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WELL-POSEDNESS OF ABSTRACT HYPERBOLIC MODELS WITH TIME DEPENDENT DAMPING

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Abstract: In this paper we study the abstract hyperbolic model with time dependent damping given by

$$\langle u''(t), v\rangle_{V',V} + d(t; u'(t), v) + a(t; u(t), v) = \langle f(t), v\rangle_{V',V} ,$$

where $V \subset V_D \subset H \subset V_D' \subset V'$ are Hilbert spaces with continuous and dense injections, here H is identified with its dual and $\langle\cdot,\cdot\rangle$ denotes the various duality products. We show that this model allows a unique solution under natural conditions on the time-dependent sesquilinear forms $a(t;\cdot,\cdot) : V \times V \to C$ and $d(t;\cdot,\cdot) : V_D \times V_D \to C$ and that the solution depends continuously on the data of the problem. This ensures good convergence properties of approximating Galerkin schemes for the numerical solution of the problem. The problem above is the ultimative weak formulation of the "strong" problem

$$u''(t, x) + D(t, x)u'(t, x) + A(t, x)u(t, x) = f(t, x),$$

for $0 < t < T < \infty, x \in \Omega \subset R^n$, modeling very abstract differential operator problems including plate, beam and shell equations with numerous kinds of damping.

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1. Introduction

Let V and H be complex Hilbert spacesforming the "Gelfand triple" $V \subset H =$

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$H' \subset V'$, where we write V' for the dual of V, with duality product $\langle \cdot, \cdot \rangle_{V',V}$. The injections are assumed to be dense and contionuous and the spaces are assumed to be separable. Moreover, we assume that there exists a Hilbert space V_D (D for damping), such that $V \subset V_D \subset H = H' \subset V_D' \subset V'$, allowing for a wide class of damping models. Thus the duality products $\langle \cdot, \cdot \rangle_{V',V}$ and $\langle \cdot, \cdot \rangle_{V_D',V_D}$ are the natural extensions by continuity of the inner product $(\cdot, \cdot)$ in H to $V' \times V$, resp. $V_D' \times V_D$. The H-norm is denoted by $\| \cdot \|$ and the V and V_D-norms are denoted $\| \cdot \|_V$, resp. $\| \cdot \|_{V_D}$. We denote $\partial_t u$ by u', where $u = u(t)$ is considered as a function of time taking values in one of the occurring Hilbert spaces, that is, $u(t,x) = u(t)(x)$, where $u(t) \in H$, for example.

Our aim is now to study the problem

$$u''(t) + D(t)u'(t) + A(t)u(t) = f(t), \quad \text{in } V', \quad t \in (0,T); \tag{1}$$

with $u(0) = u^0 \in V$ and $u'(0) = u^1 \in H$ in its weak form, that is

$$\langle u''(t), v \rangle_{V',V} + \langle D(t)u'(t), v \rangle_{V_D',V_D} + \langle A(t)u(t), v \rangle_{V',V} = \langle f(t), v \rangle_{V',V}, \tag{2}$$

for all $v \in V$ and $t \in (0,T)$, with $u(0) = u^0 \in V$ and $u'(0) = u^1 \in H$.

We will call a function $u \in L^2(0,T;V)$, with $u' \in L^2(0,T;H)$ and $u'' \in L^2(0,T;V')$ a *weak solution* of the initial-value problem (1) if it solves the equation (2) or, equivalently, solves the equation (9) below, with $u(0) = u^0 \in V$ and $u'(0) = u^1 \in H$ given.

We will assume that the operators A and D are stemming from time-dependent sesquilinear forms a and d satisfying the following natural ellipticity, coercivity and differentiability-conditions:

- $|a(t;u,v)| \le c_1 \|u\|_V \|v\|_V, \quad u, v \in V$, where c_1 is independent of t.

We assume, further, that

- $a(t;u,v)$ for $u, v \in V$ fixed is *continuously differentiable* with respect to t for $t \in [0,T]$ (T finite) and

$$|a'(t;u,v)| \le c_2 \|u\|_V \|v\|_V, \quad \forall t \in [0,T], \tag{3}$$

  c_2 once again independent of t.

We also assume that the sesquilinear form $a(t;u,v)$ is V-*elliptic*, so that:

- There exist a constant $\alpha > 0$ such that

$$|a(t;u,u)| \ge \alpha \|u\|_V^2 \text{ for all } t \in [0,T], \text{ and for all } u \in V. \tag{4}$$

Finally, we assume hermitian symmetry, that is

- $a(t, u, v) = \overline{a(t, v, u)}$ for all $u, v \in V$.

For the sesquilinear form d we assume similarly

- $|d(t; u, v)| \le c_3 \|u\|_{V_D} \|v\|_{V_D}, \quad u, v \in V_D$, where c_3 is independent of t.

We assume, further, that

- $d(t; u, v)$ for $u, v \in V_D$ fixed is *continuously differentiable* with respect to t for $t \in [0, T]$ (T finite) and

$$|d'(t; u, v)| \le c_4 \|u\|_{V_D} \|v\|_{V_D}, \quad \forall t \in [0, T], \tag{5}$$

  c_4 once again independent of t.

Then $t \to d(t; u, v)$ and $t \to a(t; u, v)$ are $C^1([0, T])$ for all $u, v \in V_D$, resp. $u, v \in V$, which implies that $d(t; u, v)$ and $a(t; u, v)$ are sufficiently well-behaved in order to have the situation for (2) to be the weak form of (1). We also assume that the sesquilinear form $d(t; u, v)$ is V_D*-coercive*, so that:

- There exist constants λ_d and $\alpha_d > 0$, such that

$$\operatorname{Re} d(t; u, u) + \lambda_d \|u\|^2 \ge \alpha_d \|u\|_{V_D}^2 \text{ for all } t \in [0, T], \text{ and for all } u \in V_D. \tag{6}$$

We know then from [4] or [5] that there exist representation operators $A(t)$ and $D(t)$

$$A(t) : V \to V', \quad D(t) : V_D \to V_D', \tag{7}$$

which for each fixed t are continuous and linear, with $a(t; u, v) = \langle A(t)u, v\rangle_{V', V}$, for all $u, v \in V$ and $d(t; u, v) = \langle D(t)u, v\rangle_{V_D', V_D}$, for all $u, v \in V_D$.

We will now consider the following problem:
Given $f \in L^2(0, T; V_D')$ (T finite) and initial conditions

$$u^0 \in V, \quad u^1 \in H,$$

we wish to find a function $u \in L^2(0, T; V), u' \in L^2(0, T; V_D)$ such that in V' we have

$$\begin{cases} u''(t) + D(t)u'(t) + A(t)u(t) = f(t), & t \in (0, T), \\ u(0) = u^0, \ u'(0) = u^1 \text{ for } t = 0, \end{cases} \tag{8}$$

i.e., for $f \in L^2(0,T;V_D')$:

$$\langle u''(t), v\rangle_{V',V} + d(t;u'(t),v) + a(t;u(t),v) = \langle f(t), v\rangle_{V',V} \quad \text{for all } v \in V. \tag{9}$$

(notice that (9) gives sense since $f(t) \in V_D' \subset V'$).

This formulation covers linear beam, plate and shell models with numerous damping models (Kelvin-Voigt, viscous, square-root, structural and spatial hysteresis) frequently studied in the litterature. The formulation above is non-standard in the sense that the damping sesquilinear form is incorporated in the variational model *and* is time-dependent. The problem without damping ($d = 0$) and $f \in L^2(0,T;H)$ was already treated by Lions in [4], see also [11]. The less general case without damping and $V = H_0^1(\Omega)$ is treated in e.g. [2]. The model above, but with $d : V_D \to C$ independent of time appears in the litterature for the first time (to our knowledge) in Banks et al, [1]. The following theorem is the time-dependent extension of the previous results.

Theorem 1. *Assume that* $(f, u^0, u^1) \in L^2(0,T;V_D') \times V \times H$. *Then there exists a unique solution* u *to (9) with* $(u,u') \in L^2(0,T;V) \times L^2(0,T;V_D)$, *and the mapping*

$$\{f, u^0, u^1\} \to \{u, u'\}, \tag{10}$$

is continuous and linear from

$$L^2(0,T;V_D') \times V \times H \to L^2(0,T;V) \times L^2(0,T;V_D). \tag{11}$$

As we will see, Theorem 1 can then be to extended to the following one.

Theorem 2. *Assume that* $(f, u^0, u^1) \in L^2(0,T;V_D') \times V \times H$. *Then there (perhaps after modifications on a set of measure zero) exists a unique solution* u *to (9) with* $(u,u') \in C(0,T;V) \times (C(0,T;H) \cap L^2(0,T;V_D))$, *and the mapping*

$$\{f, u^0, u^1\} \to \{u, u'\}, \tag{12}$$

is continuous and linear from

$$L^2(0,T;V_D') \times V \times H \to C(0,T;V) \times (C(0,T;H) \cap L^2(0,T;V_D)). \tag{13}$$

Remark 1. If we only have that the inequality (6) for the damping sesquilinear form d is satisfied with $\alpha_d = 0$, the results are still true with modifications. Now it will be necessary that $f \in L^2(0,T;H)$ and one only obtains that $u' \in L^2(0,T;H)$ – that is, the same results as if there was no damping. See e.g. [3].

2. Discussion of the Model

It is well known from the litterature that the strong formulation (1) of the problem in general causes computational problems due to irregularities stemming from non-smooth terms - typically in the force/moment terms in e.g. elasticity problems - and the weak formulation has proven advantageous both for theoretical and practical purposes, for example in the effort to estimate parameters or for control purposes. Let us also briefly discuss some of the damping models the theory applies to without going into much detail, as the pupose of this paper is to establish the validity of the abstract model. In general, as shown in e.g. [6], [7], [8] and [9], the damping term in the abstract PDE-model can be also considered as stemming from a damping term on the boundary of the spatial domain, indicating the wide range of problems covered by this formulation.

Time-dependent Kelvin-Voigt damping. Let $\Omega \subset R^n$ be a bounded domain with density $\mu(x)$ and let $\omega \subset \Omega$, with 1_ω denoting the characteristic function of ω. Let $\rho, \delta > 0$ denote material parameters and let $t \to k(t)$ denote a sufficiently smooth function. Then a $time - dependent$ damping sesquilinear form is given by

$$d(t; u, v) = \int_\Omega (\rho + \delta k(t) 1_\omega(x)) \Delta u(t, x) \Delta v(t, x) \mu(x) dx, \tag{14}$$

for $u(t), v(t) \in V_D = V$, where V can be taken e.g. as an appropriate subspace of $H^2(\Omega)$. This gives a model for e.g. a mechanical structure damped by a time-varying actuator, localized somewhere inside the structure. This could be piezocheramic patches or other “smart” devices, with the possibility of them being varying in time.

Time-dependent viscous damping. This is a velocity-proportional damping, given (with the notation from above) by the sesquilinear form

$$d(t; u, v) = \int_\Omega k(t, x) u(t, x) v(t, x) \mu(x) dx, \tag{15}$$

with e.g. $k \in C^1(0, T; L^\infty(\Omega))$ denoting the damping coefficient. We can take $V_D = H$ here.

Time-dependent spatial hysteresis damping. This model, without time-dependence, is discussed by Russel in [10], and, as mentioned in Banks [1], it has bee shown to be appropriate for models, where graphite fibers are embedded in an epoxy matrix. The time-dependent sesquilinear form that we consider here, can now be constructed with the following compact operator $K(t)$ on $L^2(\Omega)$:

$$(K(t)\varphi)(x) = \int_\Omega k(t,x,y)\varphi(y)dy, \tag{16}$$

where the nonnegative integral kernel k belongs to $C^1(0,T;L^\infty(\Omega)\times L^\infty(\Omega))$. Now, letting

$$\nu(x) = \int_\Omega \kappa(x,y)dy \tag{17}$$

denote some material property, we can define

$$d(t;u,v) = \int_\Omega ((\nu(x) - K(t))\nabla_x u(t,x)) \cdot (\nabla_x v(t,x))\mu(x)dx, \tag{18}$$

taking $V_D = H^1(\Omega)$, for example.

Time-dependent "square-root" damping. This model (without the time-dependence) has been used frequently in the litterature as a device to obtain exponentially decaying solutions of "wave-like" partial differential equations, the model we study in this paper is actually a generalized, weak formulation of such an equation. So taking e.g. $V_D = D(A^{\frac{1}{2}}(t))$, we could define the damping sesquilinear form by

$$d(t;u,v) = \int_\Omega c(t,x)(A^{\frac{1}{2}}(t,x)u(t,x))(A^{\frac{1}{2}}(t,x)v(t,x))\mu(x)dx \tag{19}$$

for some nonnegative function $c \in C^1(0,T;L^\infty(\Omega))$.

3. Proof of Theorem 1

We will follow a Galerkin approximation method (see e.g. [4], [11], [1]) with necessary, non-trivial modifications due to the presence of the time-dependent form(s). So, let $\{w_j\}_1^\infty$ denote an orthonormal basis in H that is also an orthogonal basis in V. This is possible since V is dense in H. For a fixed m we denote by V_m the finite dimensional subspace spanned by $\{w_j\}_1^m$, and we let u_m^0 and u_m^1 be chosen in V_m such that

$$u_m^0 \to u^0 \quad \text{in } V, \quad u_m^1 \to u^1 \quad \text{in } H, \text{ for } m \to \infty. \tag{20}$$

We now define the approximate solution $u_m(t)$ of order m of our problem in the following way:

$$u_m(t) = \sum_{j=1}^m g_{jm}(t)w_j, \tag{21}$$

where the $g_{jm}(t)$ are determined uniquely from the m-dimensional linear system:

$$(u_m''(t), w_j) + d(t; u_m'(t), w_j) + a(t; u_m(t), w_j) = \langle f(t), w_j \rangle_{V',V}, \qquad j = 1, 2, ..., m; \quad (22)$$

with $u_m(0) = u_m^0$ and $u_m'(0) = u_m^1$. Multiplying (22) with $\overline{g}_{jm}'(t)$ and summing over j yeilds

$$(u_m''(t), u_m'(t)) + d(t; u_m'(t), u_m'(t)) + a(t; u_m(t), u_m'(t)) = \langle f(t), u_m'(t) \rangle_{V',V}. \quad (23)$$

Now, since

$$\frac{d}{dt} a(t; u_m(t), u_m(t)) = 2 \operatorname{Re} a(t; u_m(t), u_m'(t)) + a'(t; u_m(t), u_m(t)), \quad (24)$$

we see that

$$\frac{d}{dt}\{\|u_m'(t)\|^2 + a(t; u_m(t), u_m(t))\} + 2 \operatorname{Re} d(t; u_m'(t), u_m'(t)) = a'(t; u_m(t), u_m(t)) + 2 \operatorname{Re}\langle f(t), u_m'(t)\rangle_{V',V}$$

and by integrating this equality we find

$$\|u_m'(t)\|^2 + a(t; u_m(t), u_m(t)) + \int_0^t 2 \operatorname{Re} d(t; u_m'(s), u_m'(s)) ds = \|u_m'(0)\|^2 + a(0; u_m^0, u_m^0) + \int_0^t a'(s; u_m(s), u_m(s)) ds + \int_0^t 2 \operatorname{Re}\langle f(s), u_m'(s)\rangle_{V',V} ds.$$

Using the coercivity conditions for a and d, together with the inequality (recall that $f(s) \in V_D'$)

$$|\langle f(s), u_m'(s)\rangle_{V',V}| \leq \frac{1}{4\epsilon}\|f(s)\|_{V_D'}^2 + \epsilon\|u_m'(s)\|_{V_D}^2 \quad (25)$$

we obtain, for all $\epsilon > 0$

$$\|u_m'(t)\|^2 + \alpha\|u_m(t)\|_V^2 + \int_0^t 2(\alpha_d - \epsilon)\|u_m'(s)\|_{V_D}^2 ds \leq \|u_m^1\|^2 + c_1\|u_m^0\|_V^2$$

$$+ c_2 \int_0^t \|u_m(s)\|_V^2 ds + 2\lambda_d \int_0^t \|u_m'(s)\|^2 ds + \int_0^t \frac{1}{2\epsilon} \|f(s)\|_{V_D'}^2 ds, \quad (26)$$

Since $u_m^0 \to u^0$ in V, $u_m^1 \to u^1$ in H and $f \in L^2(0,T;V_D')$, we have that, for $\epsilon > 0$ fixed and m large, there exist a constant $C > 0$, such that

$$\|u_m^1\|^2 + c_1 \|u_m^0\|_V^2 + \int_0^t \frac{1}{2\epsilon} \|f(s)\|_{V_D'}^2 ds \le C, \quad (27)$$

hence

$$\|u_m'(t)\|^2 + \alpha \|u_m(t)\|_V^2 + \int_0^t 2(\alpha_d - \epsilon) \|u_m'(s)\|_{V_D}^2 ds$$
$$\le C + c_2 \int_0^t \|u_m(s)\|_V^2 ds + 2\lambda_d \int_0^t \|u_m'(s)\|^2 ds. \quad (28)$$

Then, in particular

$$\|u_m'(t)\|^2 + \alpha \|u_m(t)\|_V^2$$
$$\le C + c_2 \int_0^t \|u_m(s)\|_V^2 ds + 2\lambda_d \int_0^t \|u_m'(s)\|^2 ds. \quad (29)$$

By Gronwall's inequality we then see that the sequence $\{u_m'\}$ is bounded in $C(0,T;H)$ and that the sequence $\{u_m\}$ is bounded in $C(0,T;V)$. From this fact together with the inequality (28) we conclude that $\{u_m'\}$ is also bounded in $L^2(0,T;V_D)$. Then it is possible to extract a subsequence $\{u_{m_k}\} \subset \{u_m\}$ and functions $u \in L^2(0,T;V)$ and $\tilde{u} \in L^2(0,T;V_D)$, such that $u_{m_k} \rightharpoonup u$, weakly in $L^2(0,T;V)$ and $u_{m_k}' \rightharpoonup \tilde{u}$, weakly in $L^2(0,T;V_D)$. But for $0 \le t < T$ we have in V, hence in V_D and H, that

$$u_{m_k}(t) = u_{m_k}(0) + \int_0^t u_{m_k}'(s) ds. \quad (30)$$

But $u_{m_k}(0) \to u^0$ in V and hence in V_D, while, for t fixed, $\int_0^t u_{m_k}'(s)ds \rightharpoonup \int_0^t \tilde{u}(s)ds$, weakly in V_D. So, by taking the weak limit in V_D in (30), we obtain in V_D the equality

$$u(t) = u^0 + \int_0^t \tilde{u}(s) ds, \quad (31)$$

from which we conclude that $u'(t)$ is in V_D *a.e.*, with $u' = \tilde{u}$ and $u(0) = u^0$.

We need now to show that u is actually a solution to the problem (9), with $u'(0) = u^1$. To see this, take a function $\varphi \in C^1([0,T])$, satisfying $\varphi(T) = 0$, and define, for $j < m$, the function φ_j by $\varphi_j(t) = \varphi(t)w_j$, where $\{w_j\}_1^m$ was the basis spanning V_m. Now, for a fixed $j < m$, we multiply (22) with $\overline{\varphi}(t)$ and integrate to obtain

$$\int_0^T ((u''_m(s), \varphi_j(s)) + d(s; u'_m(s), \varphi_j(s)) + a(s; u_m(s), \varphi_j(s)))ds = \int_0^T \langle f(s), \varphi_j(s)\rangle_{V'_D, V_D} ds.$$

Noticing that, for each t, we have that $d(t; \cdot, \varphi_j(t)) \in V'_D$ and $a(t; \cdot, \varphi_j(t)) \in V'$, we find, using the weak convergence above that for $m = m_k \to \infty$ and integration by parts in the first term, that

$$\int_0^T (-(u'(s), \varphi'_j(s)) + d(s; u'(s), \varphi_j(s)) + a(s; u(s), \varphi_j(s)))ds = \int_0^T \langle f(s), \varphi_j(s)\rangle_{V'_D, V_D} ds + (u^1, \varphi_j(0)), \tag{32}$$

for every j. Now further restrict φ to also satisfy $\varphi \in C_0^\infty(0, T)$ and write (32) as

$$\int_0^T \overline{\varphi'}(s)(-u'(s), w_j) + \int_0^T \overline{\varphi}(s)(d(s; u'(s), w_j) + a(s; u(s), w_j) - \langle f(s), w_j\rangle_{V'_D, V_D})ds = 0, \tag{33}$$

for each j. But by (33), we have

$$\frac{d}{dt}(u'(t), w_j) + d(t; u'(t), w_j) + a(t; u(t), w_j) = \langle f(t), w_j\rangle_{V'_D, V_D} \tag{34}$$

for all w_j. By density of V_m in V we conclude that $u'' \in L^2(0, T; V')$ and that for all $v \in V$

$$(u''(t), v) + d(t; u'(t), v) + a(t; u(t), v) = \langle f(t), v\rangle_{V'_D, V_D}, \tag{35}$$

which was (9). Hence the u we have constructed is indeed a solution to the equation and by (31) we have that $u(0) = u^0$. In order to verify that $u'(0) = u^1$ we integrate by parts in (32), and by application of (34) we find that, for all j:

$$-(u'(s), \varphi_j(s))|_{s=0}^{s=T} = (u^1, \varphi_j(0)), \tag{36}$$

or, equivalently

$$(u'(0), w_j)\overline{\varphi}(0) = (u^1, w_j)\overline{\varphi}(0). \tag{37}$$

Hence $u'(0) = u^1$.

In order to prove uniqueness, let u be a solution of our problem (9) corresponding to $(u^0, u^1, f) = (0, 0, 0)$, and define for a fixed $t_1 \in (0, T)$ (arbitrarily chosen) the function ψ by

$$\psi(t) = \begin{cases} -\int_t^{t_1} u(s)ds & \text{for } t < t_1, \\ 0 & \text{for } t \geq t_1, \end{cases} \tag{38}$$

so $\psi(T) = 0$. Obviously $\psi(t) \in V$ for all t, so we can take $\psi(t) = v$ in (9) which yeilds

$$\langle u''(t), \psi(t)\rangle_{V',V} + d(t; u'(t), \psi(t)) + a(t; u(t), \psi(t)) = \langle f(t), \psi(t)\rangle_{V',V}. \tag{39}$$

Since $\psi'(t) = u(t)$ for $t < t_1$ (a.e), we have that

$$\begin{aligned} &\int_0^{t_1} (\langle u''(t), \psi(t)\rangle_{V',V} + \langle u'(t), u(t)\rangle_{V',V})dt \\ &\qquad = \int_0^{t_1} \frac{d}{dt}(\langle u'(t), \psi(t)\rangle_{V',V})dt = 0, \end{aligned} \tag{40}$$

due to $\psi(t_1) = 0$ and the initial conditions. Using this and by integration of (39) we find

$$\int_0^{t_1} (\langle u'(t), u(t)\rangle_{V',V} - d(t; u'(t), \psi(t)) - a(t; u(t), \psi(t)))dt = 0; \tag{41}$$

hence

$$\begin{aligned} &\int_0^{t_1} \frac{d}{dt}(\|u(t)\|^2 - a(t; \psi(t), \psi(t)))dt \\ &= 2\int_0^{t_1} (a'(t; \psi(t), \psi(t)) + \operatorname{Re} d(t; u'(t), \psi(t)))dt. \end{aligned} \tag{42}$$

Since $\psi(t_1) = 0$ and $u(0) = u^0 = 0$ this yields

$$\|u(t_1)\|^2 + a(0; \psi(0), \psi(0)) = 2\int_0^{t_1} (a'(t; \psi(t), \psi(t)) + \operatorname{Re} d(t; u'(t), \psi(t)))dt. \tag{43}$$

From the assumptions on a and a' we arrive at

$$\|u(t_1)\|^2 + \alpha\|\psi(0)\|_V^2 \le 2\int_0^{t_1}(c_2\|\psi(t)\|_V^2 + \operatorname{Re} d(t; u'(t), \psi(t)))dt. \tag{44}$$

Now notice that

$$d(t; u'(t), \psi(t)) = \frac{d}{dt}(d(t; u(t), \psi(t))) - d'(t; u(t), \psi(t)) - d(t; u(t), u(t)), \tag{45}$$

so (from the initial conditions)

$$\int_0^{t_1} d(t; u'(t), \psi(t))dt = \int_0^{t_1}(-d'(t; u(t), \psi(t)) - d(t; u(t), u(t)))dt. \tag{46}$$

Since

$$-\operatorname{Re} d(t; u(t), u(t)) \le \lambda_D\|u(t)\|^2 - \alpha_d\|u(t)\|_{V_D}^2, \tag{47}$$

we have that

$$\begin{aligned} &\|u(t_1)\|^2 + \alpha\|\psi(0)\|_V^2 \\ &\le 2\int_0^{t_1}(c_2\|\psi(t)\|_V^2 + \lambda_D\|u(t)\|^2 - \alpha_d\|u(t)\|_{V_D}^2 + \operatorname{Re} d'(t; u(t), \psi(t)))dt. \end{aligned} \tag{48}$$

Now we introduce the function $\omega(t) = \int_0^t u(s)ds$ and use

$$\|\psi(t)\|_V^2 = \|\omega(t) - \omega(t_1)\|_V^2 \le 2\|\omega(t)\|_V^2 + 2\|\omega(t_1)\|_V^2 \tag{49}$$

to get

$$\begin{aligned} &\|u(t_1)\|^2 + (\alpha - 4c_2t_1)\|\omega(t_1)\|_V^2 \\ &\le 2\int_0^{t_1}(2c_2\|\omega(t)\|_V^2 + \lambda_D\|u(t)\|^2 - \alpha_d\|u(t)\|_{V_D}^2 + \operatorname{Re} d'(t; u(t), \psi(t)))dt. \end{aligned} \tag{50}$$

Finally we use the differentiability of the damping sesquilinear form d:

$$\begin{aligned} |d'(t; u(t), \psi(t))| &\le c_4\|u(t)\|_{V_D}\|\psi(t)\|_{V_D} \\ &\le \frac{c_4}{2}(\epsilon\|u(t)\|_{V_D}^2 + \frac{1}{\epsilon}\|\psi(t)\|_{V_D}^2) \\ &\le \frac{c_4}{2}(\epsilon\|u(t)\|_{V_D}^2 + \frac{2}{\epsilon}\|\omega(t)\|_{V_D}^2 + \frac{2}{\epsilon}\|\omega(t_1)\|_{V_D}^2) \end{aligned}$$

for all $\epsilon > 0$. We now have an inequality

$$\|u(t_1)\|^2 + (\alpha - 4c_2t_1)\|\omega(t_1)\|_V^2 - \frac{2c_4t_1}{\epsilon}\|\omega(t_1)\|_{V_D}^2$$
$$\leq 2\int_0^{t_1} (2c_2\|\omega(t)\|_V^2 + \lambda_D\|u(t)\|^2 - \alpha_d\|u(t)\|_{V_D}^2$$
$$+ \frac{c_4}{2}(\epsilon\|u(t)\|_{V_D}^2 + \frac{2}{\epsilon}\|\omega(t)\|_{V_D}^2))dt \quad (51)$$

and using here that

$$(\alpha - 4c_2t_1)\|\omega(t_1)\|_V^2 \geq (\frac{\alpha}{2} - 2c_2t_1)(\|\omega(t_1)\|_V^2 + \|\omega(t_1)\|_{V_D}^2), \quad (52)$$

we finally arrive at the formidable inequality

$$\|u(t_1)\|^2 + (\frac{\alpha}{2} - 2c_2t_1)\|\omega(t_1)\|_V^2 + (\frac{\alpha}{2} - 2c_2t_1 - \frac{2c_4t_1}{\epsilon})\|\omega(t_1)\|_{V_D}^2$$
$$\leq 2\int_0^{t_1} (2c_2\|\omega(t)\|_V^2 + \lambda_D\|u(t)\|^2 + (\frac{c_4\epsilon}{2} - \alpha_d)\|u(t)\|_{V_D}^2 + \frac{c_4}{\epsilon}\|\omega(t)\|_{V_D}^2)dt.$$

Now fix $\epsilon > 0$ such that $(\frac{c_4\epsilon}{2} - \alpha_d) < 0$ and fix $t_1 = \frac{\alpha}{8(c_2 + \frac{c_4}{\epsilon})}$ such that $(\frac{\alpha}{2} - 2c_2t_1 - \frac{2c_4t_1}{\epsilon}) = \frac{\alpha}{4}$. Then also $(\frac{\alpha}{2} - 2c_2t_1) > 0$ and the inequality above implies that for some constant $M > 0$:

$$\|u(t_1)\|^2 + \|\omega(t_1)\|_V^2 + \|\omega(t_1)\|_{V_D}^2$$
$$\leq M\int_0^{t_1} (\|u(t)\|^2 + \|\omega(t)\|_V^2 + \|\omega(t)\|_{V_D}^2)dt, \quad (53)$$

and from Gronwall's inequality we see that $u = 0$ in the interval $[0, t_1]$. Since the length of t_1 is independent of the choice of origin, we conclude that $u = 0$ on $[t_1, 2t_1]$ etc. Hence $u = 0$ and uniqueness is proved. That the solution depends continously on the data is obvious from the inequalities used to show existence; indeed, from (26) and (29) and the weak lower semicontinuity of norms we conclude that the constructed solution satisfies

$$\|u(t)\|_V^2 + \|u'(t)\|^2 + \delta\int_0^t \|u'(t)\|_{V_D}^2$$
$$\leq K(\|u^0\|_V^2 + \|u^1\|^2 + \int_0^t \|f(s)\|_{V_D'}^2 ds) \quad (54)$$

for some positive constants δ and K. Integrating from 0 to T yields the desired result (since $(u^0, u^1, f) \to (u, u')$ is linear).

This ends the proof of Theorem 1.

The proof of Theorem 2 follow from the inequality (53) and the original proof in the case $d = 0$ from [4], Vol. I, pp. 275-279, since we do not gain any additional regularity from the form d in this case.

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