International Journal of Pure and Applied Mathematics
Volume 30 No. 4 2006, 483-487

AN EIGENVALUE ESTIMATE FOR MANIFOLDS WITH PARALLEL RICCI TENSOR

Leung-Fu Cheung[1], Pui-Fai Leung[2] §

[1]Department of Mathematics
Chinese University of Hong Kong
N.T., HONG KONG, P.R. CHINA
e-mail: lfcheung@math.cuhk.edu.hk

[2]Department of Mathematics
National University of Singapore
Lower Kent Ridge Road
Singapore, 117543, SINGAPORE
e-mail: matfredl@nus.edu.sg

Abstract: We show that the first eigenvalue of a compact n-dimensional manifold which is not isometric to a sphere and which has parallel Ricci tensor and sectional curvatures bounded below by a positive constant κ is not less than $2n\kappa$.

AMS Subject Classification: 53C20
Key Words: eigenvalues, parallel Ricci tensor

1. Introduction

Let M^n be a smooth compact Riemannian manifold with all sectional curvatures bounded below by a positive constant κ. Let f be an eigenfunction of the Laplacian on M corresponding to the first non-zero eigenvalue λ. The theorem of Lichnerowicz-Obata [1] tells us that $\lambda \geq n\kappa$ and that $\lambda = n\kappa$ if and only if M is isometric to a sphere of constant curvature κ.

It is then natural to ask whether we can improve this lower bound for the case when M is not isometric to a sphere of constant curvature. In this paper, we prove that if M has parallel Ricci tensor and it is not isometric to a sphere

Received: June 23, 2006

§Correspondence author

of constant curvature, then $\lambda \geq 2n\kappa$.

We conjecture that if M has parallel Ricci tensor and it is not isometric to a sphere of constant curvature, then $\lambda \geq 2(n+1)\kappa$ and that $\lambda = 2(n+1)\kappa$ if and only if M is isometric to a real projective space of constant curvature κ.

2. Basic Formulas

We consider a smooth function $f : M^n \to R$. We shall use the following range for indices

$$1 \leq i, j, k, l, m \leq n\,,$$

and summation over repeated indices is assumed.

Take a local orthonormal frame field $\{e_i\}$ for M and denote the corresponding dual frame by $\left\{\omega^i\right\}$. Let $\left\{\omega^i_j\right\}$ be the connection 1-forms on M. So we have $\omega^i_j = -\omega^j_i$.

The structural equations on M are

$$d\omega^i = -\sum \omega^i_j \wedge \omega^j\,, \tag{1}$$

$$d\omega^i_j = -\sum \omega^i_k \wedge \omega^k_j + \frac{1}{2}\sum R^i_{jkl}\omega^k \wedge \omega^l\,, \tag{2}$$

where R^i_{jkl} 's are the curvature components on M.

Let

$$df = \sum f_i \omega^i\,. \tag{3}$$

Exterior differentiate (3) and define f_{ij} by

$$\sum f_{ij}\omega^j = df_i - \sum f_k \omega^k_i\,, \tag{4}$$

we obtain $\sum f_{ij}\omega^j \wedge \omega^i = 0$, and so

$$f_{ij} = f_{ji}\,. \tag{5}$$

Exterior differentiate (4) and define f_{ijk} by

$$\sum f_{ijk}\omega^k = df_{ij} - \sum f_{il}\omega^l_j - \sum f_{lj}\omega^l_i\,, \tag{6}$$

we obtain $\sum \left(f_{ijk} + \frac{1}{2}\sum f_m R^m_{ikj}\right)\omega^k \wedge \omega^j = 0$, and so

$$f_{ijk} = f_{ikj} + \sum f_m R^m_{ijk}\,. \tag{7}$$

Exterior differentiate (6) and define f_{ijkl} by

$$\sum f_{ijkl}\omega^l = df_{ijk} - \sum f_{ijm}\omega_k^m - \sum f_{imk}\omega_j^m - \sum f_{mjk}\omega_i^m\,, \tag{8}$$

we obtain in a similar way

$$f_{ijkl} = f_{ijlk} + \sum f_{im}R^m_{jkl} + \sum f_{mj}R^m_{ikl}\,. \tag{9}$$

Proposition 1.

$$\frac{1}{2}\triangle\left[\sum (f_{ij})^2\right] = \sum (f_{ijk})^2 + \sum f_{ij}f_{kkij} + 2\sum f_{ij}f_{km}R^m_{ijk} \\ + 2\sum f_{ij}f_{mi}R^m_{kjk} + \sum f_{ij}f_m R^m_{ijkk} + \sum f_{ij}f_m R^m_{kikj}\,,$$

where R^m_{ijkk} and R^m_{kikj} are the covariant derivatives of the curvature tensor on M.

Proof. Applying (5), (7) and (9), we have the following calculation

$$\begin{aligned}
\frac{1}{2}\triangle\left[\sum (f_{ij})^2\right] &= \sum (f_{ijk})^2 + \sum f_{ij}f_{ijkk} \\
&= \sum (f_{ijk})^2 + \sum f_{ij}\left(f_{ikj} + \sum f_m R^m_{ijk}\right)_k \\
&= \sum (f_{ijk})^2 + \sum f_{ij}f_{ikjk} + \sum f_{ij}f_{mk}R^m_{ijk} + \sum f_{ij}f_m R^m_{ijkk} \\
&= \sum (f_{ijk})^2 + \sum f_{ij}\left(f_{ikkj} + \sum f_{im}R^m_{kjk} + \sum f_{mk}R^m_{ijk}\right) \\
&\quad + \sum f_{ij}f_{mk}R^m_{ijk} + \sum f_{ij}f_m R^m_{ijkk} \\
&= \sum (f_{ijk})^2 + \sum f_{ij}f_{kikj} + \sum f_{ij}f_{im}R^m_{kjk} + 2\sum f_{ij}f_{mk}R^m_{ijk} + \sum f_{ij}f_m R^m_{ijkk} \\
&= \sum (f_{ijk})^2 + \sum f_{ij}\left(f_{kki} + \sum f_m R^m_{kik}\right)_j + \sum f_{ij}f_{im}R^m_{kjk} \\
&\quad + 2\sum f_{ij}f_{mk}R^m_{ijk} + \sum f_{ij}f_m R^m_{ijkk} \\
&= \sum (f_{ijk})^2 + \sum f_{ij}f_{kkij} + \sum f_{ij}f_{mj}R^m_{kik} + \sum f_{ij}f_m R^m_{kikj} \\
&\quad + \sum f_{ij}f_{im}R^m_{kjk} + 2\sum f_{ij}f_{mk}R^m_{ijk} + \sum f_{ij}f_m R^m_{ijkk} \\
&= \sum (f_{ijk})^2 + \sum f_{ij}f_{kkij} + 2\sum f_{ij}f_{km}R^m_{ijk} + 2\sum f_{ij}f_{mi}R^m_{kjk} \\
&\quad + \sum f_{ij}f_m R^m_{ijkk} + \sum f_{ij}f_m R^m_{kikj}\,.
\end{aligned}$$

3. Statement and Proof of Theorem

Theorem. *Let M^n be an n-dimensional compact manifold whose sectional curvatures are bounded from below by a positive constant κ. Suppose M has parallel Ricci tensor and it is not isometric to a sphere of constant curvature, then the first non-zero eigenvalue λ of M satisfies*

$$\lambda \geq 2n\kappa.$$

Proof. Let f be an eigenfuction corresponding to λ. So we have $\triangle f + \lambda f = 0$ on M. Applying Proposition 1 to f, we have

$$\begin{aligned}\frac{1}{2}\triangle\left[\sum (f_{ij})^2\right] = \sum (f_{ijk})^2 - \lambda \sum (f_{ij})^2 + 2\sum f_{ij} f_{km} R^m_{ijk} \\ + 2\sum f_{ij} f_{mi} R^m_{kjk} + \sum f_{ij} f_m R^m_{ijkk} + \sum f_{ij} f_m R^m_{kikj}\,.\end{aligned}$$

Since the Ricci tensor on M is parallel, it follows from the Bianchi identities that

$$\sum_k R^m_{kikj} = \sum_k R^m_{ijkk} = 0\,,$$

and so we have

$$\begin{aligned}&\frac{1}{2}\triangle\left[\sum (f_{ij})^2\right] \\ &\quad = \sum (f_{ijk})^2 - \lambda \sum (f_{ij})^2 + 2\sum f_{ij} f_{km} R^m_{ijk} + 2\sum f_{ij} f_{mi} R^m_{kjk}\,. \qquad (10)\end{aligned}$$

We first observe that

$$\sum (f_{ijk})^2 \geq \sum_{i,k} (f_{iik})^2 \geq \frac{1}{n}\sum_k \left(\sum_i f_{iik}\right)^2 = \frac{\lambda^2}{n}\,|df|^2\,,$$

and so

$$\int_M \sum (f_{ijk})^2 \geq \int_M \frac{\lambda^2}{n}\,|df|^2 = \int_M \frac{\lambda^3}{n} f^2\,. \qquad (11)$$

Next we shall estimate $2\sum f_{ij} f_{km} R^m_{ijk} + 2\sum f_{ij} f_{mi} R^m_{kjk}$ at an arbitrarily fixed point $x \in M$. Since (f_{ij}) is a symmetric $n \times n$ matrix we may assume it to be diagonalized at x for a suitable choice of orthonormal basis for TM_x. We may then let $f_{ij} = \mu_i \delta_{ij}$ at x. Therefore at x we have

$$2\sum f_{ij}f_{km}R^m_{ijk} + 2\sum f_{ij}f_{mi}R^m_{kjk} = 2\sum \mu_i\mu_k R^k_{iik} + 2\sum (\mu_i)^2 R^i_{kik}$$
$$= \sum_{i,k} (\mu_i - \mu_k)^2 R^i_{kik} \geq \sum_{i,k} (\mu_i - \mu_k)^2 \kappa$$
$$= \left(2n\kappa \sum (f_{ij})^2\right) - 2\kappa\lambda^2 f^2 .$$

Hence, integrating (10) over M, using Green's Theorem, (11) and the above, we obtain

$$0 \geq (2n\kappa - \lambda) \int_M \left[\sum (f_{ij})^2 - \frac{\lambda^2}{n} f^2\right] . \qquad (12)$$

Now we observe that

$$\sum (f_{ij})^2 \geq \sum_i (f_{ii})^2 \geq \frac{1}{n}\left(\sum_i f_{ii}\right)^2 = \frac{\lambda^2}{n} f^2$$

and equality holds if and only if

$$f_{ij} = \frac{\lambda}{n} f \delta_{ij}.$$

However, by a theorem of Obata [2], if M is not isometric to a sphere of constant curvature then the equation $f_{ij} = \frac{\lambda}{n} f \delta_{ij}$ does not have a non-constant solution. Therefore we must have

$$\sum (f_{ij})^2 > \frac{\lambda^2}{n} f^2$$

and by (12) this implies that $\lambda \geq 2n\kappa$. □

References

[1] M. Berger, P. Gauduchon, E. Mazet, *Le Spectre d'une Variete Riemannienne*, Springer-Verlag, 179-185.

[2] M. Obata, Certain conditions for a Riemannian manifold to be isometric with a sphere, *J. Math. Soc. Japan*, **14** (1962), 333-340.