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SOME CLASSES OF COMPLETELY MONOTONIC FUNCTIONS INVOLVING THE GAMMA FUNCTION

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Abstract: In this article, a necessary-sufficient condition is given for a class of functions involving the gamma function to be completely monotonic. A sufficient condition is provided for a class of functions to be logarithmically completely monotonic.

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1. Introduction and Main Results

Throughout the paper, $\mathcal{N}$ denotes the set of all positive integers, $\mathcal{N}_0 := \mathcal{N} \cup \{0\}$, $C(I)$ is the space of all continuous functions on the interval I and I^o denotes the interior of the interval I.

The Euler gamma function is defined and denoted for $\mathrm{Re} z > 0$ by

$$\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} dt. \tag{1}$$

The logarithmic derivative of $\Gamma(z)$, denoted by $\psi(z) := \Gamma'(z)/\Gamma(z)$, is called the psi or digamma function, and $\psi^{(k)}$, for $k \in \mathcal{N}$, are called the polygamma functions.

Recall [14] that a function f is said to be completely monotonic on an interval I, if $f \in C(I)$, has derivatives of all orders on I^o and for all $x \in I^o, n \in \mathcal{N}_0$

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$$(-1)^n f^{(n)}(x) \geq 0. \tag{2}$$

The completely monotonic functions have remarkable applications in probability and statistics [3],[8],[13], physics [2].

Recall also [10, 12] that a positive function f is said to be logarithmically completely monotonic on an interval I if $f \in C(I)$, has derivatives of all orders on I^o and

$$(-1)^n [\ln f(x)]^{(n)} \geq 0 \tag{3}$$

for all $x \in I^o$, and $n \in \mathcal{N}$.

Horn [6] proved, in terms of logarithmically completely monotonic functions, that f is logarithmically completely monotonic on $(0, \infty)$ if and only if $f \not\equiv 0$ and $\sqrt[n]{f}$ is completely monotonic on $(0, \infty)$, $n \in \mathcal{N}$. A function f such that $\sqrt[n]{f}$ is completely monotonic on $(0, \infty)$ for all $n \in \mathcal{N}$ is called infinitely divisible completely monotonic in [6]. From Horn's result, a logarithmically completely monotonic function on $(0, \infty)$ is completely monotonic on $(0, \infty)$.

In [1, 7, 10, 11, 12], functions involving the gamma, psi or polygamma functions were proved to be (logarithmically) completely monotonic. This article is a further contribution to this subject. Our main results are as follows.

Theorem 1. *Let $\alpha \neq 0, \beta$ be real parameters. Then the function*

$$g_{\alpha,\beta}(x) := \left[\frac{e^x \Gamma(x+1)}{(x+\beta)^{x+\beta}}\right]^\alpha, \quad x \in I_\beta, \tag{4}$$

where $I_\beta := (\max(0, -\beta), \infty)$, is logarithmically completely monotonic if either $\alpha > 0$ and $\beta \geq 1$ or $\alpha < 0$ and $\beta \leq 1/2$.

Theorem 2. *Let $\beta \neq 0, \alpha$ be real parameters, then the function*

$$F_{\alpha,\beta}(x) := \beta \left[x - x \ln x + \ln \Gamma(x+1) - \alpha \ln x\right], \quad x \in (0, \infty), \tag{5}$$

is completely monotonic if and only if $\alpha = 1/2$ and $\beta > 0$.

2. Lemmas

We need the following lemmas to prove our results.

Lemma 1. (see [4, p. 884], [9, Chapter 1]) *For $n \in \mathcal{N}$, $x > 0$:*

$$i) \quad \psi^{(n)}(x) = (-1)^{n+1} \int_0^\infty \frac{t^n}{1 - e^{-t}} e^{-xt} dt. \tag{6}$$

$$ii) \quad \frac{1}{x^n} = \frac{1}{\Gamma(n)} \int_0^\infty t^{n-1} e^{-xt} dt. \tag{7}$$

Lemma 2. (see [9, Chapter 1]) *As* $x \to \infty$*:*

$$i) \quad \ln\Gamma(x) = \left(x - \frac{1}{2}\right)\ln x - x + \frac{\ln(2\pi)}{2} + O\left(\frac{1}{x}\right). \tag{8}$$

$$ii) \quad \psi(x) = \ln x - \frac{1}{2x} + O\left(\frac{1}{x^2}\right). \tag{9}$$

Lemma 3. *The function*

$$\varphi(t) := \frac{1}{t}\ln\frac{e^t - 1}{t}, \tag{10}$$

is strictly increasing from $(0, \infty)$ *onto* $(1/2, 1)$.

Theorem 2 of [1] can be restated as follows.

Lemma 4. *Let* $\beta \neq 0, \alpha$ *be real parameters. Then the function*

$$f_{\alpha,\beta}(x) := \left[\frac{e^x\Gamma(x+1)}{x^{x+\alpha}}\right]^\beta, \quad x \in (0, \infty), \tag{11}$$

is logarithmically completely monotonic if and only if either $\beta > 0$ *and* $\alpha \geq 1/2$ *or* $\beta < 0$ *and* $\alpha \leq 0$.

3. Proofs of the Main Results

Proof of Theorem 1. For $x \in I_\beta$, we have

$$\ln g_{\alpha,\beta}(x) = \alpha[x + \ln\Gamma(x+1) - (x+\beta)\ln(x+\beta)], \tag{12}$$

$$[\ln g_{\alpha,\beta}(x)]' = \alpha[\psi(x+1) - \ln(x+\beta)], \text{and} \tag{13}$$

$$[\ln g_{\alpha,\beta}(x)]'' = \alpha[\psi'(x+1) - 1/(x+\beta)]. \tag{14}$$

By Lemma 1

$$[\ln g_{\alpha,\beta}(x)]'' = \alpha\int_0^\infty \delta(t)e^{-xt}dt, \tag{15}$$

where

$$\delta(t) = \frac{t}{e^t - 1} - e^{-\beta t} = e^{-t\varphi(t)} - e^{-\beta t}, \tag{16}$$

and $\varphi(t)$ is the function (10). By Lemma 3, for $t \in (0, \infty)$,

$$\delta(t)\begin{cases} > 0, & \beta \geq 1; \\ < 0, & \beta \leq 1/2. \end{cases} \tag{17}$$

From (15) and (17), we have, for $k \geq 2$,

$$(-1)^k [\ln g_{\alpha,\beta}(x)]^{(k)} > 0, \quad x \in I_\beta \tag{18}$$

if either $\alpha > 0$ and $\beta \geq 1$ or $\alpha < 0$ and $\beta \leq 1/2$. Combining (13) and (9) gives

$$\lim_{x\to\infty} [\ln g_{\alpha,\beta}(x)]' = 0. \tag{19}$$

By (18), $[\ln g_{\alpha,\beta}(x)]'$ is strictly increasing on I_β. Hence

$$[\ln g_{\alpha,\beta}(x)]' < 0, \quad x \in I_\beta \tag{20}$$

if either $\alpha > 0$ and $\beta \geq 1$ or $\alpha < 0$ and $\beta \leq 1/2$. (18) together with (20) completes the proof. □

Proof of Theorem 2. First we note that

$$F_{\alpha,\beta}(x) = \ln f_{\alpha,\beta}(x), \quad x \in (0,\infty). \tag{21}$$

Here $f_{\alpha,\beta}$ is defined by (11). If $\alpha = 1/2, \beta > 0$, by Lemma 4, for $n \in \mathcal{N}$,

$$(-1)^n F^{(n)}_{\alpha,\beta}(x) \geq 0, \quad x \in (0,\infty). \tag{22}$$

So we only need to show that

$$F_{\alpha,\beta}(x) \geq 0, \quad x \in (0,\infty). \tag{23}$$

By (8), we can show that, if $\alpha = 1/2$,

$$\lim_{x\to\infty} F_{\alpha,\beta}(x) = \beta \ln \sqrt{2\pi}. \tag{24}$$

From (22), $F_{\alpha,\beta}(x)$ is decreasing on $(0,\infty)$. Hence $F_{\alpha,\beta}(x) \geq 0$, for $x > 0$ if $\beta > 0$ and $\alpha = 1/2$.

On the other hand, if $F_{\alpha,\beta}(x)$ is completely monotonic on $(0,\infty)$, then $F_{\alpha,\beta}(x) \geq 0$ and $f_{\alpha,\beta}(x)$ is logarithmically completely monotonic on $(0,\infty)$. By Lemma 4, we have

$$\beta > 0, \quad \alpha \geq 1/2\,, \tag{25}$$

or

$$\beta < 0, \quad \alpha \leq 0\,. \tag{26}$$

For $x > 1$, from $F_{\alpha,\beta}(x) \geq 0$, we get

$$\beta \left[\frac{x - x\ln x + \ln\Gamma(x+1)}{\ln x} - \alpha \right] \geq 0. \tag{27}$$

Then by using (8)

$$\lim_{x\to\infty} \frac{x - x\ln x + \ln\Gamma(x+1)}{\ln x} = \frac{1}{2}. \tag{28}$$

Hence

$$\beta(1/2 - \alpha) \geq 0. \tag{29}$$

Combining (25), (26) and (29) yields $\beta > 0$ and $\alpha = 1/2$.

The proof is complete. □

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