, d)$. Thus, we see that x and y can be chosen with coordinates bounded by $K + \frac{K^2}{\delta}$. Hence, $B_\delta(v)$ is the image of a bounded subset of $\mathbb{C}^3 \times \mathbb{C}^3$, and so v is not in $\mathcal{R}_H$. The case, where $b_0 \neq 0$ is handled similarly, so that $\mathcal{R}_H \subseteq \{(a, b, c, d) : a = b = 0\}$.

Now fix a point of the form $v = (0, 0, c, d)$. We will consider a sequence of points $v_n = (\frac{1}{2^n}, \frac{1}{2^n}, c, d_n)$, where $d_n \to d$ (so that $v_n \to v$) and so that pre-images of the v_n are forced to be large in norm as n grows large. If $A(x, y) = v_n$, then we get the relations: $y_1 = y_2 = \frac{1}{2^n x_1}$, $y_3 = \frac{c - d_n}{x_1}$, and $x_2 + x_3 = 2^n d_n x_1$. To show that $||(x, y)||$ is large, we show that the product $(x_1 + x_2 + x_3)(y_1 + y_2 + y_3)$ is large. Using the above relations, we have:

$$\begin{aligned}(x_1 + x_2 + x_3)(y_1 + y_2 + y_3) &= (x_1 + 2^n d_n x_1)(\frac{2}{2^n x_1} + \frac{c - d_n}{x_1}) \\ &= (1 + 2^n d_n)(\frac{1}{2^{n-1}} + c - d_n).\end{aligned}$$

We claim that the above product grows unboundedly with n as long as the d_n are chosen correctly. We consider the following three cases:

Case 1. $(d = 0)$ If $d = 0$, then we may assume that $c \neq 0$, since we know $\mathbf{0}$ to be repelling. We then choose $d_n = \frac{1}{n}$, so that $(\frac{1}{2^{n-1}} + c - d_n) \to c$ as $n \to \infty$, while $(1 + \frac{2^n}{n}) \to \infty$.

Case 2. $(d \neq 0$ and $c \neq d)$ If $d \neq 0$ and $c \neq d$, then we choose $d_n = d$ for all n, and the argument is similar to Case 1.

Case 3. $(d \neq 0$ and $c = d)$ If $d \neq 0$ and $c = d$, we choose $d_n = d - \frac{1}{n}$. Then the product in question becomes $(1 + 2^n(d - \frac{1}{n}))(\frac{1}{2^{n-1}} + \frac{1}{n}) = \frac{1}{2^{n-1}} + \frac{1}{n} + 2(d - \frac{1}{n}) + \frac{2^n(d-\frac{1}{n})}{n}$ which clearly grows unboundedly with n. $\square$

4. An Open Mapping Theorem

We now give an Open Mapping Theorem for n-linear mappings, where the range is of small dimension. First, we note that Rudin's question was whether every surjective bilinear map is open at the origin. In fact, this was the most one could hope for. Unlike linear maps, open and open at the origin are not equivalent for multilinear maps. For example [2], if E is any Banach space of dimension 2 or greater, the mapping $f : \mathbb{K} \times \mathbb{E} \longrightarrow \mathbb{E}$ by $f(\alpha, v) = \alpha v$ is open at the origin, but not open at any point of the form $(0, v)$, where $v \neq 0$.

We also note that by Proposition 7, for n-homogeneous maps, *being open at the origin is equivalent to the set of repelling points being empty.*

Before stating the theorem, we need two lemmas. The proofs of the lemmas we reserve until the next section. In what follows, for an n-linear mapping f, and for $1 \leq k \leq n-1$, by a k-linear restriction of f we mean the mapping obtained from f by fixing the values of $n-k$ coordinates. For example, if $f : E_1 \times E_2 \times E_3 \longrightarrow F$ is trilinear and $z \in E_3$, we designate by $f(\cdot, \cdot, z)$ the bilinear restriction of f given by fixing the value from E_3 as z. We also mention that from here on, we assume all mappings to be continuous.

Lemma 10. *Let $A : E_1 \times \ldots \times E_n \longrightarrow F$ be a continuous n-linear mapping. For a closed subspace $W \subset F$, let $U = \{x \in E_1 \times \ldots \times E_n : A(x) \notin W\}$. Then either U is empty or it is an open dense subset of E.*

Lemma 11. *Let $A, B : E_1 \times \ldots \times E_n \longrightarrow F$ be two linearly independent n-linear mappings so that for all $1 \leq k < n$, any k-linear restrictions of A and B (by fixing the same coordinates by the same values) are linearly dependent as mappings. Then either there exists $v \in E_1 \times \ldots \times E_n$ with $A(v)$ and $B(v)$ linearly independent or A and B have the same one dimensional range.*

$*$ The proof below uses the elementary fact that an n-linear mapping into a two dimensional space is surjective if and only if its two "coordinate" forms are linearly independent. Note also that, by the definition of repelling points, if a mapping has any restriction (of any kind) which is onto and has no repelling points, then the mapping itself has no repelling points.

Theorem 12. *Let $f : E_1 \times \ldots \times E_n \longrightarrow G$ be a surjective n-linear operator and G have dimension at most 2. Then f is open at the origin.*

Proof. First, if G is one dimensional, then it is easily seen that the function m_f is lower semi-continuous. If f were not open at the origin, Proposition 8 would imply that $\mathcal{R}_f = G$, contradicting Corollary 4. Thus, we assume that G is two dimensional, and say that $f = (A, B)$, where A and B are both n-linear forms.

We now show that f has a linear restriction which is onto. This will complete the proof by the Open Mapping Theorem (and the comment $*$ above). We proceed by induction on n, the case $n = 1$ being trivial, as then f is linear.

Since f is onto, A and B must be linearly independent. Because the mapping

$$h \longrightarrow \hat{h} : \mathcal{L}^n(E_1, \ldots E_n) \longrightarrow \mathcal{L}^{(n-1)}(E_1, \ldots, E_{n-1}, E_n^*),$$

where $\hat{h}(e_1, \ldots, e_{n-1})(x) = h(e_1, \ldots, e_{n-1}, x)$ is a linear injection, $\hat{A}$ and $\hat{B}$ are linearly independent. Now, suppose there is a $v = (v_1, \ldots, v_{n-1}) \in E_1 \times \ldots \times E_{n-1}$ so that $\hat{A}(v)$ and $\hat{B}(v)$ are independent. Consider the map $\bar{f} : \{v_1\} \times \ldots \{v_{n-1}\} \times E_n \to G$ by $\bar{f}(v_1, \ldots, v_{n-1}, x) = f(v_1, \ldots, v_{n-1}, x)$. Then $\bar{f} = (\hat{A}(v), \hat{B}(v))$ is a linear restriction of f which is onto G. Thus, we assume that $\{\hat{A}(v), \hat{B}(v)\}$ is dependent for all $v \in E_1 \times \ldots \times E_{n-1}$. Now, if some k-linear restrictions (by fixing the same coordinates by the same values), of $\hat{A}$ and $\hat{B}$ are independent, then the corresponding restrictions $\tilde{A}$ and $\tilde{B}$ would be independent, and hence the corresponding restriction $\tilde{f} = (\tilde{A}, \tilde{B})$ would be onto G. The inductive hypothesis would then yield the result. Hence, by Lemma 11, we are reduced to assuming that $\hat{A}$ and $\hat{B}$ have the same one dimensional range, say $[e^*]$. For $w \in E_1 \times \ldots \times E_{n-1}$, let $\hat{A}(w) = \alpha(w)e^*$ and $\hat{B}(w) = \beta(w)e^*$. The mappings α and β are both $(n-1)-$linear forms which, since $\hat{A}$ and $\hat{B}$ are independent, must themselves be independent. Hence, the mapping (α, β) is surjective. Let $e_n \in E_n$ be such that $e^*(e_n) = 1$. Then (α, β) is the $(n-1)$-linear restriction of f to $E_1 \times \ldots \times E_{n-1} \times \{e_n\}$, which, by induction, has a linear restriction which is onto G. □

Finally, from the proof of Theorem 12, we see that a continuous n-linear operator onto a space of dimension ≤ 2 must have a linear restriction which is onto. The Bartyle-Graves Theorem guarantees that every linear surjection has a continuous right inverse. Hence, we have the following corollary.

Corollary 13. *Let $f : E_1 \times \ldots \times E_n \longrightarrow F$ be a continuous surjective n-linear mapping and F have dimension at most 2. Then f has a continuous right inverse.*

5. Proofs of Lemmas

We now give the proofs of Lemma 10 and Lemma 11.

Proof of Lemma 10. Since U is the complement of the closed set $A^{-1}(W)$, it is open. Now, we proceed by induction on n, noting that if $n = 1$, A is linear and the assertion is easily seen to be true. If U is not dense in $E_1 \times \ldots \times E_n$, then there is some open ball $B_\epsilon(v)$ in $E_1 \times \ldots \times E_n$ with $A(B_\epsilon(v)) \subseteq W$. If $v = (v_1, \ldots, v_n)$, suppose that for some j, the $(n-1)$-linear restriction $A_j = A(\cdot, \ldots, v_j, \ldots, \cdot)$ of A does not have range contained in W. Then by the inductive hypothesis, there is a dense set of points in $E_1 \times \ldots \times \{v_j\} \times \ldots \times E_n$ which are mapped into U. This, however, would contradict that $A(B_\epsilon(v)) \subseteq W$. Hence, we assume that each A_j has range contained in W. Now, any vector in the open ball $B_\epsilon(0)$ in $E_1 \times \ldots \times E_n$ can be expressed in the form $(w_1 - v_1, \ldots, w_n - v_n)$, where $w = (w_1, \ldots, w_n) \in B_\epsilon(v)$. Moreover, $A(w_1 - v_1, \ldots, w_n - v_n) = A(w_1, \ldots, w_n) + Q$, where Q is a sum of images under the mappings $A_1, \ldots, A_n$. Thus, $Q \in W$, and we are assuming $A(w_1, \ldots, w_n) \in W$, so that $A(w_1 - v_1, \ldots, w_n - v_n) \in W$. Using the n-homogeneity of A now gives that the range of A is contained in W and thus U is empty. □

Proof of Lemma 11. Let us first assume that $n \geq 2$. Assuming the hypothesis on A and B, further assume that $\{A(v), B(v)\}$ is dependent for all $v \in E_1 \times \ldots \times E_n$. Now, if A has one dimensional range, our assumptions clearly imply that B has the same one dimensional range. Thus, we assume this is not the case. Let $v = (v_1, \ldots, v_n)$ be any vector with $A(v) = f \neq 0$ and $B(v) \neq 0$ (the set of such v is open and dense). Since A does not have a one dimensional range, the set $U = \{x : A(x) \notin [f]\}$ is not empty, and hence is an open dense set by Lemma 10. Let $w = (w_1, \ldots, w_n)$ be in U. Say $B(v) = \alpha A(v)$ and $B(w) = \beta A(w)$. Let $i \neq j$ be any two indices in $\{1, \ldots, n\}$. By the assumption on the dependence of restrictions, it follows that $B(\cdot, \ldots, v_i, \ldots, \cdot) = \alpha A(\cdot, \ldots, v_i, \ldots, \cdot)$ and $B(\cdot, \ldots, w_j, \ldots, \cdot) = \beta A(\cdot, \ldots, w_j, \ldots, \cdot)$. In particular,

$$B(\cdot, \ldots, v_i, \ldots, w_j, \ldots, \cdot) = \alpha A(\cdot, \ldots, v_i, \ldots, w_j, \ldots, \cdot) = \beta A(\cdot, \ldots, v_i, \ldots, w_j, \ldots, \cdot)$$

(note here that if $n = 2$, $A(\cdot,\ldots,v_i,\ldots,w_j,\ldots,\cdot)$ and $B(\cdot,\ldots,v_i,\ldots,w_j,\ldots,\cdot)$ are constants, not $(n-2)$-linear mappings). Thus, either $\alpha = \beta$ or $A(\cdot,\ldots,v_i,\ldots,w_j,\ldots,\cdot) = B(\cdot,\ldots,v_i,\ldots,w_j,\ldots,\cdot) = 0$. Similarly, either $\alpha = \beta$ or $A(\cdot,\ldots,w_i,\ldots,v_j,\ldots,\cdot) = B(\cdot,\ldots,w_i,\ldots,v_j,\ldots,\cdot) = 0$. If $\alpha \neq \beta$, then $A(w_1+v_1, w_2+v_2,\ldots,w_n+v_n) = A(w)+A(v)$ and $B(w_1+v_1, w_2+v_2,\ldots,w_n+v_n) = B(w)+B(v) = \alpha A(w)+\beta A(v)$ are independent, contradicting our assumption. Thus, it must be that $\alpha = \beta$. As w was arbitrary, we see that $A = \alpha B$ on U. However, as U is dense in $E_1 \times \ldots \times E_n$, this implies that $A = \alpha B$ on $E_1 \times \ldots \times E_n$, contradicting that A and B are independent. Thus, either $\{A(v), B(v)\}$ is independent for some v or A and B have the same one dimensional range. Finally, if $n = 1$, then A and B are linear and the proof is a simple version of the above. □

6. Summary

We believe that the notion of repelling points is an interesting topic in the study of general operators, and may be especially useful in investigating the openness properties of multilinear operators. That surjective multilinear operators have such sets of points is in sharp contrast with their linear counterparts. As mentioned previously, for n-linear mappings, being open at the origin is equivalent to the set of repelling points being empty. We note here that this property is different than that of having a continuous right inverse [3].

Along with other fundamental questions, the topological size (in the general case) and the algebraic structure of these sets remains unknown. The Horowitz example [4], with 4-dimensional range, and Theorem 12 , together present a very interesting open problem. Namely, is there an Open Mapping Theorem for multilinear operators when the range is 3-dimensional. In other words, 3-dimensional range spaces are the only ones unaccounted for in the original question of Rudin. We make the following conjecture.

Conjecture. *A continuous n-linear mapping which is onto a space of dimension 3 is open at the origin.*

References

[1] P.J. Cohen, A counterexample to the Closed Graph Theorem for bilinear maps, *J. Functional Analysis*, **16** (1974), 235-239.

[2] A. Defant, K. Floret, *Tensor Norms and Operator Ideals*, North-Holland, p. 11.

[3] C. Fernandez, A counterexample to the Bartle-Graves selection theorem for multilinear maps, *Proceedings of the AMS*, **126** (1998), 2687-2690.

[4] C. Horowitz, An elmementary counterexample to the open mapping principle for bilinear maps, *Proceedings Of The American Mathematical Society*, **53**, No. 2 (1975), 293-294.

[5] W. Rudin, *Function Theory in Polydiscs*, Benjamin, New York (1969).