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# ASYMPTOTICS ON q-ALTERNATIVE SUMS

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**Abstract:** In this article, we extend q-Rice's formula and give asymptotic values of alternating sums involving q-binomial coefficients by the q-Rice's formula and residue theorem.

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## 1. Introduction

Rice's method has been a tremendous success in computer science, see [3] and [**?**]; the latter book devotes a whole chapter to it. It is based on the integral representation

$$\sum_{k=n_0}^{n}\binom{n}{k}(-1)^k f(k)=\frac{(-1)^n}{2\pi i}\int_{\mathbb{C}} f(z)\frac{n!dz}{z(z-1)\dots(z-n)},$$

where $\mathbb{C}$ (positively oriented) encloses the poles $1,2,\dots,n$ and no others.

Helmut Prodinger in [4] has given the follows q-Rice's formula:

$$\sum_{k=1}^{n}(-1)^{k-1}q^{\binom{k}{2}}\begin{bmatrix} n \\ k \end{bmatrix}_q f(q^{-k})=\frac{(-1)^n}{2\pi i}\int_{\mathbb{C}} f(z)\frac{(q;q)_n}{(z;q)_{n+1}},$$

where $\mathbb{C}$ encloses the poles $q^{-1},q^{-2},\dots,q^{-n}$ and no others, and used the formula

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derived some identities.

In this paper we extend q-Rice's formula in [4] and we get asymptotic values of q-alternative sums using this formula and residue theorem.

## 2. Preliminaries

**Lemma 1.** *Let $f(z)$ be analytic in a domain $D$ except for a finite number isolated singularities $z_1, z_2, \dots, z_n$, $\mathbb{C}$ be a positively oriented simple closed curve in the domain $D$ which encircles all singularities, then*

$$\oint_{\mathbb{C}} f(z)dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}[f(z), z_k]\,.$$

**Definition 1.** (see 3) q-Gamma function is defined as:

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \quad 0 < q < 1\,.$$

It satisfies the relations:

$$\frac{\Gamma_q'(x)}{\Gamma_q(x)} = -\ln(1-q) + \ln q \sum_{k=0}^{\infty} \frac{q^{k+x}}{1-q^{k+x}}\,. \tag{1}$$

**Definition 2.** (see [4]) q-binomial coefficient is defined as:

$$\left[\begin{array}{c} n \\ k \end{array}\right]_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}, \quad 0 < q < 1,$$

with $(z;q)_n = (1-z)(1-zq)\dots(1-zq^{n-1})$.

**Definition 3.** (see [3]) The modified Bell polynomials $L_m = L_m(x_1, x_2, \dots, x_m)$ are defined as

$$\exp(\sum_{k=1}^{\infty} x_k \frac{t^k}{k}) = 1 + \sum_{k=1}^{\infty} L_m t^m. \tag{2}$$

It is rather technical than difficult to proof that in general

$$L_m(x_1, x_2, \dots, x_m) = \sum_{m_1+2m_2+\dots=m} \frac{1}{m_1!m_2!\dots} \left(\frac{x_1}{1}\right)^{m_1} \left(\frac{x_2}{2}\right)^{m_2} \dots\,,$$

and to get an idea of them, we have

$$\exp(\sum_{k=1}^{\infty} x_k \frac{t^k}{k}) = 1 + x_1 t + \left(\frac{x_2}{2} + \frac{x_1^2}{2}\right) t^2 + \left(\frac{x_3}{3} + \frac{x_1 x_2}{2} + \frac{x_1^3}{6}\right) t^3$$
$$+ \left(\frac{x_4}{4} + \frac{x_1 x_3}{3} + \frac{x_2^2}{8} + \frac{x_2 x_1^2}{4} + \frac{x_1^4}{24}\right) t^4 + \cdots .$$

**Definition 4.** (see [3]) A function $f(z)$ in an unbounded domain $\Omega$ is said to have polynomial growth, if for some $r$ the formula $|f(z)| = O(|z|^r)$ holds as $z \to \infty$ in $\Omega$. We also call $r$ the degree of $f(z)$.

**Property 1.** (see [3]) If $f(z)$ is of polynomial growth (is of finite degree) in the half-plane $\Re(z) \geq c$ for some $c < n_0$, we have the alternative representation

$$\sum_{k=n_0}^{n} \binom{n}{k} (-1)^k f(k) = -\frac{(-1)^n}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(z) \frac{n! dz}{z(z-1)\dots(z-n)}, \tag{3}$$

valid for n large enough, namely as soon as $n > r + 1$.

## 3. Main Results

**Theorem 1.** *Let $f(z)$ be an analytic function defined in a neighborhood $\Omega$ of the positive real axis $[0, \infty)$. Let $\mathbb{C}$ be a contour enclosing $q^{-n_0}, q^{-n_0-1} \dots, q^{-(np+h)}$ ($n_0 \geq 1, p > 0, h \geq 0$ are integers) but does not include any of the integers $q^0, q^{-1} \dots, q^{-(n_0-1)}$ and not include singularity of $f(z)$. Then*

$$\sum_{k=n_0}^{n} (-1)^{k-1} q^{\binom{k}{2}} \left[ \begin{array}{c} np+h \\ k \end{array} \right]_q f(q^{-k}) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{(q;q)_{np+h}}{(z;q)_{np+h+1}} f(z) dz . \tag{4}$$

*Proof.* We apply Residue Theorem. Taking into account contribution of the simple poles at the $q^{-n_0}, q^{-n_0-1} \dots, q^{-(np+h)}$. The integral equals the sum of the residues of the integrand multiply $2\pi i$. Then we have:

$$\mathrm{Res}_{z=q^{-k}} \frac{(q;q)_{np+h}}{(z;q)_{np+h+1}} f(z) = \mathrm{Res}_{z=q^{-k}} \frac{1}{1 - zq^k}$$
$$\times \left( \frac{f(z)(q;q)_{np+h}}{(1-z)(1-zq)\dots(1-zq^{k-1})(1-zq^{k+1})\dots(1-zq^{np+h})} \right)$$
$$= (-1)^{k-1} q^{\binom{k}{2}} \left[ \begin{array}{c} np+h \\ k \end{array} \right]_q f(q^{-k}) .$$

Simple summation for $k = n_0, \dots, n$, which completes the proof. It is the conclusion of [4] when $p = 1$, $h = 0$. $\square$

Next, we consider the basic case of a rational function.

**Theorem 2.** *Let $f(z)$ be a rational function which is analytic in a neighborhood of $[n_0, +\infty)$. If $n$ is big enough we have*

$$\sum_{k=n_0}^{n} (-1)^{k-1} q^{\binom{k}{2}} \begin{bmatrix} np+h \\ k \end{bmatrix}_q f(q^{-k}) = -\sum_{z} \mathrm{Res} \frac{(q;q)_{np+h}}{(z;q)_{np+h+1}} f(z)\,, \tag{5}$$

*where the sum is extended to all poles $z$ of $\frac{f(z)(q;q)_{np+h}}{(z;q)_{np+h+1}}$ not in*

$$\{q^{-n_0}, q^{-n_0-1} \dots, q^{-(np+h)}\}.$$

*Proof.* The proof is verbatim as in [3]: A rational function is of polynomial growth, i.e. $f(z) = O(|z|^r)$ for some $r$. If $n > r+1$, then the integral in (4), taken over a circle of radius $R$ (avoiding poles of the integrand), tends to zero for $R \to \infty$. By the Reside Theorem, the integral also equals $\sum_{k=n_0}^{n} (-1)^{k-1} q^{\binom{k}{2}} \begin{bmatrix} np+h \\ k \end{bmatrix}_q f(q^{-k})$ plus the sum of the residues of (5) at the other poles of the integrand.

As a next step, we try to express the residues. As every rational function can be expressed as a linear combination of terms of the form $B(z-b)^{-r}$, where $r \in N_0$, we only have to consider function of this type.

**Proposition.** *$\alpha \neq q^{-k} (k = 0, 1 \dots np+h)$ be a complex number, then*

$$T(\alpha) = \mathrm{Res}_\alpha \frac{1}{(z-\alpha)^r} \frac{(q;q)_{np+h}}{(z;q)_{np+h+1}}\,,$$

*has the following asymptotic (as $n \to \infty$)*

$$\begin{aligned} T(\alpha) &= \frac{(q;q)_{np+h}}{(\alpha;q)_{np+h+1}} L_{r-1}(H_{1q}^{np+h+1}(\alpha), H_{2q}^{np+h+1}(\alpha), \dots) \\ &= \exp(\sum_{k=1}^{\infty} \frac{1}{k} (\frac{\alpha^k - q^k}{1-q^k}) (1 + O(q^{np})) L_{r-1}( \frac{\ln(1-\alpha)}{\alpha \ln q} + \frac{1}{2} \frac{1}{1-\alpha} + O(\frac{1}{n}), \\ &\qquad H_{2q}^{np+h+1}(\alpha), \dots) = \exp\left( \sum_{k=1}^{\infty} \frac{1}{k} \frac{\alpha^k - q^k}{1-q^k} \right) \\ &\qquad \times \frac{1}{(r-1)!} \left( \frac{\ln(1-\alpha)}{\alpha \ln q} + \frac{1}{2} \frac{1}{1-\alpha} \right)^{r-1} (1 + O(\frac{1}{n}))\,. \end{aligned}$$

*Proof.* The residue computation reduces to coefficient extractions:

$$\begin{aligned}
T(\alpha) &= [(z-\alpha)^{r-1}]\frac{(q;q)_{np+h}}{(1-z)(1-zq)\dots(1-zq^{np+h})} \\
&= (q;q)_{np+h}[t^{r-1}]\frac{1}{(1-\alpha-t)(1-\alpha q-tq)\dots(1-\alpha q^{np+h}-tq^{np+h})} \\
&= \frac{(q;q)_{np+h}}{(\alpha;q)_{np+h+1}}[t^{r-1}]\exp(-\sum_{i=1}^{np+h+1}\ln(1-\frac{q^{i-1}t}{1-\alpha q^{i-1}})) \\
&= \frac{(q;q)_{np+h}}{(\alpha;q)_{np+h+1}}[t^{r-1}]\exp(\sum_{m=1}^{\infty}H_{mq}^{np+h+1}(\alpha)\frac{t^m}{m}) \\
&= \frac{(q;q)_{np+h}}{(\alpha;q)_{np+h+1}}L_{r-1}\left(H_{1q}^{np+h+1}(\alpha),H_{2q}^{np+h+1}(\alpha),\dots\right),
\end{aligned}$$

where $H_{mq}^{np+h+1}(\alpha)=\sum_{i=1}^{np+h+1}(\frac{q^{i-1}}{1-\alpha q^{i-1}})^m$. The final form results from asymptotic values of $H_{1q}^{np+h+1}(\alpha)$ and $\frac{(q;q)_{np+h}}{(\alpha;q)_{np+h+1}}$.

In fact, by Euler-Maclaurin summation formula

$$H_{1q}^{np+h+1}(\alpha)=\frac{\ln(1-\alpha)}{\alpha\ln q}+\frac{1}{2}\frac{1}{1-\alpha}+O(\frac{1}{n}).$$

But

$$\ln(q;q)_{np+h}=-\sum_{m=1}^{\infty}(\sum_{i=1}^{np+h}q^{im})\frac{1}{m}=-\sum_{k=1}^{\infty}\frac{1}{k}\frac{q^k}{1-q^k}+O(q^{np}).$$

Therefore

$$(q;q)_{np+h}=\exp(-\sum_{k=1}^{\infty}\frac{1}{k}\frac{q^k}{1-q^k})(1+O(q^{np})).$$

Using the same methods we get

$$(\alpha;q)_{np+h+1}=\exp(-\sum_{k=1}^{\infty}\frac{1}{k}\frac{\alpha^k}{1-q^k})(1+O(q^{np})).$$ □

**Example 1.** The asymptotic analysis of q-alternative sums

$$s(m)=\sum_{k=1}^{n}\frac{(-1)^{k-1}q^{\binom{k}{2}+km}}{1-q^k}\begin{bmatrix}np+h\\k\end{bmatrix}_q.$$

By setting $f(z)=\frac{1}{(z-1)^m}$ in Theorem 2 and using Euler-Maclaurin summation formula:

$$s(m)=-\mathrm{Res}_{z=1}(\frac{1}{(z-1)^m}\frac{(q;q)_{np+h}}{(z;q)_{np+h+1}})=[t^m]\frac{(q;q)_{np+h}}{(1-zq)\dots(1-zq^{np+h})}$$
$$=[t^m]\exp(\sum_{k=1}^{\infty}\sum_{i=1}^{np+h}(\frac{q^i}{1-q^i})^m\frac{t^m}{m})=[t^m]\exp(\sum_{m=1}^{\infty}H_{mq}^{np+h}\frac{t^m}{m})$$
$$=L_m(H_{1q}^{np+h},H_{2q}^{np+h},\dots)=\frac{1}{m!}\left(\frac{\ln(1-q)}{\ln q}\right)^m(1+O(\frac{1}{n}).$$

In particular, when $m=1$ by (1) or Euler-Maclaurin summation formula we get:

$$\sum_{k=1}^{n}\frac{(-1)^{k-1}q^{\binom{k+1}{2}}}{1-q^k}\left[\begin{array}{c}np+h\\k\end{array}\right]_q=\frac{\ln(1-q)}{\ln q}+\frac{1}{\ln q}\frac{\Gamma_q'(1)}{\Gamma_q(1)}+O(q^{np}).$$

**Example 2.** The asymptotic analysis of q-alternative sums

$$s=\sum_{k=1}^{n}\frac{(-1)^{k-1}q^{\binom{k+1}{2}+km}}{1-q^{k+m}}\left[\begin{array}{c}np+h\\k\end{array}\right]_q,$$

where $m\neq 0,1,\dots,np+h$ is positive integer.

Setting $f(z)=\frac{1}{z-q^m}$ in Theorem 2, we obtain:

$$s=-\mathrm{Res}_{z=1}(\frac{1}{z-q^m}\frac{(q;q)_{np+h}}{(z;q)_{np+h+1}})-\mathrm{Res}_{z=q^m}(\frac{1}{z-q^m}\frac{(q;q)_{np+h}}{(z;q)_{np+h+1}})$$
$$=\frac{1}{1-q^m}-\frac{(q;q)_{np+h}}{(q^m;q)_{np+h+1}}=\frac{1}{1-q^m}-\exp\left(\sum_{k=1}^{\infty}\frac{1}{k}\frac{q^{mk}-q^k}{1-q^k}\right)(1+O(q^{np})).$$

## References

[1] L. Comtet, *Advanced Combinatorics*, D. Reidel Publishing Co., Dordrecht (1974).

[2] W. Edwin Clar, Mourad E.H. Ismail, Binomial and q-binomial coefficient inequalities related to the Hamiltonicity of the Kneser graphs and their q-analogues, *Journal of Cominatorial Theory*, Series A, **76** (1996), 83-98.

[3] P. Flajolet, R. Sedgewick, Mellin transforms asymptotics: Finite differences and Rice's integrals, *Theoretical Computer Science*, **144** (1995), 101-124.

[4] P. Helmut, Some applications of the $q$-Rice formula, Analysis of algorithms, *Random Structures Algorithms*, **19** (2001), 552-557.

[5] A. Odlyzko, Asymptotic enumeration methods, In: *Handbook of Combinatorics* (Ed-s: L. Lovasz, R. Graham, M. Grötschel), Amsterdam, The Netherlands, Elsevier (1995).