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ON UNIFORM ASYMPTOTIC STABILITY OF LINEAR IMPULSIVE DIFFERENTIAL SYSTEMS WITH DELAY

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Abstract: In this paper, it is shown that if a linear impulsive differential system with delay satisfies Perron condition then its trivial solution is uniformly asymptotically stable.

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1. Introduction

It is well known in the theory of differential equations (see [5, p. 120]) that if for every continuous function $f(t)$ bounded on $[0, \infty)$, the solution of the system

$$x'(t) = A(t)x(t) + f(t),$$

satisfying $x(0) = 0$ is bounded on $[0, \infty)$, then the trivial solution of the corresponding homogeneous system

$$x'(t) = A(t)x(t)$$

is uniformly asymptotically stable. This result is referred as Perron Theorem [11]. Later, Perron Theorem has been extended to delay differential systems [5, page 371]. Indeed, it was shown that if for every continuous function $f(t)$ bounded on $[0, \infty)$, the solution of the system

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$$x'(t) = A(t)x(t) + B(t)x(t-\tau) + f(t),\ t > 0$$

satisfying $x(t) = 0$ for $t \in [-\tau, 0]$ is bounded on $[0, \infty)$, then the trivial solution of the system

$$x'(t) = A(t)x(t) + B(t)x(t-\tau),$$

is uniformly asymptotically stable. For more related matrials, we refer the reader to the papers [1, 15, 19].

Our aim in this article is to carry out the above result to linear impulsive differential systems with delay of the form

$$\begin{aligned} x'(t) &= A(t)x(t) + B(t)x(t-\tau) + f(t), \quad t \neq \theta_k, \\ \Delta x(\theta_k) &= A_k x(\theta_k) + B_k x(\theta_k - \tau) + f_k, \quad k \in \{1, 2, 3, \ldots\}, \end{aligned} \tag{1}$$

where $\Delta x(t) := x(t^+) - x(t^-)$ and $x(t^+) := \lim_{r \to t^+} x(r)$, $x(t^-) := \lim_{r \to t^-} x(r)$. It is to be noted that the impulsive conditions in (1) also involve delay. Therefore, the solution value at any discontinuity point will also depend on the past data.

Impulsive differential systems with delay have attracted the interests of many researchers in the last two decades [3, 4, 8, 9, 13] since they provide a natural description of the motion of several real world processes which, on one hand, depends on the processes history that often turns out to be the cause of phenomena substantially affecting the motion and, on other hand, is subject to short time perturbations whose duration is almost negligible. Such processes are often investigated in various fields of science and technology, such as physics, population dynamics, ecology, biological systems, optimal control, etc., (for more details we refer the reader to the papers [10, 12, 14, 16, 17, 18, 20] and the list of reference quoted therein).

2. Preliminaries

Before proceeding, we shall set forth some notations and preliminaries that will be used throughout the rest of the paper. Let $\mathbb{R}^n$ be the space of n-dimensional column vectors $x = \mathrm{col}(x_1, \ldots, x_n)$ with the norm $\|x\| = \max_{1 \le i \le n} |x_i|$. By the same symbol $\|\cdot\|$ we denote the corresponding matrix norm in the space $\mathbb{R}^{n\times n}$. Let $\mathbb{R}^+ := [0, \infty)$ and $\mathbb{N} := \{1, 2, 3, \ldots\}$, we denote by $CB(\mathbb{R}^+, \mathbb{R}^n)$ the set of all continuous and bounded functions $\psi : \mathbb{R}^+ \to \mathbb{R}^n$ and $S(\mathbb{N}, \mathbb{R}^n)$ the set of all bounded sequences $\nu_i : \mathbb{N} \to \mathbb{R}^n$. For system (1) we assume the following conditions:

(H1) $\{\theta_k\}$ is a real sequence such that $\theta_k < \theta_{k+1}$ and $\lim_{k\to\infty}\theta_k = \infty$, $k \in \mathbb{N}$.

(H2) $A, B \in CB(\mathbb{R}^+, \mathbb{R}^{n\times n})$ and $\tau > 0$;

(H3) $A_k, B_k \in S(\mathbb{N}, \mathbb{R}^{n\times n})$, $k \in \mathbb{N}$;

(H4) $f(t) \in CB(\mathbb{R}^+, \mathbb{R}^n)$ and $f_k \in S(\mathbb{N}, \mathbb{R}^n)$.

By a solution of (1) on an interval J, we mean a function x defined on J such that x is continuous on J except possibly at $\theta_k \in J$ for $k \in \mathbb{N}$, where $x(\theta_k^+)$ and $x(\theta_k^-)$ exist, $x(\theta_k^-) := x(\theta_k)$, and that x satisfies (1) on J. Let $PLC([-\tau, 0], \mathbb{R}^n)$ denote the set of piecewise left continuous functions $\phi : [-\tau, 0] \to \mathbb{R}^n$ having a finite number of discontinuity points of the first kind. Then the initial value problem of (1) is to find a solution $x(t)$ of (1) such that

$$x(t+\sigma) = \phi(t), \quad t \in [-\tau, 0], \tag{2}$$

where

(H5) $\phi \in PLC([-\tau, 0], \mathbb{R}^n)$ and $\sigma \geq 0$ is given.

Under the above conditions (even less), one can easily show that the initial value problem (1), (2) has a unique solution which belongs to the set $PLC([\sigma - \tau, \infty), \mathbb{R}^n)$. This follows rather easily by the method of steps [6].

Consider the homogeneous system corresponding to (1)

$$\begin{aligned} x'(t) &= A(t)x(t) + B(t)x(t-\tau), \quad t \neq \theta_k, \\ \Delta x(\theta_k) &= A_k x(\theta_k) + B_k x(\theta_k - \tau), \quad k \in \mathbb{N}. \end{aligned} \tag{3}$$

Let $\tau_k = \theta_k - \tau$, $\{\delta_k\} = \{\theta_k\} \cup \{\tau_k\}$ and set

$$\alpha_k = \begin{cases} A_j, & \text{if} \quad \delta_k = \theta_j \\ 0, & \text{if} \quad \delta_k \neq \theta_j \end{cases}, \quad \text{for every } j \in \mathbb{N},$$

$$\beta_k = \begin{cases} B_j, & \text{if} \quad \delta_k = \theta_j \\ 0, & \text{if} \quad \delta_k \neq \theta_j \end{cases}, \quad \text{for every } j \in \mathbb{N},$$

and

$$F_k = \begin{cases} f_j, & \text{if} \quad \delta_k = \theta_j \\ 0, & \text{if} \quad \delta_k \neq \theta_j \end{cases}, \quad \text{for every } j \in \mathbb{N}.$$

Then (1) and (3) are equivalent to the systems

$$\begin{aligned} x'(t) &= A(t)x(t) + B(t)x(t-\tau) + f(t), \quad t \neq \delta_k, \\ \Delta x(\delta_k) &= \alpha_k x(\delta_k) + \beta_k x(\delta_k - \tau) + F_k, \quad k \in \mathbb{N}, \end{aligned} \tag{4}$$

and

$$\begin{aligned} x'(t) &= A(t)x(t) + B(t)x(t-\tau), \quad t \neq \delta_k, \\ \Delta x(\delta_k) &= \alpha_k x(\delta_k) + \beta_k x(\delta_k - \tau), \quad k \in \mathbb{N}, \end{aligned} \tag{5}$$

respectively.

Define the matrices

$$C_k = \begin{cases} B_j, & \text{if} \quad \delta_k + \tau = \theta_j \\ 0, & \text{if} \quad \delta_k + \tau \neq \theta_j \end{cases}, \quad \text{for every } j \in \mathbb{N},$$

and consider the system

$$\begin{aligned} y'(t) &= -A^T(t)y(t) - B^T(t+\tau)y(t+\tau), \quad t \neq \delta_k, \\ \Delta y(\delta_k) &= -\alpha_k^T y(\delta_k^+) - C_k^T y(\delta_k^+ + \tau), \quad k \in \mathbb{N}, \end{aligned} \tag{6}$$

where by $"T"$ we mean the transposition. Then, by employing the function

$$\begin{aligned} < y(t), x(t) >= y^T(t)x(t) \quad &+ \int_t^{t+\tau} y^T(s)B(s)x(s-\tau)\mathrm{d}s \\ &+ \sum_{t \leq \theta_j < t+\tau} y^T(\theta_j^+)B_j x(\theta_j - \tau), \end{aligned} \tag{7}$$

where $x(t)$ is a solution of (3) (or (5)) and $y(t)$ is a solution of (6), it has been shown that

$$< y(t), x(t) >= \text{constant}. \tag{8}$$

The proof of relation (8) can be found in [2]. In virtue of (8), one may introduce the following fundamental definition.

Definition 1. System (6) is said to be an adjoint of (3) (or (5)) with respect to (7).

It is easy to verify also that the adjoint of (6) is (3) (or (5)), i.e. the systems are mutually adjoint of each other.

Function (7) resembles the one used by Halanay in [5] for differential systems with delay. Indeed, in case of absence of impulses in the considered systems, (7) reduces to

$$< y(t), x(t) >= y^T(t)x(t) + \int_t^{t+\tau} y^T(s)B(s)x(s-\tau)\mathrm{d}s.$$

Thus, we may say that function (7) generalizes a fundamental result to impulsive differential systems with delay.

Definition 2. A matrix solution $X(t,\alpha)$ of (3) satisfying $X(\alpha,\alpha)=I$ and $X(t,\alpha)=0$ for $t<\alpha$ is called a fundamental matrix of (3).

Definition 3. A matrix solution $Y(t,\alpha)$ of (6) satisfying $Y(\alpha,\alpha)=I$ and $Y(t,\alpha)=0$ for $t>\alpha$ is said to be a fundamental matrix of (6).

3. Some Essential Lemmas

In this section, we shall state and prove some essential lemmas that will be used in proving the main results. The first two lemmas are results of Bainov and Simeonov [2]. We shall present these two results without proofs.

Lemma 4. (see [2]) *Let $X(t,\alpha)$ be a fundamental matrix of (3) and $\sigma\geq 0$ be a real number. If $x(t)$ is a solution of (1), then*

$$\begin{aligned} x(t) &= X(t,\sigma)x(\sigma)+\int_{\sigma}^{\sigma+\tau}X(t,\alpha)B(\alpha)x(\alpha-\tau)\mathrm{d}\alpha+\int_{\sigma}^{t}X(t,\alpha)f(\alpha)\mathrm{d}\alpha \\ &+ \sum_{\sigma\leq\theta_j<\sigma+\tau}X(t,\theta_j^+)B_jx(\theta_j-\tau)+\sum_{\sigma\leq\theta_j<t}X(t,\theta_j^+)f_j, \quad t\geq\sigma. \end{aligned} \tag{9}$$

Lemma 5. (see [2]) *Let $X(t,\alpha)$ be a fundamental matrix of (3) and $Y(t,\alpha)$ be a fundamental matrix of (6). Then*

$$X(t,s)=Y^T(s,t). \tag{10}$$

In what follows, we are concerned with the boundedness of the fundamental matrix $X(t,\alpha)$ of system (3).

Definition 6. System (3) is said to verify Perron condition if for every function $f(t)\in CB(\mathbb{R}^+,\mathbb{R}^n)$ and every sequence $f_k\in S(\mathbb{N},\mathbb{R}^n)$, the solution of (1) with $x(t)=0$ for $t\in[-\tau,0]$, is bounded for $t\in\mathbb{R}^+$.

Lemma 7. *If system (3) verifies Perron condition, then there exists a constant c such that*

$$\int_0^t\|X(t,\alpha)\|\mathrm{d}\alpha+\sum_{0\leq\theta_j<t}\|X(t,\theta_j^+)\|<c \quad \text{for} \quad t\geq 0. \tag{11}$$

Proof. Define the space $\Pi=CB\times S$ whose elements are represented by the pair (f,ϵ), $\epsilon=\{f_k\}, f_k\in\mathbb{R}^n, k\in\mathbb{N}$, supplied by the norm $\|(f,\epsilon)\|=\sup_{t\in\mathbb{R}^+}\|f(t)\|+\sup_{k\in\mathbb{N}}\|f_k\|$. Consider the operator $U:\Pi\to PLC$ defined by

$$U(f,f_k)=\int_0^tX(t,\alpha)f(\alpha)\mathrm{d}\alpha+\sum_{0\leq\theta_j<t}X(t,\theta_j^+)f_k.$$

Clearly, Π is a Banach space and PLC is a normed space. Then, by applying Banach Steinhaus Theorem [7], the proof can be completed by following the same arguments used in [5]. □

Lemma 8. *Let $X(t,\alpha)$ be the fundamental matrix of (3). If there is a constant $c>0$ such that*

$$\int_0^t \|X(t,\alpha)\|\mathrm{d}\alpha + \sum_{0\le\theta_j<t}\|X(t,\theta_j^+)\| < c \qquad \textit{for} \quad t\ge 0,$$

then

$$\|X(t,\alpha)\| < M \quad \textit{for} \quad t\ge\alpha\ge 0.$$

Proof. Consider system (6) which verifies $Y^T(\alpha,t)$. Integrating both sides from σ to t, we obtain

$$\begin{aligned} Y^T(\sigma,t) &= I + \int_\sigma^t Y^T(\alpha,t)A(\alpha)\mathrm{d}\alpha + \int_\sigma^t Y^T(\alpha+\tau,t)B(\alpha+\tau)\mathrm{d}\alpha \\ &- \sum_{\sigma\le\delta_k<t}\Delta Y^T(\delta_k,t). \end{aligned} \tag{12}$$

However,

$$\sum_{\sigma\le\delta_k<t}\Delta Y^T(\delta_k,t) = -\sum_{\sigma\le\theta_j<t} Y^T(\theta_j^+,t)A_j - \sum_{\sigma+\tau\le\theta_j<t} Y^T(\theta_j^+,t)B_j.$$

In view of (12), it follows that

$$\|Y^T(\sigma,t)\| \le 1 + 2\gamma\int_0^t \|Y^T(\alpha,t)\|\mathrm{d}\alpha + 2\rho\sum_{0\le\theta_j<t}\|Y^T(\theta_j^+,t)\|,$$

where

$$\gamma = \max\{\sup_\alpha \|A(\alpha)\|, \sup_\alpha\|B(\alpha)\|\},$$

and

$$\rho = \max\{\sup_j \|A_j\|, \sup_j\|B_j\|\}.$$

Using relation (10) and the hypothesis of the lemma, we finish the proof. □

4. Main Results

Let $\|\phi\|_0 = \sup_{t\in[-\tau,0]} \|\phi(t)\|$.

Theorem 9. *If system (3) verifies Perron condition then its trivial solution is uniformly stable.*

Proof. Let $x(t;\sigma,\phi)$ denote the solution of (3) satisfying (2). From (9), we have

$$\begin{aligned} x(t;\sigma,\phi) &= X(t,\sigma)\phi(\sigma) + \int_{-\tau}^{0} X(t,\alpha+\tau+\sigma)B(\alpha+\tau+\sigma)\phi(\alpha+\sigma)\mathrm{d}\alpha \\ &+ \sum_{-\tau\le\theta_j<0} X(t,\theta_j^{+}+\tau+\sigma)B_{j+\tau+\sigma}\phi(\theta_j+\sigma). \end{aligned}$$

Perron condition implies that inequality (11) holds and hence by Lemma 8 we conclude that $\|X(t,r)\| < M$. Therefore

$$\|x(t;\sigma,\phi)\| \le M_1\|\phi\|_0,$$

where

$$M_1 = M(1+\tau\gamma+\tau\rho).$$

Thus, the zero solution of (3) is uniformly stable. □

Theorem 10. *If system (3) verifies Perron condition then its trivial solution is uniformly asymptotically stable.*

Proof. In view of Theorem 9, it remains to prove that

$$\lim_{t\to\infty} x(t;\sigma,\phi) = 0, \tag{13}$$

uniformly with respect to σ and ϕ. For our purpose, let $\mu \ge \sigma$, it is clear that $x(t) = x(t;\sigma,\phi)$ satisfies

$$\begin{aligned} x(t;\sigma,\phi) &= X(t,\mu)x(\mu;\sigma,\phi) + \int_{\mu}^{\mu+\tau} X(t,\alpha)B(\alpha)x(\alpha-\tau;\sigma,\phi)\mathrm{d}\alpha \\ &+ \sum_{\mu\le\theta_j<\mu+\tau} X(t,\theta_j^{+})B_j x(\theta_j-\tau;\sigma,\phi). \end{aligned}$$

Integrating both sides and changing the order of integrations and the order of sum and integral, we have

$$(t-\sigma)x(t;\sigma,\phi) = \int_{\sigma}^{\sigma+\tau}\int_{\sigma}^{\alpha} X(t,\alpha)B(\alpha)x(\alpha-\tau;\sigma,\phi)\mathrm{d}\mu\mathrm{d}\alpha$$

$$
\begin{aligned}
&+ \int_{\sigma+\tau}^{t}\int_{\alpha-\tau}^{\alpha} X(t,\alpha)B(\alpha)x(\alpha-\tau;\sigma,\phi)\mathrm{d}\mu\mathrm{d}\alpha \\
&+ \int_{\sigma}^{t} X(t,\mu)x(\mu;\sigma,\phi)\mathrm{d}\mu \\
&+ \sum_{\sigma\le\theta_j<\sigma+\tau}\int_{\sigma}^{\theta_j} X(t,\theta_j^+)B_j x(\theta_j-\tau;\sigma,\phi)\mathrm{d}\mu \\
&+ \sum_{\sigma+\tau\le\theta_j<t}\int_{\theta_j-\tau}^{\theta_j} X(t,\theta_j^+)B_j x(\theta_j-\tau;\sigma,\phi)\mathrm{d}\mu.
\end{aligned}
$$

The above expression is a consequence of the facts that $X(t,s)=0$ for $t<s$ and $X(t,\theta_r)=0$ for $t<\theta_r$.

It follows that

$$
\begin{aligned}
(t-\sigma)\|x(t;\sigma,\phi)\| &\le \gamma\tau^2 MM_1\|\phi\|_0 + \rho\tau^2 MM_1\|\phi\|_0 \\
&\quad + M_1\max\{\tau\gamma,\tau\rho,1\}\|\phi\|_0\Big[\int_0^t \|X(t,s)\|_0\mathrm{d}s + \sum_{0\le\theta_r<t}\|X(t,\theta_r^+)\|\Big].
\end{aligned}
$$

In view of Lemma 7, the right side of the above inequality is bounded. Dividing both sides by $t-\sigma$, we obtain

$$
\|x(t;\sigma,\phi)\| \le \frac{M_2}{t-\sigma}\|\phi\|_0, \tag{14}
$$

where M_2 is chosen as

$$
M_2 < MM_1(\gamma\tau^2+\rho\tau^2) + M_1\max\{\tau\gamma,\tau\rho,1\}c.
$$

Obviously, (13) follows from (14). The proof is complete. □

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