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A NOTE ON MEAN ERGODIC THEORY FOR WEIGHTED LEBESGUE SPACES

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Abstract: In this paper the mean ergodic theorem and two allied lemmas (see [1], pp. 664-667) in Lebesgue spaces are extended to weighted Lebesgue spaces.

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1. Introduction

In [1], pp. 662-667, Dunford and Schwartz discussed the behaviour of the averages of iterates of a linear operator and then attempted to throw some light upon the problems of statistical mechanics and probability. The conditions of an operator T in an arbitrary complex Banach space Y were given which are necessary and sufficient for the convergence in Y of the averages

$$A(n) = \frac{1}{n}\sum_{j=0}^{n-1} T^j$$

of the iterates of T. The strong convergence of the averages $A(n)$ in $L^p(X, \Sigma, \mu) = L^p(\mu)$ was presented in [1]. Also, these general conditions were interpreted

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for operators in Lebesgue spaces $L^p(X,\Sigma,\mu)$ by Dunford and Schwartz which are encountered in statistical mechanics. The purpose of this paper is to give a formal proof of a mean ergodic theorem for averages of iterates of T in weighted Lebesgue spaces $L^p_w(X,\Sigma,\mu) = L^p_w(\mu)$, where w is a weight function, i.e., w is a measurable function and $w \geq 1$. The spaces $L^p_w(\mu)$ are defined by $L^p_w(\mu) = \{f \mid fw \in L^p(\mu)\}$, whose elements are $Y-$valued and measurable. It is a Banach space under the norm $\|f\|_{p,w} = \|fw\|_p$, see [2].

2. Results

Lemma 1. *Let $Y \neq 0$ be a complex Banach space, (X,Σ,μ) a finite positive measure space and φ a map of X into itself for which $\varphi^{-1}(\Sigma) \subseteq \Sigma$ and there exists a constant $c \geq 1$ satisfying $w(x) \leq w(\varphi(x)) \leq cw(x)$. Then, for $1 \leq p < \infty$, the linear operator T defined in the complex linear space Y^X of all functions on X to Y by*

$$Tf(x) = f(\varphi(x)) \quad x \in X, \quad f \in Y^X \tag{1.1}$$

maps $L^p_w(\mu)$ into itself if and only if

$$\sup_{E\in\Sigma} \frac{\mu\left(\varphi^{-1}(E)\right)}{\mu(E)} = M < \infty. \tag{1.2}$$

Furthermore, when this condition is satisfied T is a continuous linear map in $L^p_w(\mu)$ whose norm satisfies $\frac{1}{c}M^{\frac{1}{p}} \leq \|T\| \leq M^{\frac{1}{p}}$.

Remark 2. If $\mu(E) = 0$ then the ratio $\frac{\mu\left(\varphi^{-1}(E)\right)}{\mu(E)}$ is taken to be zero.

Proof. It is easy to see that when f is a measurable function and φ is defined as above, equality (1.1) makes Tf measurable. The symbol $\|T\|_{p,w}$ will be used for the norm of the operator T, when it is considered as operating in $L^p_w(\mu)$. By the definition of the spaces $L^p_w(\mu)$, we know that $f \in L^p_w(\mu)$ implies $fw \in L^p(\mu)$. Since

$$|f(x)| = |f(x)w(x)| \frac{1}{w(x)} \leq |f(x)w(x)| \quad \text{for all } x \in X,$$

it follows that $f \in L^p(\mu)$ and so $L^p_w(\mu) \subseteq L^p(\mu)$. Besides, for any $f \in L^p_w(\mu)$ we have $\|f\|_{p,w} \geq \|f\|_p$. Suppose T maps $L^p_w(\mu)$ into itself and let $f \in L^p(\mu)$. Then $\frac{f}{w} \in L^p_w(\mu)$, so $T\frac{f}{w} \in L^p_w(\mu)$ and $\left(T\frac{f}{w}\right)w \in L^p(\mu)$ for all $f \in L^p(\mu)$. Then

$$|(Tf)(x)| = \left|\frac{f(\varphi(x))}{w(\varphi(x))}w(\varphi(x))\right| \leq \left|\left(T\frac{f}{w}\right)(x)\,cw(x)\right| = c\left|\left(\left(T\frac{f}{w}\right)w\right)(x)\right|$$

is found. By this inequality $\left\|T\frac{f}{w}\right\|_{p,w} < \infty$ implies that $\|Tf\|_p \leq \left\|c\left(T\frac{f}{w}\right)w\right\|_p = c\left\|T\frac{f}{w}\right\|_{p,w} < \infty$. It follows that $Tf \in L^p(\mu)$. Thus T maps $L^p(\mu)$ into itself.

Conversely, suppose T maps $L^p(\mu)$ into itself and let $f \in L^p_w(\mu)$. Then $fw \in L^p(\mu)$ and so $T(fw) \in L^p(\mu)$ and $\|T(fw)\|_p < \infty$. Hence $\frac{T(fw)}{w} \in L^p_w(\mu)$. Since

$$|(Tf)(x)| = |f(\varphi(x))| \leq \left|\frac{f(\varphi(x))\,w(\varphi(x))}{w(x)}\right| = \left|\frac{T(fw)}{w}(x)\right|,$$

it follows that $\|Tf\|_{p,w} \leq \left\|\frac{T(fw)}{w}\right\|_{p,w} = \|T(fw)\|_p < \infty$. Thus T maps $L^p_w(\mu)$ into itself. We see that T maps $L^p_w(\mu)$ into itself if and only if T maps $L^p(\mu)$ into itself. Consequently, since T maps $L^p_w(\mu)$ into itself implies that T maps $L^p(\mu)$ into itself, we obtain such a constant M by [1, Lemma 7, p. 664]. that proves the first part of the Lemma 1.

For the other side, suppose there exists a constant $M < \infty$ in Lemma 1. Then by [1, Lemma 7, p. 664], $\|T\|_p \leq M^{\frac{1}{p}}$ while T is operating on $L^p(\mu)$. Now let us take any $f \in L^p_w(\mu)$. Then $fw \in L^p(\mu)$ and so

$$\|T(fw)\|_p \leq M^{\frac{1}{p}}\|fw\|_p = M^{\frac{1}{p}}\|f\|_{p,w}.$$

Since $w(x) \leq w(\varphi(x))$, it follows that

$$\begin{aligned}\|Tf\|_{p,w} &= \left[\int_X |f(\varphi(x))|^p w^p(x)\,d\mu(x)\right]^{\frac{1}{p}} \\ &\leq \left[\int_X |f(\varphi(x))|^p w^p(\varphi(x))\,d\mu(x)\right]^{\frac{1}{p}} \\ &= \|T(fw)\|_p \leq M^{\frac{1}{p}}\|f\|_{p,w}, \quad f \in L^p_w(\mu).\end{aligned}$$

Hence $\|T\|_{p,w} \leq M^{\frac{1}{p}} < \infty$, and T is therefore a continuous linear operator in $L^p_w(\mu)$. Now take any $E \in \Sigma$ and a unit vector $u \in Y$. Then $\frac{\chi_E}{w}u \in L^p_w(\mu)$ and so $T\left(\frac{\chi_E}{w}u\right) = \frac{\chi_E(\varphi)}{w(\varphi)}u \in L^p_w(\mu)$. Then

$$\left\|\frac{\chi_E(\varphi)}{w(\varphi)}u\right\|_{p,w} = \left\|T\left(\frac{\chi_E}{w}u\right)\right\|_{p,w} \leq \|T\|_{p,w}\left\|\frac{\chi_E}{w}u\right\|_{p,w},$$

or

$$\int_X \left| \frac{\chi_E(\varphi(x))}{w(\varphi(x))} u \right|^p w^p(x)\, d\mu(x) \le \|T\|_{p,w}^p \int_X \left| \frac{\chi_E(x)}{w(x)} u \right|^p w^p(x)\, d\mu(x),$$

or

$$\int_{\varphi^{-1}(E)} \frac{w^p(x)}{w^p(\varphi(x))} d\mu(x) \le \|T\|_{p,w}^p \int_E d\mu(x) = \|T\|_{p,w}^p \mu(E).$$

This shows that $\mu\left(\varphi^{-1}(E)\right) = 0$ whenever $\mu(E) = 0$, and that

$$M_{p,w} = \sup_{E\in\Sigma} \frac{\int_{\varphi^{-1}(E)} \frac{w^p(x)}{w^p(\varphi(x))} d\mu(x)}{\mu(E)} \le \|T\|_{p,w}^p \tag{1.3}$$

(this ratio is taken to be zero if $\mu(E) = 0$).

Consider now any $f \in L_w^p(\mu)$. Find a sequence of simple functions $\{s_n\}$ such that, for all $x \in X$, $0 \le s_1(x) \le s_2(x) \le .... \le |f(x)|\ ^p w^p(x)$ and $s_n(x) \to |f(x)|\ ^p w^p(x)$. For a fixed n, let $s_n(x) = \sum_{i=1}^{k} \alpha_i \chi_{E_i}(x)$, where $\alpha_i \ge 0$ and $E_1,...., E_k$ are pairwise disjoint members of Σ. Clearly,

$$s_n(\varphi(x)) = \sum_{i=1}^{k} \alpha_i \chi_{E_i}(\varphi(x)) = \sum_{i=1}^{k} \alpha_i \chi_{\varphi^{-1}(E_i)}(x)$$

and we have

$$\begin{aligned} \int_X \frac{s_n(\varphi(x))\, w^p(x)}{w^p(\varphi(x))} d\mu(x) &= \sum_{i=1}^{k} \alpha_i \int_{\varphi^{-1}(E)} \frac{w^p(x)}{w^p(\varphi(x))} d\mu(x) \\ &\le \sum_{i=1}^{k} \alpha_i \mu(E_i)\, M_{p,w} = \int_X s_n(x)\, d\mu(x). \end{aligned}$$

Making $n \to \infty$, by the Monotone Convergence Theorem (see [3]), we get

$$\int_X |f(\varphi(x))|^p w^p(\varphi(x)) \frac{w^p(x)}{w^p(\varphi(x))} d\mu(x) \le M_{p,w} \int_X |f(x)|^p w^p(x)\, d\mu(x),$$

or,

$$\|T\|_{p,w}^p \le M_{p,w} \|f\|_{p,w}^p,$$

for all $f \in L_w^p(\mu)$. Hence $\|T\|_{p,w} \le M_{p,w}^{\frac{1}{p}}$ which by (1.3) gives $\|T\|_{p,w} = M_{p,w}^{\frac{1}{p}}$. Since $\frac{w(x)}{w(\varphi(x))} \ge \frac{1}{c}$, we have

$$M_{p,w}\mu(E) \ge \int_{\varphi^{-1}(E)} \frac{w^p(x)}{w^p(\varphi(x))} d\mu(x) \ge \frac{1}{c^p} \mu\left(\varphi^{-1}(E)\right)$$

for any $E \in \Sigma$ and this implies, by definition of M and $M_{p,w}$, that $M_{p,w} \geq \frac{1}{c^p} M$. Thus,

$$\frac{1}{c^p} M^{\frac{1}{p}} \leq M_{p,w}^{\frac{1}{p}} = \|T\|_{p,w} \leq M^{\frac{1}{p}}. \qquad \square$$

Lemma 3. *Let (X, Σ, μ) be a finite positive measure space, and φ a map of X into itself for which $\varphi^{-1}(\Sigma) \subseteq \Sigma$ and there exists a constant $c \geq 1$ satisfying $w(x) \leq w(\varphi(x)) \leq cw(x)$ for all $x \in X$. Suppose that there is a constant M for which*

$$\frac{1}{n} \sum_{j=0}^{n-1} \mu\left(\varphi^{-j}(E)\right) \leq M\mu(E) \qquad , E \in \Sigma \quad n = 1, 2, \ldots. \tag{3.1}$$

Then, for every p with $1 \leq p < \infty$, the operator T defined by the equation (1.1) maps $L_w^p(\mu)$ into itself, and the averages $A(n)$ are uniformly bounded as operators in $L_w^p(\mu)$.

Proof. By [1, Lemma 8, pp. 666-667], while T maps $L^p(\mu)$ into itself, we observed that $\|A(n)\|_p \leq M^{\frac{1}{p}}$ when $A(n)$ is operating on $L^p(\mu)$ for all n. If $f \in L_w^p(\mu)$ then $fw \in L^p(\mu)$ and we have, taking a unit vector $u \in Y$,

$$\begin{aligned}
\|A(n) f\|_{p,w}^p &= \int_X \left| \frac{1}{n} \sum_{j=0}^{n-1} f\left(\varphi^{-j}(x)\right) \right|^p w^p(x)\, d\mu(x) \\
&\leq \int_X \left[\frac{1}{n} \sum_{j=0}^{n-1} \left| f\left(\varphi^{-j}(x)\right) \right| w(x) \right]^p d\mu(x) \\
&\leq \int_X \left[\frac{1}{n} \sum_{j=0}^{n-1} \left| f\left(\varphi^{-j}(x)\right) \right| w\left(\varphi^{-j}(x)\right) \right]^p d\mu(x) \\
&= \int_X \left| \left[\frac{1}{n} \sum_{j=0}^{n-1} \left| f\left(\varphi^{-j}(x)\right) \right| w\left(\varphi^{-j}(x)\right) u \right] \right|^p d\mu(x) \\
&= \int_X \left| \left[A(n)\left(|fw|\, u \right) \right](x) \right|^p d\mu(x) \\
&= \|A(n)\left(|fw|\, u\right)\|_p^p \leq \|A(n)\|_p^p \, \|\left(|fw|\, u\right)\|_p^p = \|A(n)\|_p^p \, \|f\|_{p,w}^p
\end{aligned}$$

for all $f \in L_w^p(\mu)$. Therefore

$$\|A(n)\|_{p,w} \leq \|A(n)\|_p \leq M^{\frac{1}{p}} \text{ for all } n.$$

Consequently, the averages of iterates $A(n)$ are uniformly bounded as operators in $L_w^p(\mu)$. □

Theorem 4. (Mean Ergodic Theorem) *Let (X, Σ, μ) be a finite positive measure space, and φ be a mapping of X into itself for which $\varphi^{-1}(\Sigma) \subseteq \Sigma$ and there exists a constant $c \geq 1$ satisfying $w(x) \leq w(\varphi(x)) \leq cw(x)$ for all $x \in X$. If the inequality*

$$\frac{1}{n}\sum_{j=0}^{n-1}\mu\left(\varphi^{-j}(E)\right) \leq M\mu(E)\,, \quad E \in \Sigma\,,\ n = 1, 2, ... \tag{4.1}$$

holds while M is a constant independent of E and n, then for every p with $1 \leq p < \infty$, the operator T defined by (1.1) is a continuous linear map in $L_w^p(\mu)$ and the averages $A(n)$ are strongly convergent, as operators of $L_w^p(\mu)$ to $L_w^p(\mu)$.

Proof. If the inequality (4.1) is thought for $n = 2$, we see that

$$\frac{1}{2}\sum_{j=0}^{1}\mu\left(\varphi^{-j}(E)\right) \leq M\mu(E)\,,$$

or,

$$\frac{1}{2}\left[\mu(E) + \mu\left(\varphi^{-1}(E)\right)\right] \leq M\mu(E)\,,$$

or,

$$\frac{\mu\left(\varphi^{-1}(E)\right)}{\mu(E)} \leq (2M - 1)$$

can be written. Since it holds for all $E \in \Sigma$, by Lemma 1 the operator T defined like (1.1) maps $L_w^p(\mu)$ into itself. Since the conditions of Lemma 1 are satisfied, T is a continuous linear map on $L_w^p(\mu)$. If we think of the space of continuous and linear operators from $L_w^p(\mu)$ into $L_w^p(\mu)$, then it is easily seen that the averages $A(n)$ are members of this complete space. Since the averages $A(n)$ are strongly convergent while operating on $L^p(\mu)$ ([1],pp.667,Theorem 9), we can write that the sequence $\{A(n)f\} \subset L^p(\mu)$ converge for all $f \in L^p(\mu)$. By the way, when the averages $A(n)$ are operating on $L_w^p(\mu)$, we showed that $A(n)f \in L_w^p(\mu)$ for all n and for each $f \in L_w^p(\mu)$. Let us take any $f \in L_w^p(\mu)$, then $fw \in L^p(\mu)$ and $\{A(n)(fw)\} \subset L^p(\mu)$ converge for all $f \in L_w^p(\mu)$, since the averages $A(n)$ are strongly convergent in $L^p(\mu)$. It is to say for any $f \in L_w^p(\mu)$ and for each n,

$$\|A(n)f\|_{p,w} = \|(A(n)f)w\|_p \leq \|A(n)(fw)\|_p$$

is found. Hence, it is easy to see that the averages $A\,(n)$ are strongly convergent, as operators of $L_w^p\,(\mu)$ to $L_w^p\,(\mu)$. □

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