International Journal of Pure and Applied Mathematics

Volume 30 No. 4 2006, 441-465

ELLIPTIC CURVES ARISING FROM NUMERICAL RANGES

Mao-Ting Chien[1], Hiroshi Nakazato[2 §]

[1]Department of Mathematics
Soochow University
Taipei, 11102, TAIWAN, R.O.C.
e-mail: mtchien@scu.edu.tw

[2]Department of Mathematical Sciences
Faculty of Science and Technology,
Hirosaki University
Hirosaki, 036-8561, JAPAN
e-mail: nakahr@cc.hirosaki-u.ac.jp

Abstract: We deal with n-by-n matrices whose numerical ranges have elliptic boundary generating curves. The elliptic boundary generating curves are parametrized and classified for small n. Examples are constructed for each class. From a graph theoretical view point, we give a blockwise nilpotent Toeplitz matrix whose boundary generating curve has genus 1, the curve is elliptic but not rational.

AMS Subject Classification: 15A60, 14H52
Key Words: numerical ranges, elliptic curves, boundary generating curves, genus

1. Introduction

Let $A \in M_n$. The *numerical range* of A is the set of complex numbers

$$W(A) = \{x^*Ax : x \in \mathbf{C}^n, |x| = 1\}.$$

It is a well-known result due to Toeplitz and Hausdorff that the numerical range $W(A)$ is always a convex set. Kippenhahn [16] introduced the homogeneous polynomial

Received: June 6, 2006 © 2006, Academic Publications Ltd.

§Correspondence author

$$F_A(t,x,y) = \det(tI_n + (x/2)(A+A^*) + (-i\,y/2)(A-A^*)).$$

The equation $F_A(t,x,y) = 0$ defines an algebraic curve

$$C_F = \{[(t,x,y)] \in \mathbf{CP}^2 : F_A(t,x,y) = 0\}$$

on the complex projective plane, where $[(t,x,y)]$ is the equivalence class containing $(t,x,y) \in \mathbf{C}^3\backslash(0,0,0)$ under the relation $(t_1,x_1,y_1) \sim (t_2,x_2,y_2)$ if $(t_2,x_2,y_2) = k(t_1,x_1,y_1)$ for some nonzero complex number k. The dual curve of C_F is denoted by

$$C_F^\wedge = \{[(T,X,Y)] \in \mathbf{CP}^2 : T\,t + X\,x + Y\,y = 0 \text{ is a tangent line of } C_F\},$$

The intersection C of the curve $C_F^\wedge$ and the real affine plane $\{(1,U,V) : U, V \text{are real numbers}\}$ is called the *boundary generating curve* of $W(A)$. The most interesting result obtained in [16] is that $W(A)$ is the convex hull of the curve C.

Assume that F is irreducible and $P_1, \cdots, P_m$ are singular points of the curve C_F, and $r_1, \ldots, r_m$ are their respective multiplicity. Suppose that each singular point P_j of the curve C_F is an ordinary multiple point. The *genus* of C_F is defined by

$$g(C_F) = (1/2)(n-1)(n-2) - (1/2)\sum_{j=1}^{m} r_j(r_j - 1). \tag{1.1}$$

An irreducible curve $\Lambda : f(x,y) = 0$ is *rational* if there exist rational functions $\phi(\lambda), \psi(\lambda) \in \mathbf{C}[\lambda]$ such that:

(1) For all but a finite set of $\lambda, (\phi(\lambda), \psi(\lambda))$ is a point of Λ;

(2) For every, but a finite set of points, $(x,y) \in \Lambda$ there is a unique λ such that $x = \phi(\lambda), y = \psi(\lambda)$.

If an irreducible curve C_F has no singular points other than ordinary ones, the rationality of the curve C_F is equivalent to the condition that $g(C_F) = 0$. The curve $C_F^\wedge$ is irreducible if and only if the curve C_F is irreducible. Under the assumption that C_F is irreducible, the genus of $C_F^\wedge$ coincides with the genus of C_F, see [21].

From the viewpoint of boundary generating curve, the numerical range is treated in many papers [6, 7, 12, 16, 19, 20]. The authors of this paper discussed the rationality of the boundary generating curves, and showed that the boundary generating curve of the numerical range of a nilpotent matrix associated to a circulant graph is rational, see [7]. The boundary generating curve is related with many interesting geometric properties of the numerical range of a matrix. For example, the numerical range of a matrix with flat portion is discussed in [2].

For 3×3 matrices, Kippenhahn [16] classified the shapes of the numerical ranges according to the factorability of the homogeneous polynomial F_A, and constructed examples for the corresponding classes. Rodman and Spitkovsky [15] gave a series of tests to determine the shapes of the numerical ranges falling in each class.

In this paper, using elliptic functions method, we parametrize the elliptic boundary generating curves of the numerical ranges of certain 3×3 matrices in Section 2. In Section 3, we classify 4×4 matrices A that the boundary generating curve of $W(A)$ is an irreducible elliptic curve, and we construct several examples for each class. The classification is based on the singularity of the elliptic boundary generating curves of the numerical ranges. Elliptic curves appear also in the generalized numerical range of a matrix polynomial [8, 9].

In Section 4 and Section 5, we construct and prove an irreducible boundary generating curve with genus 1, the curve is ellitpic but not rational. Section 6 contains different property of two boundary generating curves associated with directed graphs.

2. Parametrization

We recall some results from the theory of elliptic functions [1, 3, 13]. For an arbitrary pair of complex numbers g_2, g_3 with $g_2^3 - 27g_3^2 \neq 0$, we consider the following differential equation

$$(\frac{d}{du}\,\mathbf{p}(u))^2 = 4\,\mathbf{p}(u)^3 - g_2\,\mathbf{p}(u) - g_3 \tag{2.1}$$

on the Gaussian plane $\mathbf{C}$. If $\mathbf{p}(u)$ is a solution of (2.1), so is its translation $\mathbf{p}(u+\alpha)$. Every non-constant solution $\mathbf{p}(u)$ of (2.1) is doubly periodic and has a pole of order 2 at a point. Choose $\mathbf{p}(u)$ so that the lattice Γ formed by the poles of $\mathbf{p}(u)$ contains the origin 0. Γ is a subgroup of the additive group $\mathbf{C}$. Such a function $\mathbf{p}(u)$ is called a *Weierstrass* $\mathbf{p}$ *function* and denoted by $\mathbf{p}(u : g_2, g_3)$. The lattice Γ formed by poles of $\mathfrak{p}(u : g_2, g_3)$ has generators $2\,\omega$, $2\,\mu$ with $\Im(\mu/\omega) > 0$:

$$\Gamma = \{2m\,\omega + 2n\,\mu :\, n,\, m \in \mathbf{Z}\}.$$

The quantities ω, μ are called *half periods* of $\mathbf{p}(u:g_2,g_3)$. The function $\mathbf{p}(u)$ and its derivative $\mathbf{p}'(u)$ are expressed in terms of ω and μ

$$\begin{aligned}\mathbf{p}(u) &= \frac{1}{u^2}+\sum_{n,m\in\mathbf{Z},(n,m)\neq(0,0)}\left(\frac{1}{(u-2n\omega-2m\mu)^2}-\frac{1}{(2n\omega+2m\mu)^2}\right),\\ \mathbf{p}'(u) &= -2\sum_{(n,m)\in\mathbf{Z}}\frac{1}{(u-2n\omega-2m\mu)^3}.\end{aligned}$$

With these relations, we write $\mathbf{p}(u:g_2,g_3)=\mathbf{p}(u|\omega,\mu)$.

The following lemma is fundamental to parametrize the boundary of the numerical range of a generic 3×3 matrix.

Lemma 1. (see [13]) *Suppose that $e_1>e_2>e_3$ are three distinct real numbers and g_2,g_3 are real coefficients satisfying*

$$4x^3-g_2\,x-g_3=4(x-e_1)(x-e_2)(x-e_3).$$

Then the Weierstrass $\mathbf{p}$ *function* $\mathbf{p}(u:g_2,g_3)$ *has half periods* ω,μ *with* $\omega>0,\mu/i>0$, *such that*

$$\begin{aligned}&\{(x,y)\in\mathbf{R}^2: e_3\le x\le e_2,\ y^2=4x^3-g_2\,x-g_3\}\\ &=\{(\mathbf{p}(u+\mu),\mathbf{p}'(u+\mu)):-\omega\le u\le\omega\},\end{aligned}$$

and

$$\{(x,y)\in\mathbf{R}^2: x\ge e_1,\ y^2=4x^3-g_2\,x-g_3\}=\{(\mathbf{p}(u),\mathbf{p}'(u)):0<u<2\omega\}.$$

Suppose C is an irreducible complex projective curve on $\mathbf{CP}^2$. The curve C is called an *elliptic curve* if there exists a birational transformation $T:(x,y)\mapsto(X,Y)$ such that the curve $T(C)$ is a cubic elliptic curve:

$$X=\mathbf{p}(u:g_2,g_3),\ Y=\mathbf{p}'(u:g_2,g_3).$$

This is equivalent to say that the curve $T(C)$ satisfies

$$Y^2=4X^3-g_2X-g_3$$

for some $g_2,g_3\in\mathbf{C}$ with $g_2^3-27g_3^2\neq 0$. We note that an irreducible curve C is rational if $g(C)=0$, and is elliptic if $g(C)=1$.

The boundary of the numerical range of a generic 3×3 complex matrix is an elliptic curve of degree at most 6. It can be parametrized by elliptic functions. However, the explicit form of such a parametrization is rather complicated in general. In this section, we introduce elliptic functions theory, and parametrize the boundary of the numerical range of certain 3×3 matrices. The boundary of the numerical range of a 3×3 matrix may be plotted by the elliptic function method. Matrix theoretical approch of the numerical ranges of 3×3 matrices is discussed in [15].

Theorem 2. *Let*

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & a & a-3 \end{pmatrix},$$

where a is an arbitrary real number. Then the boundary generating curve $U(u)+i\,V(u)$ of $W(A)$ is birationally equivalent to the curve $y^2 = 4x^3 - g_2\,x - g_3$, and is parametrized by

$$U(u) = -\frac{144\mathbf{p}(u : g_2, g_3)^2 - 12g_2}{Q(u : g_2, g_3)}, \tag{2.2}$$

$$V(u) = \frac{24\sqrt{a^2 - 2a + 3}\,\mathbf{p}'(u : g_2, g_3)}{Q(u : g_2, g_3)}, \tag{2.3}$$

where

$$g_2 = \frac{1}{12}(a^4 + 4a^3 + 10a^2 - 36a + 153), \tag{2.4}$$

$$g_3 = \frac{1}{216}(a^6 + 6a^5 + 21a^4 - 28a^3 + 135a^2 + 270a - 189), \tag{2.5}$$

and

$$\begin{aligned} Q(u : g_2, g_3) &= 48\mathbf{p}(u : g_2, g_3)^3 + 12(a^2 + 2a + 3)\mathbf{p}(u : g_2, g_3)^2 \\ &\quad + 12g_2\mathbf{p}(u : g_2, g_3) + 24g_3 - (a^2 + 2a + 3)\,g_2. \end{aligned}$$

Furthermore, the boundary $\partial W(A)$ coincides with

$$\{U(u + \mu) + i\,V(u + \mu) : -\omega \le u \le \omega\}$$

and the "curved triangle" part of the boundary generating curve coincides with

$$\{U(u) + i\,V(u) : 0 < u < 2\omega\}.$$

Proof. First we compute the homogeneous polynomial

$$\begin{aligned} F_A(t, x, y) &= 4\det(t\,I_3 + (x/2)(A + A^*) - i\,(y/2)(A - A^*)) \\ &= 4t^3 + (-12 + 4a)x\,t^2 + (-3 - 2a - a^2)x^2\,t \end{aligned}$$

$$+4x^3 - (3-2a+a^2)y^2\, t.$$

Consider the affine curve C: $F_A(1,x,y)=0$. We set

$$x = X + \frac{1}{12}(a^2+2a+3), \tag{2.6}$$

$$y = \frac{1}{\sqrt{a^2-2a+3}}\, Y. \tag{2.7}$$

According to the transformations (2.6) and (2.7), the curve C is expressed in the Weierstrass canonical form (cf. [3, 13, 14]): $Y^2 = 4X^3 - g_2\, X - g_3$, where g_2 and g_3 are given by (2.4) and (2.5). Then the discriminant $D = g_2^3 - 27g_3^2$ of the polynomial

$$4X^3 - g_2 X - g_3 = 2(a^6+2a^5+8a^4-52a^3+369a^2-702a+1026)$$

is always positive. By Lemma 1, the dual curve $C^\wedge$ of the boundary generating curve C is parametrized by

$$x = x(u) = \mathbf{p}(u:g_2,g_3) + \frac{1}{12}(a^2+2a+3),$$

$$y = y(u) = \frac{1}{\sqrt{a^2-2a+3}}\, \mathbf{p}'(u:g_2,g_3).$$

By the formula in [22], p. 469,

$$\mathbf{p}''(u:g_2,g_3) = 6\mathbf{p}(u:g_2,g_3)^2 - \frac{1}{2}g_2,$$

we have that

$$x'(u) = \mathbf{p}'(u:g_2,g_3), \tag{2.8}$$

$$y'(u) = \frac{1}{\sqrt{a^2-2a+3}}\,(6\mathbf{p}(u:g_2,g_3)^2 - g_2/2). \tag{2.9}$$

By [4], the dual curve $C = (C^\wedge)^\wedge$ of $C^\wedge$ is parametrized by

$$U(u) = \frac{-y'(u)}{x(u)\,y'(u) - y(u)x'(u)}, \quad V(u) = \frac{x'(u)}{x(u)\,y'(u) - y(u)\,x'(u)}.$$

Using (2.8) and (2.9), we compute the denominator

$$\begin{aligned} &\sqrt{a^2-2a+3}(x(u)\,y'(u) - y(u)\,x'(u)) \\ = \;& 6\mathbf{p}(u)^3 - (g_2/2)\,\mathbf{p}(u) + (1/2)(a^2+2a+3)p(u)^2 \end{aligned}$$

$$\begin{aligned} & -(g_2/24)(a^2+2a+3) - \mathbf{p}'(u)^2 \\ = \; & 2\mathbf{p}(u)^3 + (1/2)(a^2+2a+3)\mathfrak{p}(u)^2 + (g_2/2)\,\mathbf{p}(u) \\ & + g_3 - (g_2/24)(a^2+2a+3). \end{aligned}$$

The representations (2.2) and (2.3) then follow, and the last conclusion follows immediately from Lemma 1. □

Remark. A classical graphics of the "curved triangle" of the numerical range can be found in [16]. A rather fine inner approximation is obtained in [4].

3. Elliptic Curves

Let $A \in M_4$ and let

$$F_A(t,x,y) = \det(t\,I_4 + (x/2)(A+A^*) - i\,(y/2)(A-A^*)),$$

and its algebraic curve

$$C_F = \{[(t,x,y)] \in \mathbf{CP}^2 : F_A(t,x,y) = 0\}.$$

Suppose that $P_0 = (t_0, x_0, y_0) \in \mathbf{C}^3 \backslash \{(0,0,0)\}$ is a point of C_F of multiplicity m. We assume that $t_0 \neq 0$. Since $[(t_0,x_0,y_0)] = [(1, x_0/t_0, y_0/t_0)]$, we may further assume that $t_0 = 1$. Partition $F_A(1,\, x_0+x,\, y_0+y)$ into two sums

$$F_A(1,\, x_0+x,\, y_0+y) = \sum_{j=0}^{m} a_{j,m-j}\, x^j\, y^{m-j} + \sum_{j+k>m} a_{j,k}\, x^j\, y^k,$$

and define the complex homogeneous polynomial

$$T_{P_0}(x,y) = \sum_{j=0}^{m} a_{j,m-j}\, x^j\, y^{m-j} = \prod_{k=1}^{m} (\alpha_k\, x + \beta_k\, y),$$

where $(\alpha_k,\, \beta_k) \in \mathbf{C}^2 \backslash \{(0,0)\}$. The straight line $\{[(1,\, x,\, y)] : \alpha_k(x-x_0) + \beta_k(y-y_0) = 0\}$ is called a *tangent* at P_0 of C_F. If $m=1$, P_0 is called a non-singular point, and the curve C_F has a unique tangent. If $m \geq 2$, the point P_0 is called a *singular* point, or *multiple* point of C_F. If the tangent lines of C_F at P_0 are mutually distinct, that is, $\alpha_j\,\beta_k - \alpha_k\,\beta_j \neq 0$, for $1 \leq j < k \leq m$, the point P_0 is called an *ordinary* m-ple point.

We give the following classification of 4×4 matrices A for which the boundary generating curve C of $W(A)$ is an irreducible elliptic curve, and we construct examples of matrices falling in each class.

Class 1. The curve C_F has a unique singular point on $\mathbf{CP}^2$. The singular point is a tacnode of order 2 and lies on the real projective plane $\mathbf{RP}^2$.

Class 2. The singular points of C_F are a pair of imaginary conjugate ordinary double points.

Class 3. The singular points of C_F are a pair of imaginary conjugate simple cusps.

Class 4. The singular points of C_F are two real ordinary double points.

Class 1. Consider the curve $\{(t,x,y) \in \mathbf{CP}^2 : F(t,x,y) = -t^2\, y^2 + x^4 + a_1\, t\, x^3 + a_2\, t^2\, x^2 + a_3\, t^3\, x + a_4\, t^4\}$. The polynomial

$$G_4(x) = x^4 + a_1\, x^3 + a_2\, x^2 + a_3\, x + a_4$$

has four distinct real roots. By the Newton-Puiseux method, we find that the equation $F(t,x,1) = 0$ has two series solutions:

$$t = \phi(x) = x^2 + \frac{a_1}{2}\, x^3 + (\frac{a_2}{2} + \frac{a_1^2}{8})\, x^4 + \ldots,$$

$$t = -\phi(-x) = -x^2 + \frac{a_1}{2}\, x^3 - (\frac{a_2}{2} + \frac{a_1^2}{8})\, x^4 + \ldots.$$

Thus the point $(t,x,y) = (0,0,1)$ is a tacnode of order 2 of the curve. This point is the unique singular point of the curve. By [13], we choose a root ℓ of $G_4(x) = 0$, and set

$$x = \frac{1}{x_1} + \ell, \tag{3.1}$$

$$y = -\frac{y_1}{x_1^2}. \tag{3.2}$$

Then the equation $y^2 = G_4(x)$ can be rewritten as

$$\begin{aligned} y_1^2 &= x_1^4\, y^2 \\ &= (1 + \ell\, x_1)^4 + a_1\, x_1 (1 + \ell\, x_1)^3 + a_2\, (1 + \ell\, x_1)^2 + a_3\, x_1^3\, (1 + \ell\, x_1) + a_4\, x_1^4 \\ &= (4\ell^3 + 3a_1\, \ell^2 + 2a_2 \ell + a_3)\, x_1^3 + (6\ell^2 + 3a_1\, \ell + a_2)\, x_1^2 + (4\ell + a_1)\, x_1 + 1 \\ &= b_1\, x_1^3 + b_2\, x_1^2 + b_3\, x_1 + b_4 = G_3(x_1). \end{aligned}$$

The polynomial G_3 is characterized by

$$G_3(x) = x^4\, G_4(\ell + 1/x).$$

Hence G_3 has three distinct real roots different from 0. Thus by the real birational transformations (3.1) and (3.2), the curve $y^2 = G_4(x)$ is equivalent to the curve $y_1^2 = G_3(x_1)$. Moreover we set

$$x_1 = \frac{4}{4\ell^3 + 3a_1\,\ell^2 + 2a_2\ell + a_3}\left(x_2 - \frac{1}{12}(6\ell^2 + 3a_1\,\ell + a_2)\right), \tag{3.3}$$

$$y_1 = \frac{4}{4\ell^3 + 3a_1\,\ell^2 + 2a_2\ell + a_3}y_2. \tag{3.4}$$

Then we have the equation

$$y_2^2 = 4x_2^3 - g_2\,x_2 - g_3,$$

where the quantities g_2, g_3 are given by the following:

$$g_2 = \frac{1}{12}(a_2^2 - 3a_1\,a_3 + 12a_4),$$

$$g_3 = \frac{1}{432}(-2a_2^3 + 9a_1\,a_2\,a_3 - 27a_3^2 - 27a_1^2\,a_4 + 72\,a_2\,a_4).$$

Now we consider the companion matrix

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & b & 2 & -b \end{pmatrix}$$

associated to the monic polynomial $t^4 + b\,t^3 - 2t^2 - b\,t + 1$, b is an arbitrary real number. We compute that

$$\begin{aligned} F(t,x,y) &= 4\det(t\,I_4 + (x/2)(B + B^*) - i\,(y/2)(B - B^*)) \\ &= -(4 + b^2)\,t^2\,y^2 + 4x^4 + 4b\,t\,x^3 - (12 + b^2)\,t^2\,x^2 - 4b\,t^3\,x + 4t^4. \end{aligned}$$

By the Strum method [11], we find that the polynomial

$$G_b(x) = 4x^4 + 4b\,x^3 - (12 + b^2)\,x^2 - 4b\,x + 4$$

has four distinct roots for every $b \in \mathbf{R}$. For simplicity we treat the case $b = 0$. Then the polynomial $G_0(x)$ is given by

$$\begin{aligned} G_0(x) &= 4x^4 - 12x^2 + 4 = 4\,(x^4 - 3x^2 + 1) = 4(x^2 + x - 1)\,(x^2 - x - 1) \\ &= 4(x - (1/2)(\sqrt{5} + 1))(x - (1/2)(\sqrt{5} - 1))(x - (1/2)(1 - \sqrt{5})) \end{aligned}$$

$$\times\,(x + (1/2)(\sqrt{5} + 1)). \quad (3.5)$$

We take a root $\ell = (1/2)(1+\sqrt{5})$ of the polynomial (3.5), and apply the general principle to the curve $y^2 = G_4(x) = x^4 - 3x^2 + 1$. In this case the mapping from x, y to x_2, y_2 according to (3.1), (3.2), (3.3) and (3.4) is described by

$$x = \frac{(5+\sqrt{5})(4\sqrt{5}x_2 + 5 - 2\sqrt{5})}{10(4x_2 - 2 - \sqrt{5})}, \quad y = \frac{-4\,(5+\sqrt{5})y_2}{(4x_2 - 2 - \sqrt{5})^2}.$$

Solving these equations for x_2, y_2 we have

$$x_2 = \frac{(4+2\sqrt{5})x + 3 - \sqrt{5}}{4(2x - 1 - \sqrt{5})}, \quad y_2 = \frac{-(5+\sqrt{5})y}{(2x - 1 - \sqrt{5})^2}.$$

The equation $y^2 = x^4 - 3x^2 + 1$ can be rewritten as $y_2^2 = 4x_2^3 - (7/4)x_2 + (3/8) = 4\,(x_2 - 1/2)(x_2 - 1/4)(x_2 + 3/4)$. Under the above transformations, the connecting component of $\{(t, x, y) \in \mathbf{RP}^2 : t^2\,y^2 = x^4 - 3x^2\,t^2 + t^4\}$ containing $(t, x, y) = (0, 0, 1)$ corresponds to the curve $\{(x_2, y_2) \in \mathbf{R}^2 : y_2^2 = 4x_2^3 - (7/4)\,x_2 + (3/8),\ x_2 \geq 1/2\}$. The component $\{(x, y) \in \mathbf{R}^2 : y^2 = x^4 - 3x^2 + 1, -(1/2)(\sqrt{5} - 1) \leq x \leq (1/2)(\sqrt{5} + 1)\}$ corresponds to the curve $\{(x_2, y_2) \in \mathbf{R}^2 : y_2^2 = 4x_2^3 - (7/4)\,x_2 + (3/8),\ -3/4 \leq x_2 \leq 1/4\}$.

Another example of the this class for which the tacnode lies on the affine plane $\{(x, y) : x, y \in \mathbf{R}\}$ is given in [18].

Class 2. We construct two examples falling in this class. First consider

$$A = \begin{pmatrix} 0 & 1+i & 0 & 0 \\ 1 & 0 & 0.1 - i & 0 \\ 0 & 1 & 0 & 1+i \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

We compute that

$$\begin{aligned} F(t, x, y) &= 400\det(t\,I_4 + (x/2)(A + A^*) - i\,(y/2)(A - A^*)) \\ &= 400\,t^4 - 1221\,t^2\,x^2 + 625x^4 - 400\,t^2\,x\,y + 1000\,x^3\,y \\ &\quad -381\,t^2\,y^2 + 650\,x^2\,y^2 + 200\,x\,y^3 + 25y^4. \end{aligned}$$

The curve $\{(t, x, y) \in \mathbf{CP}^2 : F(t, x, y) = 0\}$ has a pair of conjugate imaginary ordinary double points $(t, x, y) = (0, 1, -2 + i), (0, 1, -2 - i)$. The curve has no other singular points. Hence the polynomial F is irreducible in $\mathbf{C}[t, x, y]$ and the genus of the curve is 1. If we replace the entry $0.1 - i$ by $-i$ in the (2,3)-position of A, then the corresponding polynomial is decomposed into the product of two factors of degree 2 and thus the corresponding numerical range is the convex hull of two ellipses.

Remark. This example given in Class 2 was studied by the first author of this paper in [5]. The result answers the question raised by Professor C. R. Johnson during the 3rd WONRA in 1996, namely, is the numerical range of the tridiagonal matrix the convex hull of two ellipses?

For the second example of the this class, we set

$$B = \begin{pmatrix} \sqrt{3}+\sqrt{2} & -1 & -\sqrt{2} & 0 \\ 1 & \sqrt{3}-\sqrt{2} & 0 & -\sqrt{2} \\ \sqrt{2} & 0 & -\sqrt{3}+\sqrt{2} & -1 \\ 0 & \sqrt{2} & 1 & -\sqrt{3}-\sqrt{2} \end{pmatrix}.$$

We compute that

$$\begin{aligned} G(t,x,y) &= \det(t\, I_4 + (x/2)(B+B^*) - i\,(y/2)(B-B^*)) \\ &= t^4 - 10\,t^2\,x^2 + x^4 - 6t^2\,y^2 + 2x^2\,y^2 + y^4. \end{aligned}$$

The singular points of the curve $G(t,x,y) = 0$ are a pair of conjugate imaginary ordinary double points $(t,x,y) = (0,1,i),(0,1,-i)$. We take a simple point $(t,x,y) = (1,0,\sqrt{2}-1)$. Using [22], pp. 490-493, we consider the 2-dimensional linear series of conic curves passing through the three points $(0,1,i),(0,1,i),(1,0,\sqrt{2}-1)$. The series corresponds to the following family

$$\{\lambda F_1(t,x,y) + \mu F_2(t,x,y) + \nu F_3(t,x,y) : (\lambda,\mu,\nu) \in \mathbf{C}^3\},$$

where

$$\begin{aligned} F_1(t,x,y) &= x^2 + y^2 + (-3+2\sqrt{2})t^2, \\ F_2(t,x,y) &= t\,y + (1-\sqrt{2})t^2, \\ F_3(t,x,y) &= t\,x. \end{aligned}$$

We use affine coordinates and change variables

$$x_1 = \frac{F_1(1,x,y)}{F_3(1,x,y)} = \frac{x^2+y^2+(-3+2\sqrt{2})}{x}, \tag{3.6}$$

$$y_1 = \frac{F_2(1,x,y)}{F_3(1,x,y)} = \frac{y+(1-\sqrt{2})}{x}. \tag{3.7}$$

Eliminating x, y from $G(1,x,y) = 0$ by (3.6) and (3.7), we obtain that

$$K(1,x_1,y_1) = 0,$$

where

$$K(t, x_1, y_1)$$
$$= x_1^3 + (2 - 2\sqrt{2})x_1^2\, y_1 - 4\sqrt{2}x_1\, y_1^2 + (-4 - 4\sqrt{2})\, t^2\, x_1 + (-8 + 8\sqrt{2})\, t^2\, y_1.$$

Solving (3.6) and (3.7) for x, y, we have that

$$x = \frac{x_1 + (2 - 2\sqrt{2})\, y_1}{1 + y_1^2}, \quad y = \frac{x_1\, y_1 + (1 - \sqrt{2})\, y_1^2 + (-1 + \sqrt{2})}{1 + y_1^2}.$$

The real curve $\{(x_1, y_1) \in \mathbf{R}^2 : K(1, x_1, y_1) = 0\}$ has a canonical form $Y^2 = (X - \alpha)(X - \beta)(X - \gamma)$ for some distinct $\alpha, \beta, \gamma \in \mathbf{R}$.

Class 3. Consider the matrix

$$S = \begin{pmatrix} 2 & a & 0 & 0 \\ -a & 10/9 & 0 & b \\ 0 & 0 & -2 & c \\ 0 & -b & -c & -2/9 \end{pmatrix},$$

where $a = (4/3)\sqrt{7/15}$, $b = (16/3)\sqrt{2/35}$ and $c = (8/3)\sqrt{5/21}$. We compute that

$$\begin{aligned} H(t, x, y) &= 729 \det(t\, I_4 + (x/2)(S + S^*) - i\,(y/2)(S - S^*)) \\ &= 729t^4 + 648t^3\, x - 3096t^2\, x^2 - 2592t\, x^3 + 720x^4 \\ &\quad -3024t^2\, y^2 - 2496t\, x\, y^2 + 1728x^2\, y^2 + 1024y^4. \end{aligned}$$

The singular points of the curve $H(t, x, y) = 0$ are a pair of conjugate imaginary simple cusps $(t, x, y) = (1, -3/2, 3i/2), (1, -3/2, -3i/2)$. We take a simple point $(t, x, y) = (1, 1/2, 0)$. Again, using [22], pp. 490-493, we consider the 2-dimensional linear series of conic curves passing through the three points $(1, -3/2, 3i/2), (1, -3/2, -3i/2),\ (1, 1/2, 0)$. The series corresponds to the following family

$$\{\lambda H_1(t, x, y) + \mu H_2(t, x, y) + \nu H_3(t, x, y) : (\lambda, \mu, \nu) \in \mathbf{C}^3\}$$

where

$$H_1(t, x, y) = 6x^2 + 8y^2 - 3t\, x, \quad H_2(t, x, y) = 4x^2 + 4t\, x - 3t^2,$$

$$H_3(t, x, y) = -2x\, y - 3t\, y.$$

We use affine coordinates and change variables

$$x_1 = \frac{H_1(1,x,y)}{H_3(1,x,y)} = \frac{-6x^2 - 8y^2 + 3x}{(2x+3)\,y}, \tag{3.8}$$

$$y_1 = \frac{H_2(1,x,y)}{H_3(1,x,y)} = \frac{1-2x}{y}. \tag{3.9}$$

Eliminating x, y from $H(1,x,y) = 0$ by (3.8) and (3.9), we obtain that

$$L(1, x_1, y_1) = 0,$$

where

$$L(t, x_1, y_1) = 128x_1^3 + 240x_1^2\, y_1 - 216x_1\, y_1^2 - 1920\, t^2\, x_1 - 243y_1^3 - 1296t^2\, y_1.$$

Solving (3.8) and (3.9) for x, y, we have that

$$x = \frac{3x_1\, y_1 + 8}{16 - 2x_1\, y_1 + 3y_1^2}, \quad y = \frac{-8x_1 + 3y_1}{16 - 2x_1\, y_1 + 3y_1^2}.$$

The real curve $\{(x_1, y_1) \in \mathbf{R}^2 : L(1, x_1, y_1) = 0\}$ has a canonical form $Y^2 = (X-\alpha)(X-\beta)(X-\gamma)$ for some distinct $\alpha, \beta, \gamma \in \mathbf{R}$.

Class 4. Consider the matrix

$$T = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We compute that

$$\begin{aligned} F(t,x,y) &= 4\det(t\, I_4 + (x/2)(T+T^*) - i\,(y/2)(T-T^*)) \\ &= 4t^4 - 5t^2\, x^2 + 2t\, x^3 - 5t^2\, y^2 + 2t\, x\, y^2 + x^2\, y^2 + y^4. \end{aligned}$$

The curve $F(t,x,y) = 0$ has two real ordinary double points $(1,1,1)$ and $(1,1,-1)$. The curve has no other singular points. Hence F is irreducible in $\mathbf{C}[t,x,y]$. We take its simple point $(t,x,y) = (0,1,0)$. Consider the 2-dimensional linear series of conic curves passing through $(1,1,1)$, $(1,1,-1)$, $(0,1,0)$. The series correspond to the 2-dimensional projective space $\lambda F_1 + \mu F_2 + \nu F_3$, where

$$F_1(t,x,y) = y^2 - t^2,\ F_2(t,x,y) = x\, y - t\, y,\ F_3(t,x,y) = t\, x - t^2.$$

Set

$$x_1 = \frac{F_1(1,x,y)}{F_3(1,x,y)} = \frac{y^2-1}{x-1}, \quad y_1 = \frac{F_2(t,x,y)}{F_3(1,x,y)} = \frac{y\,(x-1)}{x-1} = y.$$

Solving these equations for x, y, we obtain

$$x = \frac{y_1^2 + x_1 - 1}{x_1}, \quad y = y_1.$$

The equation $F(1,x,y) = 0$ is transformed into $M(1,x_1,y_1) = 0$, where

$$M(t,x_1,y_1) = -2t^3 + t^2\,x_1 + 4t\,x_1^2 + x_1^3 + 2t\,y_1^2 + x_1\,y_1^2.$$

Consider the images of the two connected components of the curve $\{(x_1,y_1) \in \mathbf{R}^2 : M(1,x_1,y_1) = 0\}$ under the mapping $(x_1,y_1) \mapsto (x,y)$. We find that the images intersect at the points $(x,y) = (1,1), (1,-1)$. Taking into account of this fact, we then use both components to parametrize the dual curve of the curve $\partial W(T)$.

We note that for a 3×3 nilpotent matrix, Marcus and Pesce [17] gave an algebraic characterization for the numerical range to be a circular disc. A more general condition was given in [10].

4. Construction of Genus 1

It is shown in [7], the boundary generating curve of $W(A)$ for the 4×4 matrix

$$A = \begin{pmatrix} 0 & b & a & b \\ 0 & 0 & b & a \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad a, b \in \mathbf{R},$$

is rational. It is also mentioned there that the boundary generating curve of the numerical range of a blockwise nilpotent Toeplitz matrix may not be rational. We construct such a matrix and give a complete proof of irreducibility. Indeed we construct a 8×8 matrix such that its boundary generating curve is irreducible and elliptic with genus 1.

Theorem 3. *Let A be the 8×8 real nilpotent matrix defined by*

$$A = \begin{pmatrix} 0_2 & B & H & B \\ 0_2 & 0_2 & B & H \\ 0_2 & 0_2 & 0_2 & 0_2 \end{pmatrix},$$

where 0_2, B, H are 2×2 Hermitian matrices defined by

$$0_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then the boundary generating curve of $W(A)$ is irreducible and has genus 1.

Proof. Consider the polynomial

$$F(t,x,y) = 256\det(t\,I_8 + (x/2)(A+A^*) + (-i\,y/2)(A-A^*)).$$

Then $F(t,x,y) = G(t^2,\, x^2, y^2)$, where

$$\begin{aligned} G(T,X,Y) = {} & 256T^4 - 1280T^3\,X + 608T^2\,X^2 - 80T\,X^3 + X^4 \\ & - 1280T^3\,Y + 2240T^2\,X\,Y - 240T\,X^2\,Y + 4X^3\,Y \\ & + 1632T^2\,Y^2 - 240T\,X\,Y^2 + 6X^2\,Y^2 - 80T\,Y^3 + 4X\,Y^3 + Y^4. \end{aligned}$$

We analyse the singularities of the curve $C_F : F(t,x,y) = 0$. For this purpose we use birational transformations. The genus of an algebraic curve is invariant under such transformations (cf. [21], pp. 75-86, [22], pp. 489-493). First we use a projective transformation:

$$t = t, \quad q = x - i\,y - t, \quad r = x + i\,y - t.$$

Then

$$t = t, \quad x = \frac{1}{2}(q + r + 2t), \quad -i\,y = \frac{1}{2}(q - r).$$

Denote $H(t,q,r) = F(t,(1/2)(q+r+2t),\,(i/2)(q-r))$. Secondly we take a Cremona transformation. The algebraic (quadratic) transformation of C_H is given by $K_0(T,Q,R) = 0$, where $K_0(T,Q,R) = H(Q\,R,\,T\,R,\,T\,Q)$. We have

$$K_0(T,Q,R) = -Q^4\,R^4\,K(T,Q,R),$$

where K is a homogeneous polynomial of degree 8 and does not contain a factor of the form $T^i\,Q^j\,R^k$ for $i+j+k > 0$. Then every singular point of the curve $C_K : K(T,Q,R) = 0$ is an ordinary mutiple point. The curve C_K has two ordinary quartic points $(T,Q,R) = (0,1,0)$ and $(0,0,1)$. The remaining singular points of C_K are the following eight ordinary double points:

$$(1,0,-1),\ (1,-1,0), (1,1,1), (3,-1,-1),$$

$$(-21 - 8\epsilon\sqrt{6},\ 1 - \eta 2i\,(\sqrt{3} + \epsilon\sqrt{2}),\ 1 + \eta 2i\,(\sqrt{3} + \epsilon\sqrt{2})),$$

$(\epsilon, \eta = 1, -1)$. Hence K is square free in $\mathbf{C}[T, Q, R]$. We shall prove the irreducibility of K in $\mathbf{C}[T, Q, R]$ in Section 5 by using a combinatorial method. Under the irreducibility of K, the genus of the curve C_K is computed by the genus formula (1.1):

$$\frac{1}{2} \times 7 \times 6 - \frac{1}{2} \times 2 \times 4 \times 3 - \frac{1}{2} \times 8 \times 2 \times 1 = 21 - 12 - 8 = 1. \qquad \square$$

5. Irreducibility

In this section we prove that the polynomial K is irreducible in $\mathbf{C}[T, Q, R]$. First we prove that K contains no linear factors. We denote by $T_j(T, Q, R) = 0$ $(j = 1, 2, \ldots, 8)$ the eight tangents of C_K at the four points

$$P_{\epsilon,\eta} = (-21 - 8\eta\sqrt{6},\ 1 - 2i\epsilon(\sqrt{3} + \eta\sqrt{2}), 1 + 2i\epsilon(\sqrt{3} + \eta\sqrt{2}))$$

for $\epsilon, \eta \in \{1, -1\}$. Denote by $A_0(T, Q, R)$ the product of the eight linear homogeneous polynomials T_j. We compute that

$$\begin{aligned}
A_0(T, Q, R) = {}& 25344958401\, Q^8 - 12169324440\, Q^7\, R + 2793694428\, Q^6\, R^2 \\
&+ 12489149400\, Q^5\, R^3 - 50416726202\, Q^4\, R^4 + 12489149400\, Q^3\, R^5 \\
&+ 2793694428\, Q^2\, R^6 - 12169324440\, Q\, R^7 + 25344958401\, R^8 \\
&+ 87672627504\, Q^7\, T - 90751640112\, Q^6\, R\, T + 81773791344\, Q^5\, R^2\, T \\
&- 85160178544\, Q^4\, R^3\, T - 85160178544\, Q^3\, R^4\, T + 81773791344\, Q^2\, R^5\, T \\
&- 90751640112\, Q\, R^6\, T + 87672627504\, R^7\, T + 106677046224\, Q^6\, T^2 \\
&- 128846469024\, Q^5\, R\, T^2 + 129070842672\, Q^4\, R^2\, T^2 \\
&- 233641317056\, Q^3\, R^3\, T^2 + 129070842672\, Q^2\, R^4\, T^2 \\
&- 128846469024\, Q\, R^5\, T^2 + 106677046224\, R^6\, T^2 \\
&+ 62361534144\, Q^5\, T^3 - 47450982976\, Q^4\, R\, T^3 \\
&- 6330296448\, Q^3\, R^2\, T^3 - 6330296448\, Q^2\, R^3\, T^3 - 47450982976\, Q\, R^4\, T^3 \\
&+ 62361534144\, R^5\, T^3 + 26446532704\, Q^4\, T^4 - 14455403648\, Q^3\, R\, T^4 \\
&+ 10608528960\, Q^2\, R^2\, T^4 - 14455403648\, Q\, R^3\, T^4 + 26446532704\, R^4\, T^4 \\
&+ 8017829120\, Q^3\, T^5 + 1475079936\, Q^2\, R\, T^5 + 1475079936\, Q\, R^2\, T^5 \\
&+ 8017829120\, R^3\, T^5 + 1931832576\, Q^2\, T^6 \\
&+ 986317312\, Q\, R\, T^6 + 1931832576\, R^2\, T^6
\end{aligned}$$

$$+ 299881472\, Q\, T^7 + 299881472\, R\, T^7 + 29073664\, T^8.$$

By using this identity we obtain that

$$K(T,-1,-1) = 495 - 600\, T - 1196\, T^2 + 888\, T^3 + 522\, T^4 - 424\, T^5 + 52\, T^6 + 8\, T^7 - T^8,$$

$$A_0(T,-1,-1) = 6500229376 + 12930799616\, T - 19838477312\, T^2 - 17160509440\, T^3 + 34590787072\, T^4 - 18985818112\, T^5 + 4849982464\, T^6 - 599762944\, T^7 + 29073664\, T^8,$$

and find that the resultant of the above polynomials does not vanish. Thus the polynomial K does not contain a linear factor L for which C_T passing through a point $P_{\epsilon,\eta}$.

Denote by $A_1(T,Q,R)$ the product of the four linear factors corresponding to the tangents at $(0,1,0)$. We compute that

$$A_1(T,Q,R) = 495R^4 + 300R^3\, T - 118R^2\, T^2 + 332R\, T^3 - T^4.$$

The product $A_2(T,Q,R)$ of the four linear factors corresponding to the tangents at $(0,0,1)$ satisfies

$$A_2(T,Q,R) = A_1(T,R,Q).$$

The product $A_3(T,Q,R)$ of the two linear factors corresponding to the tangents at $(1,1,1)$ and the product $A_4(T,Q,R)$ of the two linear factors corresponding to the tangents at $(3,-1,-1)$ are computed as follows

$$A_3(T,Q,R) = (Q-T)^2 + (R-T)^2 = (Q^2+R^2) - 2(Q+R)\,T + 2T^2,$$
$$A_4(T,Q,R) = (3Q+T)^2 + (3R+T)^2 = 9(Q^2+R^2) + 6(Q+R)\,T + 2T^2.$$

By a similar argument, we can prove that, for other cases of A_j ($j = 1,2,3,4$), $K(T,Q,R)$ has no common factors. The tangents at $(1,0,-1)$ are $Q=0$, $R+T=0$. The tangents at $(1,-1,0)$ are $R=0$, $Q+T=0$. It is easy to see that none of Q, R, $Q+T$, $R+T$ is a factor of K. Thus K contains no linear factor.

Denote by P_j ($j=1,2,\ldots,8$) the eight ordinary double points of C_K. Set $P_9 = (0,1,0)$, $P_{10} = (0,0,1)$. The curve C_K has no singular points other than $P_1,\ldots,P_{10}$. Suppose that $K = K_1\, K_2 \cdots K_p$ is an irreducible decomposition of K and $p \geq 2$. Denote by $m(k,j)$ the multiplicity of C_{K_k} at the point P_j. If $K_k(P_j) \neq 0$, we set $m(k,j) = 0$. Then we have the following equations:

$$\sum_{k=1}^{p} m(k,j) = 2$$

for $j = 1, 2, \ldots, 8$, and

$$\sum_{k=1}^{p} m(k,j) = 4$$

for $j = 9, 10$. Denote by $n(k)$ the degree of K_k. Then by Bezout Theorem,

$$\sum_{j=1}^{10} m(k,j)\, m(\ell,j) = n(k)\, n(\ell)$$

for $1 \leq k < \ell \leq p$. Moreover by the irreducibility of K_k, the following inequality holds:

$$\sum_{j=1}^{10} m(k,j)\,(m(k,j)-1) \leq (n(k)-1)\,(n(k)-2).$$

Since K contains no linear factors, we have the following possible irreducible decompositions of K:

Case 0. $p = 1$, K itself is irreducible;
Case 1. $p = 4$, $n(1) = n(2) = n(3) = n(4) = 2$;
Case 2. $p = 3$, $n(1) = n(2) = 3$, $n(3) = 2$;
Case 3. $p = 2$, $n(1) = 4$, $n(2) = 4$;
Case 4. $p = 2$, $n(1) = 5$, $n(2) = 3$;
Case 5. $p = 2$, $n(1) = 6$, $n(2) = 2$.

We prove the irreducibility of K by showing the invalidation of *Case 1-Case 5.*

Case 1. If this is the case, each C_{K_k} has no singular points. Then we have $m(k,j) = 1$ or $m(k,j) = 0$ for every $1 \leq k \leq 4$, $1 \leq j \leq 10$. Hence we have

$$m(1,9) = m(2,9) = m(3,9) = m(4,9) = 1,$$

$$m(1,10) = m(2,10) = m(3,10) = m(4,10) = 1.$$

By the Bezout's Theorem, we have

$$\sum_{j=1}^{10} m(k,j)\, m(\ell,j) = 4 \text{ and } \sum_{j=1}^{8} m(k,j)\, m(\ell,j) = 2$$

for $1 \le k < \ell \le 4$. These relations imply that for every pair $1 \le k < \ell \le 4$, there exists a pair of points P_{j_1}, P_{j_2} with $1 \le j_1 < j_2 \le 8$ for which

$$m(k, j_1) = m(\ell, j_2) = 1, \quad m(k, j_2) = m(k, j_2) = 1,$$

$$m(s, j_1) = m(s, j_2) = 0,$$

for $1 \le s \le 4$, $s \ne j_1, s \ne j_2$. Since $2 \times {}_4C_2 = 2 \times 6 = 12 > 8$, the case 1 does not occur.

Case 2. If this is the case, C_{K_3} has no singular points. Hence for each $1 \le j \le 10$, $m(3, j) = 1$ or $m(3, j) = 0$. Since $n(1) = n(2) = 3$, the numbers $m(1, 9)$, $m(1, 10)$, $m(2, 9)$, $m(2, 10)$ are less than or equal to 2, and the numbers $m(1, 9) + m(1, 10)$, $m(2, 9) + m(2, 10)$ are less than or equal to 3. By the symmetry $K(T, Q, R) = K(T, R, Q)$, the factors K_1 and K_2 satisfy $K_1(T, Q, R) = K_2(T, R, Q)$ or $K_1(T, Q, R) = K_1(T, R, Q)$, $K_2(T, Q, R) = K_2(T, R, Q)$. We may assume that

$$m(1, 9) = 2, \quad m(1, 10) = 1, \quad m(2, 9) = 1, \quad m(2, 10) = 2,$$

$$m(3, 9) = m(3, 10) = 1.$$

By the Bezout Theorem, we have

$$\sum_{j=1}^{8} m(1, j)\, m(2, j) = 9 - 2 - 2 = 4, \tag{5.1}$$

$$\sum_{j=1}^{8} m(1, j)\, m(3, j) = 6 - 2 - 1 = 3, \tag{5.2}$$

$$\sum_{j=1}^{8} m(2, j)\, m(3, j) = 6 - 1 - 2 = 3. \tag{5.3}$$

From the relation (5.1), we assume that

$$m(1, 1) = m(1, 2) = m(1, 3) = m(1, 4) = m(1, 5) = 1,$$

$$m(2, 1) = m(2, 2) = m(2, 3) = m(2, 4) = m(2, 5) = 1,$$

$$m(3, 1) = m(3, 2) = m(3, 3) = m(3, 4) = m(4, 5) = 0.$$

By the condition (5.2), we conclude that

$$m(1, 6) = m(1, 7) = m(1, 8) = 1, \; m(3, 6) = m(3, 7) = m(3, 8) = 1.$$

But these imply that $m(2,6) = m(2,7) = m(2,8) = 0$ and the condition (5.3) does not hold.

Case 5. If this is the case, C_{K_2} has no singular points. Hence for each $1 \le j \le 10$, $m(2,j) = 1$ or $m(2,j) = 0$. By the Bezout Theorem we have

$$\sum_{j=1}^{10} m(1,j)\, m(2,j) = 12.$$

Then we have

$$\sum_{j=1}^{8} m(1,j)\, m(2,j) \le 8,$$

$$m(1,9)\, m(2,9) \le 3, \quad m(1,10)\, m(2,10) \le 3.$$

Hence $m(2,9) = m(2,10) = 1$, $m(1,9) = m(1,10) = 3$, and

$$\sum_{j=1}^{8} m(1,j)\, m(2,j) = 6.$$

Thus the number of points P_j $(1 \le j \le 8)$ for which $K_2(P_j) = 0$ is just 6. Since $K(T,Q,R) = K(T,R,Q)$ is symmetry, the factor K_2 also satisfies $K_2(T,Q,R) = K_2(T,R,Q)$. Set

$$\begin{aligned}
P_1 &= (1,-1,0),\\
P_2 &= (1,0,-1),\\
P_3 &= (-21-8\sqrt{6}, 1-2i(\sqrt{3}+\sqrt{1}), 1+2i(\sqrt{3}+\sqrt{2})),\\
P_4 &= (-21-8\sqrt{6}, 1+2i(\sqrt{3}+\sqrt{2}), 1-2i(\sqrt{3}+\sqrt{2})),\\
P_5 &= (-21+8\sqrt{6}, 1-2i(\sqrt{3}-\sqrt{2}), 1+2i(\sqrt{3}-\sqrt{2})),\\
P_6 &= (-21+8\sqrt{6}, 1+2i(\sqrt{3}-\sqrt{2}), 1-2i(\sqrt{3}-\sqrt{2})),\\
P_7 &= (1,1,1),\\
P_8 &= (3,-1,-1).
\end{aligned}$$

By the symmetry of K_2, we obatain that

$$K_2((0,1,0)) = K_2((0,0,1)) = 0\,, \tag{5.4}$$

$$K_2(P_1) = K_2(P_3) = K_3(P_5) = 0. \tag{5.5}$$

By the equations (5.4) and (5.5), we may set

$$K_2(T,Q,R) = b\,Q\,R + e\,(Q+R)\,T + f\,T^2.$$

Since $K_2(P_1) = 0$, we have $e = f$. By the condition $K_2(P_3) = K_2(P_5) = 0$, we have $b - 2e + 42f = 0$, $7b - 14e + 275f = 0$. Then $b = e = f = 0$ and this implies $K_2 = 0$.

Case 4. In this case, we have, by Bezout theorem,

$$\sum_{j=1}^{10} m(1,j)m(2,j) = 15.$$

If $m(2,9) \leq 1$ and $m(2,10) \leq 2$ then

$$m(1,9)m(2,9) + m(1,10)m(2,10) \leq 6.$$

But this is impossible since $m(1,j)m(2,j) \leq 1$ for $1 \leq j \leq 8$. Since the factor K_2 satisfies the symmetry $K_2(T,Q,R) = K_2(T,R,Q)$, it follows that $m(2,9) = m(2,10) = 2$. But this is again impossible since

$$\sum_{j=1}^{10} m(2,j)(m_(2,j) - 1) \leq (n(2)-1)(n(2)-2) = 2 \times 1 = 2.$$

Case 3. We prove that this case does not occur. Otherwise, the Bezout theorem shows that

$$\sum_{j=1}^{10} m(1,j)m(2,j) = 16.$$

Since $\sum_{j=1}^{8} m(1,j)m(2,j) \leq 8$, it follows that $m(1,9)m(2,9) + m(1,10)m(2,10) \geq 8$. By $n(1) = n(2) = 4$, the numbers $m(1,9), m(2,9), m(1,10), m(2,10)$ are less than or equal to 3, and

$$m(1,9) + m(1,10) \leq 4, \; m(2,9) + m(2,10) \leq 4.$$

Thus

$$m(1,9) = m(1,10) = m(2,9) = m(2,10) = 2,$$

and that

$$\begin{aligned} m(1,1) &= m(1,2) = m(1,3) = m(1,4) = m(1,5) = m(1,6) \\ &= m(1,7) = m(1,8) = 1, \\ m(2,1) &= m(2,2) = m(2,3) = m(2,4) = m(2,5) = m(2,6) \\ &= m(2,7) = m(2,8) = 1. \end{aligned}$$

The conclusion then follows from the following lemma.

Lemma 4. *Let $L(T,Q,R) \in \mathbf{C}[T,Q,R]$ be a nonzero quartic homogeneous polynomial and satisfy the condition*

$$L((0,1,0)) = L_T((0,1,0)) = L_Q((0,1,0)) = L_R((0,1,0)) = 0,$$

$$L((0,0,1)) = L_T((0,0,1)) = L_Q((0,0,1)) = L_R((0,0,1)) = 0,$$
$$L((1,1,1)) = L((3,-1,-1)) = 0,$$
$$L((-21-8\eta\sqrt{6},\, 1-2i\epsilon(\sqrt{3}+\eta\sqrt{2}), 1+2i\epsilon(\sqrt{3}+\eta\sqrt{2})) = 0$$

for every $\epsilon, \eta \in \{1,-1\}$. Then L is reducible and

$$L(T,Q,R) = \alpha\,(2QR+QT+RT)\,(3QR-QT-RT-T^2),$$

for some non-zero $\alpha \in \mathbf{C}$, and the curve C_L passes through the point $(1,0,0)$.

Remark. The irreducible conic curve $3QR-QT-RT-T^2=0$ passes through the points $(T,Q,R) = (0,1,0), (0,0,1), (1,1,1), (3,-1,-1)$. The curves $2QR+QT+RT=0$ and $3QR-QT-RT-T^2=0$ intersect at the points

$$(T,Q,R) = (0,1,0), (0,0,1), (5,-1+2i,-1-2i), (5,-1-2i,-1+2i).$$

6. A Mysterious Phenomenon

We end up this paper with two interesting boundary generating curves associated with directed graphs. Consider the directed graph Γ_1 in Figure 1.

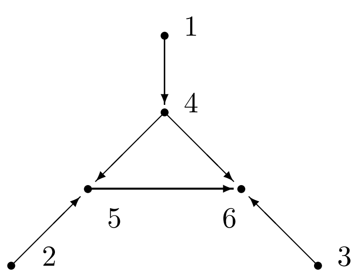

Figure 1.

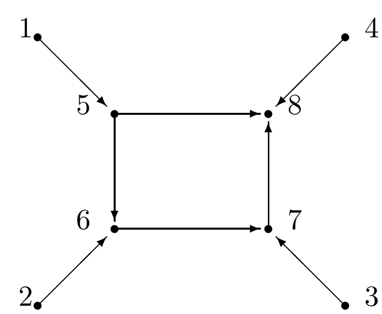

Figure 2.

The adjacency matrix of the directed graph Γ_1 is

$$A = \begin{pmatrix} 0_3 & I_3 \\ 0_3 & N_3 \end{pmatrix}, \quad \text{where} \quad N_3 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

It can be shown that the boundary generating curve of A is irreducible and rational. On the other hand, we consider the directed graph Γ_2 in Figure 2.

The adjacency matrix of the directed graph Γ_2 is

$$B = \begin{pmatrix} 0_4 & I_4 \\ 0_4 & N_4 \end{pmatrix}, \quad \text{where} \quad N_4 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Although the structure of Γ_2 is an extension of that Γ_1, but the boundary generating curve of B is elliptic, not rational.

Acknowledgments

The first author thanks Taiwan National Science Council for support of this work.

References

[1] L.V. Ahlfors, *Complex Analysis*, 3rd Ed., McGraw-Hill, Auckland (1979).

[2] E.S. Brown, I.M. Spitkovsky, On flat portions on the boundary of the numerical range, *Linear Algebra and Its Applications*, **390** (2004), 75-109.

[3] K. Chandrasekharan, *Elliptic functions*, Springer, (Grundehren der Mathematischen wissenschaften, vol. 281), New York, 1991.

[4] M.T. Chien, Numerical computation and circularity of higher numerical range, *Soochow Journal of Mathematics*, **20** (1994), 555-567.

[5] M.T. Chien, On the numerical range of tridiagonal operators, *Linear Algebra and Its Applications*, **246** (1996), 203-214.

[6] M.T. Chien, The envelope of the generalized numerical range, *Linear and Multilinear Algebra*, **43** (1998), 363-376.

[7] M.T. Chien, H. Nakazato, Boundary generating curves of the c-numerical range, *Linear Algebra and Its Applications*, **294** (1999), 67-84.

[8] M.T. Chien, H. Nakazato,The numerical range of linear pencils of 2-by-2 matrices, *Linear Algebra and Its Applications*, **341** (2002), 69-100.

[9] M.T. Chien, H. Nakazato, P. Psarrakos, The q-numerical range and the Davis-Wielandt shell of reducible 3-by-3 matrices, *Linear and Multilinear Algebra*, **54** (2006), 79-112.

[10] M.T. Chien, B.S. Tam, Circularity of the numerical range, *Linear Algebra and Its Applications*, **201** (1994), 113-133.

[11] A.J. E. Leal Duartes, O Theorem de Tarski e suas Appliçöes em Teoria de Matrices (in Portugese), Doctor Thesis, Coimbra University, Coimbra, 1993.

[12] M. Fiedler, Geometry of the numerical range of matrices, *Linear Algebra and Its Applications*, **37** (1981), 81-96.

[13] A. Hurewitz, R. Courant, *Vorlesungen über allegemeine funktiontheorie und elliptische funktionen*, Springer-Verlag, New York, 1964.

[14] K. Iwasawa, *Algebraic functions*, (Translations of Mathematical Monographs, vol. 1118), American Mathematical Society, Providence, 1993.

[15] D.S. Keeler, L. Rodman, I.M. Spitkovsky, The numerical range of 3×3 matrices, *Linear Algebra and Its Applications*, **252** (1997), 112-139.

[16] R. Kippenhahn: Über den Wertevorrat einer matrix, *Mathematische Nachrichten*, **6** (1951), 193-228.

[17] M. Marcus, C. Pesce, Computer generated numerical ranges and some resulting theorems, *Linear and Multilinear Algebra*, **20** (1987), 121-157.

[18] H. Nakazato, Quartic curves associated to 4×4 matrices, *Science Reports of the Hirosaki University*, **43** (1996), 209-221.

[19] H. Nakazato, Quartic curves associated with some hyperbolic ternary forms, *Bulletin of the Faculty of Science, Technology Hirosaki University*, **6** (2004), 7-15.

[20] H. Shapiro, A conjecture of Kippenhahn about characteristic polynomial of a pencil generated by two hermitian matrices II, *Linear Algebra and Its Applications*, **45** (1982), 97-108.

[21] R.J. Walker, *Algebraic Curves*, Dover Publ. New York (1950).

[22] Z.X. Wang, D.R. Guo, *Special functions*, World Scientific, Singapore (1989).