International Journal of Pure and Applied Mathematics

Volume 30 No. 4 2006, 497-506

WEIL UNIFORMITIES IN FUZZY SET THEORY

Maryam Yaghoubi[1 §], S.B. Nimse[2]

[1]Department of Mathematics
University of Pune
C/6, Mantri 1, Panchawati, Pune, 411008, INDIA
e-mail: yaghoobim@yahoo.com

[2]College of Commerce and Science
Ahmednagar, INDIA
e-mail: dr.sbnimse@rediffmail.com

Abstract: In this paper we present a theory of fuzzy uniform space in the style of Weil which is equivalent to the ones of Hutton [5] and we show that every (weil) uniformity ($\mathcal{U}$) on a set X corresponds a fuzzy uniformity ($\mathcal{E}$) and we show that to every fuzzy weil uniformity ($\mathcal{E}$) on X corresponds a uniformly ($\mathcal{U}$).

AMS Subject Classification: 54A40, 06D20
Key Words: Weil entourages, fuzzy uniformity, fuzzy Weil uniformity

1. Introduction

Fuzzy uniformity was studied by four authors: B. Hutton, R. Lowen, U. Höhle and A.K. Kastara. While U. Höhle [3, 4] and R. Lowen [9] starting point took a certain counterpart of the filter approach to based on uniform covers (see Tukey [15]). B. Hutton [5] and A.K. Katsara [8] presented an equivalent notion of fuzzy uniformity in terms of certain families of maps from the L^X into itself. Recently, other authors continued this concept as an approach to fuzzy L-uniform space.

The content of this paper is summarized as follows: In the Section 2 we recall some preliminary ideas. In Section 3, we present a theory of fuzzy uniform space via entourage in the style of Weil that is called fuzzy Weil uniform space, in

Received: June 26, 2006

§Correspondence author

Section 4 we show the relation between fuzzy Weil uniformity on a set X and Hutton uniformity on a set X. We show that every (Weil) uniformity ($\mathcal{U}$) on a set X corresponds a fuzzy Weil uniformity ($\mathcal{E}$) and every fuzzy Weil uniformity ($\mathcal{E}$) on a set X corresponds a uniformly ($\mathcal{U}$) in Section 5.

2. Preliminaries

Let X be a nonempty set. Let a complete lattice $L = (L, \leq, \vee, \wedge, ')$ be a completely distributive lattice (frame) with an order − reversing involution $'$, 0 and 1 denote the least and the greatest element in L.

Definition 1.1. (see [5]) Hutton generalized the concept of uniformity to fuzzy case as follows. Let $H(L,X)$ be the set of maps $e : L^X \to L^X$ which satisfies: (1) $e(\phi) = \phi$, (2) $e(\mu) \geq \mu$, (3) $e(\vee_{i\in\Gamma}\mu_i) = \vee_{i\in\Gamma}e(\mu_i)$ for $\{\mu_i\}_{i\in\Gamma} \subset L^X$. For every $e, h \in H(L,X)$, we have the following properties:

(1) for all $\mu \in L^X, e^{-1}(\mu) = \wedge\{\rho \in L^X / e(\rho') \leq \mu'\}$. Then $e^{-1} \in H(L,X)$.

(2) for all $\mu \in L^X, e \wedge h(\mu) = \wedge\{e(\mu_1) \vee h(\mu_2)/\mu_1 \vee \mu_2 = \mu\}$ then $e \vee h \in H(L,X)$.

(3) for all $\mu \in L^X, eoh(\mu) = e(h(\mu))$. Then $eoh \in H(L,X)$.

A Hutton uniformity on X is a subset D of $H(L,X)$ such that it satisfies the following axioms:

HU_1) $D \neq \emptyset$.

HU_2) $e \in D$ and $e \leq h$ implies $h \in D$.

HU_3) $e \in D$ and $h \in D$ implies $e \wedge h \in D$.

HU_4) $e \in D$ implies there exist $h \in D$ such that $hoh \leq e$.

HU_5) $e \in D$ implies $e^{-1} \in D$.

Definition 1.2. Let $(X,D), (X',D')$ be two Hutton uniform spaces A mapping $f : X \to X'$ is said to be uniformly homomorphism such that for every $e \in D$, there exists $g \in D'$ with $g\cdot \overrightarrow{f} \leq \overrightarrow{f} \cdot e$.

Definition 1.3. (see [7] Let L be a frame. Recall [7] that the coproduct of the frame L by itself

$$L \overset{u_1^L}{\to} L \oplus L \overset{u_2^L}{\to} L$$

can be constructed as follows:

Take the Cartesian product $L \times L$ with the usual order A down set A of $L \times L$ is a C − ideal if $(\{x\} \times S \subseteq A \Rightarrow (x, \vee S) \in A)$ and $(S \times \{y\} \subseteq A \Rightarrow (\vee S, y) \in$

A). Put $L \oplus L$ as the frame of all C-ideals of $L \times L$. Observe that the case $S = \phi$ implies that every C-ideal contains the set $\mathbb{O} := \downarrow \{(1,0)\} \cup \downarrow \{(0,1)\}$. Obviously, each $\downarrow \{(x,y)\} \cup \mathbb{O}$ is a C-ideal. It is denoted by $x \oplus y$. Finally put $u_1^L(x) = x \oplus 1$ and $u_2^L(y) = 1 \oplus y$.

The following clear facts are useful:

— For every $A \in L \oplus L, A = \vee\{x \oplus y/(x,y) \in A\}$ and so every element of $L \oplus L$ is join-generated by some family of elements $x \oplus y$.

— $\mathbb{O} \neq x \oplus y \subseteq z \oplus w$ implies $x \leq z$ and $y \leq w$.

For any frame homomorphism $f : L \to M$, we write $f \oplus f : L \oplus L \to M \oplus M$ for the frame homomorphism given by $(f \oplus f) \cdot u_i^L = u_i^M \cdot f$ $(i = 1,2)$. Obviously, $(f \oplus f)(\vee_\gamma(x_\gamma \oplus y_\gamma) = \vee_\gamma(f(x_\gamma) \oplus f(y_\gamma))$.

Given A, B in the lattice $D(L \times L)$ of all down-sets $L \times L$ we denoted by $k(A) = \cap\{B \in L \oplus L/A \subseteq B\}$ the C-ideal generated by A and by AoB the C-ideal generated by $A \cdot B = \{(x,y) \in L \times L/\exists z \in L \setminus \{0\} : (x,z) \in A, (z,y) \in B\}$, that is, $\vee\{x \oplus y/\exists z \in L \setminus \{0\} : (x,z) \in A, (z,y) \in B\}$.

Lemma 1.4. (see [10]) *For any* $A, B \in D(L \times L) \cdot k(A)ok(B) = AoB$.

Definition 1.5. (see [10]) For $A \subset L$ and $x, y \in L$ we define A is called a cover of L if $\vee \ A = 1$

$$st(x, A) = \vee\{y \in L/(y,y) \in A, y \wedge x \neq 0\}.$$

3. Fuzzy Weil Uniform Space

Since L^X is a frame [5] $\overline{0}$ and $\overline{1}$ denote the least and the greatest element in L^X we can define Weil entourage on L^X.

Definition 2.1. $E \in L^X \oplus L^X$ is Weil entourage of L^X if and only if $\{\mu \in L^X/(\mu,\mu) \in E\}$ is a cover of L^X. That is $\vee\{\mu \in L^X/(\mu,\mu) \in E\} = \overline{1}$.

The collection $wE_n(L^X)$ of all Weil entourage of L^X may be partially ordered by inclusion.

Definition 2.2. We define the composition of fuzzy Weil entourage as follows:

$$EoF = \vee\{f \oplus g/\exists h \in L^X \setminus \overline{0}, (f,h) \in E, (h,g) \in F\}$$

the inverse of a fuzzy Weil entourage E has the natural definition $E^{-1} = \{(g,f)/(f,g) \in E\}$.

We also consider a new partial order in L^X, induced by a family $\mathcal{E}$ of fuzzy Weil entourages:

$g \overset{\mathcal{E}}{\Delta} f$ (g is $\mathcal{E}$ - strongly below f) if there is $E \in \mathcal{E}$ such that $Eo(g \oplus g) \subseteq f \oplus f$.

When $\mathcal{E}$ is symmetric ($E \in \mathcal{E}$ implies $E^{-1} \in \mathcal{E}$) this is equivalent to saying there is $E \in \mathcal{E}$ such that $(f \oplus f)oE \subseteq g \oplus sg$.

Definition 2.3. Let X be a nonempty set and $\mathcal{E} \subset wEnt(L^X)$ we say $(X, \mathcal{E})$ is a fuzzy Weil uniform space if

$FwE1$) $\mathcal{E}$ is a filter of $(wEnt(L^X), \subseteq)$.

$FwE2$) for each $E \in \mathcal{E}$ there is $F \in \mathcal{E}$ such that $FoF \subseteq E$.

$FwE3$) for any $E \in \mathcal{E}$, E^{-1} is also in $\mathcal{E}$.

Definition 2.4. Let $(X, \mathcal{E}), (X', \mathcal{E}')$ be two fuzzy Weil uniform space. A mapping $f : X \to X'$ is said to be uniformly homomorphic such that $(\overrightarrow{f} \oplus \overrightarrow{f})(E) \in \mathcal{E}'$ whenever $E \in \mathcal{E}$, $\overrightarrow{f}: L^X \to L^Y$, $\overrightarrow{f}(\mu)(y) = \vee_{f(x)=y}\mu(x)$.

We will denote by F Weil - UNIF the category whose objects are F Weil uniform spaces and morphisms are uniformly homomorphisms mappings.

4. Relation Between Hutton Uniformities and Fuzzy Weil Uniformities

We want to show that the category H-UNIF is isomorphic to a category F Weil - UNIF. The functor $\phi : F$ Weil - UNIF $\to$ Hutt-UNIF.

Theorem 3.1. *Let $\mathcal{E}$ be a fuzzy Weil uniformity on a space X defined for each $E \in \mathcal{E}$.*

$e_E : L^X \to L^X$ by $e_E(\mu) = St(\mu, E)$ and denote the set $\{e_E/E \in \mathcal{E}\}$ by $D_{\mathcal{E}}$. Then $D_{\mathcal{E}}$ is a Hutton uniformity on a space X.

Proof. $D_{\mathcal{E}}$ satisfies the axiom (1)-(3).

(1) for all $E \in \mathcal{E}$ $\;e_E(\phi) = st(\phi, E) = \phi$.

(2) for all $E \in \mathcal{E}$ $\;e_E(\mu) = st(\mu, E) \geq \mu$.

(3) for all $E \in \mathcal{E}$ $\;e_E(\vee_i\mu_i) = st(\vee_i\mu_i, E) = \vee_i st(\mu_i, E)$.

HU1) $D_{\mathcal{E}} \neq \phi$ since $\mathbb{D} \in \mathcal{E}$ then $E_{\mathbb{D}} \in D_{\mathcal{E}}$.

HU2, HU3) $e_E, e_F \in D_{\mathcal{E}}$ then $E, F \in \mathcal{E}$ in order to prove that $D_{\mathcal{E}}$ is a filter basis just take for some Weil entourage G such that $G \subseteq E \cap F$.

HU_4) $e_E \in D_{\mathcal{E}}$ consider $F \in \mathcal{E}$ such that $F^2 \subseteq E$ we observe $st(st(\mu, F), F) \leq st(\mu, F^2)$

$$st(st(\mu, F), F) = \vee\{\lambda \in L^X/(\lambda, \lambda) \in F, \lambda \wedge st(\mu, F) \neq \overline{0}\}$$

Consider $\lambda \in L^X$ with $(\lambda, \lambda) \in F$ and $\lambda \wedge st(\mu, F) \neq \overline{0}$. Then there is $\gamma \in L^X$ such that $(\gamma, \gamma) \in F, (\gamma \wedge \mu) \neq \overline{0}$ and $(\gamma \wedge \lambda) \neq \overline{0}$, therefore $(\lambda, \lambda \wedge \gamma) \in F$ and $(\lambda \wedge \gamma, \gamma) \in F$ thus $(\lambda, \gamma) \in F^2$. Similarly $(\gamma, \lambda) \in F^2$. Also $(\lambda, \lambda), (\gamma, \gamma) \in F^2$. But F^2 is a C-ideal so $(\lambda \vee \gamma, \lambda \vee \gamma) \in F^2$. In conclusion $(\lambda \vee \gamma, \lambda \vee \gamma) \in F^2$ and

$(\lambda \vee \gamma) \wedge \mu \geq \gamma \wedge \mu \neq \overline{0}$, hence $\lambda \leq st(\mu, F^2)$ and $st(st(\mu, F), F) \leq st(\mu, F^2)$. Hence $e_F o e_F \leq e_{F^2} \leq e_E$.

$HU_5)$ $e_E \in D_{\mathcal{E}}$ we have that

$$\lambda \wedge e_E(\mu) = \lambda \wedge st(\mu, E) = \overline{0} \Leftrightarrow \vee\{\lambda \wedge \gamma/(\gamma,\gamma) \in E, \gamma \wedge \mu \neq \overline{0}\} = \overline{0}\,,$$

$$e_E(\lambda) \wedge \mu = st(\lambda, E) \wedge \mu = \overline{0} \Leftrightarrow \vee\{\gamma \wedge \mu/(\gamma,\gamma) \in E, \gamma \wedge \lambda \neq \overline{0}\} = \overline{0}.$$

These two formula are equivalent, then we have

$$e_E^{-1}(\mu) = \wedge\{\lambda/e_E(\lambda') \leq \mu'\} = \wedge\{\lambda/e_E(\lambda') \wedge \mu \leq \overline{0}\}$$

$$= \wedge\{\lambda/\lambda' \wedge e_E(\mu) \leq \overline{0}\} = \wedge\{\lambda/e_E(\mu) \leq \lambda\} = e_E(\mu).$$

If $\mathcal{E}$ is a fuzzy Weil uniformity on L, then $\phi(\mathcal{E})$ denotes the Hutton uniformity by $D_{\mathcal{E}}$. The correspondence $(X, \mathcal{E}) \to (X, \phi(\mathcal{E}))$ is functorial. □

Theorem 3.2. *Let $(X, \mathcal{E}), (X', \mathcal{E}')$ be fuzzy Weil uniform space and let $f : (X, \mathcal{E}) \to (X', \mathcal{E}')$ be a fuzzy Weil uniform homomorphism then $f : (X, \phi(\mathcal{E})) \to (X', \phi(\mathcal{E}'))$ is a Hutton uniform homomorphism.*

Proof. Let $e_E \in D_{\mathcal{E}}$, where $E \in \mathcal{E}$. Take a symmetric $F \in \mathcal{E}$ such that $F^2 \subseteq E$. Since f is a Weil uniform homomorphism, $(\overrightarrow{f} \oplus \overrightarrow{f})(F) \in \mathcal{E}'$. In order to show that $f : (X, \phi(\mathcal{E})) \to (X', \phi(\mathcal{E}'))$ is uniform it suffices to show that

$$e_{(\overrightarrow{f} \oplus \overrightarrow{f})(F)} \cdot \overrightarrow{f} \leq \overrightarrow{f} \cdot e_E.$$

So, fix $\alpha \in L^X$ and take $\beta \in L^{X'}$ such that $(\beta, \beta) \in (\overrightarrow{f} \oplus \overrightarrow{f})(F)$ and $(\beta \wedge \overrightarrow{f}(\alpha)) \neq \overline{0}, \beta < e^{f(x)}_{(\overrightarrow{f} \oplus \overrightarrow{f})(F)}$. Then $(\beta, \beta \wedge \overrightarrow{f}(\alpha)) \in (\overrightarrow{f} \oplus \overrightarrow{f})(F)$ and $(\beta \wedge \overrightarrow{f}(\alpha), \overrightarrow{f}(\alpha)) \in \overrightarrow{f}(\alpha) \oplus \overrightarrow{f}(\alpha)$ and consequently $(\beta, \overrightarrow{f}(\alpha)) \in ((\overrightarrow{f} \oplus \overrightarrow{f})(F) o (\overrightarrow{f}(\alpha) \oplus \overrightarrow{f}(\alpha))$. Further, since F is of the form $\vee_{\gamma \in \Gamma}(a_\gamma \oplus b_\gamma)$ for some subset $\{(a_\gamma, b_\gamma)/\gamma \in \Gamma\}$ of $L^X \times L^X$.

$$(\overrightarrow{f} \oplus \overrightarrow{f})(F) o (\overrightarrow{f}(\alpha) \oplus \overrightarrow{f}(\alpha)) = (\overrightarrow{f} \oplus \overrightarrow{f})(\vee(a_\gamma \oplus b_\gamma)) o \overrightarrow{f}(\alpha) \oplus \overrightarrow{f}(\alpha)$$
$$= k(\cup_{\gamma \in \Gamma} \overrightarrow{f}(a_\gamma) \oplus \overrightarrow{f}(b_\gamma)) o k(\downarrow (\overrightarrow{f}(\alpha), \overrightarrow{f}(\alpha)).$$

By Lemma 1.4

$$= \cup(\overrightarrow{f}(a_\gamma) \oplus \overrightarrow{f}(b_\gamma) o (\downarrow (\overrightarrow{f}(\alpha), \overrightarrow{f}(\alpha)) \subset (\overrightarrow{f} \cdot e_E)(\alpha) \oplus \overrightarrow{f}(\alpha)$$

For any $(a,b) \in (\cup_{\gamma\in\Gamma} \overrightarrow{f}(a_\gamma) \oplus \overrightarrow{f}(b_\gamma)) o \downarrow (\overrightarrow{f}(\alpha), \overrightarrow{f}(\alpha)) \setminus \emptyset \exists c \in L^X \setminus \overline{0}$ and $\gamma \in \Gamma$ $(a,c) \leq (\overrightarrow{f}(a_\gamma), \overrightarrow{f}(b_\gamma))$ and $(c,b) \leq (\overrightarrow{f}(\alpha), \overrightarrow{f}(\alpha))$ it follows that $a \leq \overrightarrow{f}(a_\gamma \vee b_\gamma)$. Therefore $a \leq (\overrightarrow{f} \cdot e_E)(\alpha)$. Indeed $(a_\gamma \vee b_\gamma) \wedge \alpha \neq \overline{0}$ because $\overrightarrow{f}(b_\gamma \wedge \alpha) \geq c \neq \overline{0}$ and by the symmetry of $F(a_\gamma \vee b_\gamma, a_\gamma \vee b_\gamma) \in F^2 \subseteq E$ inclusion $a \leq \overrightarrow{f}(a_\gamma \vee b_\gamma) < \overrightarrow{f}(e_E(\alpha)) \Rightarrow (a,b) \in (f \cdot e_E)(\alpha) \oplus f(\alpha)$. We have that $(B, f(\alpha)) \in (\overrightarrow{f} \oplus \overrightarrow{f})(F) o (\overrightarrow{f}(\alpha) \oplus \overrightarrow{f}(\alpha)) \subseteq (\overrightarrow{f} \cdot e_E)(\alpha) \oplus \overrightarrow{f}(\alpha)$. Hence $B \leq (\overrightarrow{f} \cdot e_E)(\alpha)$ which implies that $e_{(\overrightarrow{f} \oplus \overrightarrow{f})(F)} \overrightarrow{f}(\alpha) \leq \overrightarrow{f}(e_E)(\alpha)$. Then $e_{(\overrightarrow{f} \oplus \overrightarrow{f})(F)} \overrightarrow{f} \leq \overrightarrow{f} \cdot e_E$. □

Definition 3.3. Let $e : L^X \to L^X, \mu \in L^X$ then μ is e-small if $\mu \leq e(\lambda)$ whenever $\mu \wedge \lambda \neq \overline{0}$.

The functor ψ : Hutt - UNIF $\to$ F. Weil - UNIF.

Theorem 3.4. *Let D be a Hutton uniform on a space X defined for each $e \in D, E_e$ by $E_e = \cup\{\alpha \oplus \alpha / \alpha \in \cup_e\}$ such that $\cup_e$ be the cover of all e-small elements of L^X and denote the set $\{E_e / e \in D\}$ by $\mathcal{E}_D$, then $\mathcal{E}_D$ is a fuzzy Weil uniform on a space X.*

Proof. For all $e \in D, E_e$ is Weil entourage because the set of all e-small elements of L^X is a cover. $\vee\{f \in L^X / (f,f) \in E_e\} = \overline{1}$ because $\vee\{\alpha, \alpha \in \cup_e\} = \overline{1}$.

FwE_1) $E_e, E_f \in \mathcal{E}_D$ then $e, f \in D$. Take $g \in D$ such that $g \leq e \wedge f$ clearly $E_g \subseteq E_e \cap E_f$. Thus $\mathcal{E}_D$ is a filter basis of Weil entourages of L^X.

FwE_2) Let $E_e \in \mathcal{E}_D$ take $f \in D$ such that $f^3 \leq e$.

By Lemma 1.4, we have $E_f o E_f = (\cup_{\alpha \in \cup_f} \alpha \oplus \alpha) o (\cup_{\alpha \in U_f} \alpha \oplus \alpha)$. Let $(a,c) \in E_f o E_f$. Then $(a,b) \leq (\alpha, \alpha)$ and $(b,c) \leq (\beta, \beta)$, where $\alpha, \beta \in U_f$. Then $a < \alpha < st(\alpha, E_f)$ $c < \beta < st(\alpha, E_f)$.

We prove $st(\alpha, E_f)$ is e-small.

Let $\lambda \wedge st(\alpha, E_f) \neq \overline{0}, (\gamma, \gamma) \in E_f$ with $\gamma \wedge \alpha \neq \overline{0}$ and $\gamma \wedge \lambda \neq \overline{0}$. Then α, γ is f-smallness then $\gamma < f(\lambda), \alpha < f(\gamma)$ then $\alpha < f^2(\lambda)$. Therefore, for every $(\gamma', \gamma') \in E_f$ such that $\gamma' \wedge \alpha \neq \overline{0}$ we have $\gamma' \leq f(\alpha) < f^3(\lambda) < e(\lambda)$. Then $st(\alpha, E_f)$ is e-small then $st(\alpha, E_f) \oplus st(\alpha, E_f) \in E_e$. Then $(a,c) \in E_e$.

FwE_3) The symmetry condition (HU_5) is obviously satisfied since each E_e is symmetric. □

Theorem 3.5. *Let $(X,D), (X',D')$ be Hutton uniform space and let $f : (X,D) \to (X',D')$ be a fuzzy Weil uniform homomorphism. Then $f : (X, \psi(D)) \to (X', \psi(D'))$ is a Hutton uniform homomorphism.*

Proof. Let $E_e \in \mathcal{E}_D$, where $e \in D$ and let $U_{e^3} \leq U$

$$(\overrightarrow{f} \oplus \overrightarrow{f})(E_e) = (\overrightarrow{f} \oplus \overrightarrow{f})(\cup_{\alpha \in U_e} \alpha \oplus \alpha) = \cup_{\alpha \in U_e} \overrightarrow{f}(\alpha) \oplus \overrightarrow{f}(\alpha).$$

We prove that $\overrightarrow{f}(U_e)$ is the cover of all g-small elements of $L^{X'}$. Since f is fuzzy uniform homomorphism then $e \in D$ there is $g \in D'$ such that $g \cdot \overrightarrow{f} \leq \overrightarrow{f} \cdot e$. Let γ be g – small element of $L^{X'}$. Since $\overrightarrow{f}(U_e)$ is a cover of $L^{X'}$ there is $\beta \in U_e$ such that $\gamma \wedge \overrightarrow{f}(\beta) \neq \overline{0}$ and $\gamma \leq g \cdot \overrightarrow{f}(\beta) \leq \overrightarrow{f} \cdot e(\beta)$ since β is e-small then $e(\beta)$ is e^3-small. We have $U_g \leq \overrightarrow{f}[U_{e^3}] \leq \overrightarrow{f}(U)$ then $\overrightarrow{f}(U)$ is the cover of all g-small elements of $L^{X'}$. □

5. Relation Between Weil Uniformities [16] and Fuzzy Weil Uniformities

We want to show that every fuzzy (Weil) uniformity ($\mathcal{E}$) on X corresponds a uniformly ($\mathcal{U}$).

The functor $\psi : F$ Weil-UNIF $\rightarrow$ UniF .

Theroem 4.1. *Let $\mathcal{E}$ be a fuzzy Weil uniformity on a space X defined for each $E \in \mathcal{E}, V_E \subset X \times X$ by $V_E = \{(x, y) \in X \times X | \mu(x) \leq st(\mu, E)(y)\}$ and denote the set $\{V_E/E \in \mathcal{E}\}$ by $\mathcal{U}_\mathcal{E}$, then $\mathcal{U}_\mathcal{E}$ is Weil uniformity on a space X.*

Proof. We know for all $x \in X$ $f(x) \leq st(f, E)(x)$ then V_E contains the diagonal $\Delta(X)$.

UW_1) $V_E, V_F \in U_\mathcal{E}$ then $E, F \in \mathcal{E}$ in order to prove that $\mathcal{U}_\mathcal{E}$ is a filter basis just take for some Weil entourage G such that $G \subseteq E \cap F$.

UW_2) $V_E \in U_\mathcal{E}$ consider $F \in \mathcal{E}$ such that $F^2 \subseteq E$. We prove that $V_F o V_F \subset V_E$.

Let $(x, z) \in V_F o V_F$. There is y s.t. $(x, y) \in V_F, (y, z) \in V_F$ $(x, y) \in V_F$. Then $\mu(x) < st(st(\mu, F), F)(y)$ by (HU4) from Theorem 3.1 $st(st(\mu, F), F)(y)$ $\leq st(\mu, F^2)(y)$. Then $\mu(x) < st(\mu, F^2)(y) < st(\mu, E)(y)$ then $(x, y) \in V_E$.

UW_3) The symmetry condition (FwU_3) is obviously satisfied since each V_E is symmetric. □

Theorem 4.2. *Let $(X, \mathcal{E})$ and $(X', \mathcal{E}')$ be a uniform spaces and let $f : X \rightarrow X'$ be uniformly continuous then f is also uniformly continuous as a map between the uniform spaces $(X, \psi(\mathcal{E}))$ and $(X', \psi(\mathcal{E}'))$.*

Proof. Let $V' \in \psi(\mathcal{E}')$ and let $E' \in \mathcal{E}'$ such that $\psi(E') \in V'$. Since f is $(\mathcal{E}, \mathcal{E}')$ - continuous, we have $E \in (\overleftarrow{f} \oplus \overleftarrow{f})(E') \in \psi$. Let $V \in \psi(E)$ we will show that $(f(x), f(y)) \in V'$ whenever $(x, y) \in V$. In fact let $(x, y) \in V_E$ and $\mu \in L^Y$

then we have $\rho = \overleftarrow{f}(\mu)$ and $\rho(x) \leq st(\rho, E)(y)$. Since $\rho(x) = \overleftarrow{f}(\mu)(x) = \mu(f(x))$ we have

$$\mu(f(x)) = \rho(x) < st(\rho, E)(y) = \vee\{\alpha(y)/(\alpha, \alpha) \in E, \alpha \wedge \rho \neq \overline{0}\}.$$

Since $(\alpha, \alpha) \in E$ then there is $(\alpha', \alpha') \in E'$ such that $\alpha = \overleftarrow{f}(\alpha')$. Then we have $\mu(f(x)) < \vee\{\overleftarrow{f}(\alpha')(y)/(\alpha', \alpha') \in E', \alpha' \wedge \overleftarrow{f}(\rho) \neq \overline{0}\} = \vee\{\alpha'(f(y))/ (\alpha', \alpha') \in E', \alpha' \wedge \mu \neq \overline{0}\}$ which implies that $(f(x), f(y)) \in \psi$. □

We want to show that every (Weil) uniformity $(\mathcal{U})$ on a set X corresponds a fuzzy Weil uniformity $(\mathcal{E})$.

The functor θ : UNIF→ F Weil - UINIF.

Theorem 4.3. *Let $\mathcal{U}$ be a uniformity on a space X defined for each $V \in U, E_V = U\{\alpha \oplus \alpha/\alpha \in U_V\}$ and*

$$U_V = \{\alpha \in L^X/\alpha(x) < \vee\{\beta(y)/(x, y) \in V, \alpha \wedge \beta \neq \overline{0}\}$$

be a cover of L^X and denote the set $\{E_V/V \in \mathcal{U}\}$ by $\mathcal{E}_u$. $\mathcal{E}_u$ is fuzzy Weill uniform on a space X.

Proof. It is trivial that for all $V \in \mathcal{U}, E_V$ is Weil entourage.

FwU_1) $E_V, E_U \in \mathcal{E}_{\mathcal{U}}$ we consider $W \subset U \cap V$

$$E_W \subset E_{U\cap V} = \cup\{\alpha \oplus \alpha/\alpha \in U_{U\cap V}\}.$$

It is trival

$$\cup\{\alpha \oplus \alpha/\alpha \in U_U \cap U_V\} \subset \cup\{\alpha \oplus \alpha/\alpha \in U_U\} \cap \cup\{\alpha \oplus \alpha/\alpha \in U_V\} \subset E_U \cap E_V.$$

FwU_2) $E_V \in \mathcal{E}_{\mathcal{U}}$ take $U \in \mathcal{U}$ such that $U^3 \subset V$. By Lemma 1.4 we have

$$E_U o E_U = (\cup_{\alpha\in U_U}(\alpha \oplus \alpha))o(\cup_{\alpha\in U_U}(\alpha \oplus \alpha))$$

Let $(a, c) < \alpha \oplus \alpha, (c, b) < B \oplus \beta$.

$a < \alpha < st(\alpha, E_U)$ $b < \beta < st(\alpha, E_U)$.

We prove that $st(\alpha, E_U) \in U_V$. Let $\lambda \wedge st(\alpha, E_U) \neq \overline{0}$. Then there is $\gamma \in L^X, (\gamma, \gamma) \in E_U$. Then $\lambda \wedge \gamma \neq \overline{0}$ and $\alpha \wedge \gamma \neq \overline{0}$. Then $\alpha, \gamma \in U_u$. Then we have $\alpha(x) < \vee\{\gamma(y)|(x, y) \in U, \alpha \wedge \gamma \neq \overline{0}\} < \vee\{\lambda(z)|(y, z) \in U, (x, y) \in U, \alpha \wedge \gamma \neq \overline{0}, \gamma \wedge \lambda \neq \overline{0}\} < \vee\{\lambda(z)|(x, z) \in UoU, \alpha \wedge \lambda \neq \overline{0}\}$. Then $\alpha \in U_{U^2}$. Therefore, for every $(\gamma', \gamma') \in E_U$ such that $\gamma' \wedge \alpha \neq \overline{0}$ we have $\gamma'(x) < \vee\{(\alpha(y)|(x, y) \in U, \gamma' \wedge \alpha \neq \overline{0}\} < \vee\{\lambda(z)(x, z) \in U^3(\lambda \wedge \alpha) \neq \overline{0}\} < \vee\{\lambda(z)|(x, z) \in V(\lambda \wedge \alpha) \neq \overline{0}\}$ then $st(\alpha, E_U) \in U_V$ then $st(\alpha, E_U) \oplus st(\alpha, E_U) \in E_V$ then $(a, b) \in E_V$.

FwE_3) It is trivial each E_V is symmetric. □

Theorem 4.4. *Let* $(X,\mathcal{U})$ *and* $(X',\mathcal{U}')$ *be a uniform spaces and let* f *be a function from* $X \to Y$*. Then* f *is uniformly continuous iff* f *is uniformly continuous as a map between the fuzzy Weil uniform space* $(X,\theta(\mathcal{U})) \to (X',\theta'(\mathcal{U}'))$*.*

Proof. Suppose f is $(\mathcal{U},\mathcal{U}')$- continous and let $E' \in \theta(\mathcal{U}')$ there exist $V' \in \mathcal{U}'$ s.t. $E' = E_{V'} = \cup\{\alpha \oplus \alpha/\alpha \in U_{V'}\}$. Let $V \in \mathcal{U}$ such that $(f(x), f(y)) \in V'$ whenever $(x,y) \in V$. If $\theta(V) = F$ we have $(\overleftarrow{f} \oplus \overleftarrow{f})(E_{V'}) = (\overleftarrow{f} \oplus \overleftarrow{f})(\cup_{\alpha\in U_{V'}}\alpha\oplus \alpha) = \cup_{\alpha\in U_{V'}} \overleftarrow{f}(\alpha)\oplus \overleftarrow{f}(\alpha)$. We prove that $U_V \subset \overleftarrow{f}(U_{V'})$. It is trivial $\overleftarrow{f}(U_{V'})$ is a cover of L^X. Let $\alpha \in U_V \alpha(x) < \vee\{(\beta(y)/(x,y) \in V, \alpha\wedge\beta \neq \overline{0}\}$ then $\overrightarrow{f}(\alpha)(f(x)) < \vee\{\overrightarrow{f}(\beta)(f(y))/(f(x), f(y)) \in f(V), \overrightarrow{f}(\alpha)\wedge \overrightarrow{f}(\beta) \neq \overline{0}\}$ therefore $\overrightarrow{f}(\alpha) \in U_{V'}$ then $\alpha \in \overleftarrow{f}(U_{V'})$ we conclude that $F \subset (\overleftarrow{f} \oplus \overleftarrow{f})(E)$ then $(\overleftarrow{f} \oplus \overleftarrow{f})(E) \in \theta(\mathcal{U})$.

Conversely, let f be a $(\theta(\mathcal{U}), \theta(\mathcal{U}'))$ continuous and let $V' \in \mathcal{U}'$ we have $\theta(V') \in \theta(\mathcal{U}')$. Then $(\overleftarrow{f} \oplus \overleftarrow{f})(\theta(V')) \in \theta(\mathcal{U})$. There exist $V \in u$ such that $\theta(V) \leq \overleftarrow{f} \oplus \overleftarrow{f}(\theta(V'))$. Then we have $\alpha \in U_V$ then $f(\alpha) \in U_{V'}$. It is trivial $(f(x), f(y)) \in V'$ whenever $(x,y) \in V$. □

References

[1] C.H. Dowker, D. Strauss, Sums in the category of frames, *Houston J. Math.*, **3** (1977), 7-15.

[2] J. Gutirrez Garcia, M.A. Deprada Vicente, A.P. Šostak, A unified approach to the concept of a fuzzy L-uniform space, *Preprint*.

[3] U. Höhle, Probabilistic uniformization of fuzzy uniformities, *Fuzzy Sets and Systems*, **2** (1978), 311-332.

[4] U. Höhle, Probabilistic topologies induced by L-fuzzy uniformities, *Manuscripta Math.*, **38** (1982), 289-323.

[5] B. Hutton, Uniformities on fuzzy topological spaces, *J. Math. Anal. Appl.*, **58** (1977), 559-571.

[6] J. Isbell, Atomless parts of Spaces, *Math. Scand.*, **31** (1972), 5-32.

[7] P.T. Johnstone, *Stone Spaces*, Cambridge Studies in Advanced Mathematics 3, Cambridge University Press (1982).

[8] A.K. Katsaras, On fuzzy uniform spaces, *J. Math. Anal. Appl.*, **101** (1984), 97-113.

[9] R. Lowen, Fuzzy uniform spaces, *J. Math. Anal. Appl.*, **82** (1981), 370-385.

[10] J. Picado, *Weil Entourages in Pointfree Topology*, Ph.D. Thesis, University of Coimbra (1995).

[11] J. Picado, Weil uniformities for frames, *Comment. Math. Univ. Carolin*, **36** (1995), 357-370.

[12] A. Pultr, Pointless uniformities I. Complete regularity, *Comment. Math. Unive. Carolin.*, **25** (1984), 91-104.

[13] A. Pultr, Pointless uniformities II. (Dia) Metrization, *Comment. Math. Univ. Carolin.*, **25** (1984), 105-120.

[14] S.E. Rodabaugh, A theory of fuzzy uniformitie with applications to the fuzzy real lines, *J. Math. Anal. Appl.*, **129** (1988), 37-70.

[15] J.W. Tukey, *Convergence and Uniformity in Topology*, Ann. Math. Studies, No. 1, Princeton University Press (1940).

[16] A. Weil, *Sur les Spaces ã Structure Uniforme et sur la Topologie Général*, Act. Sci. et. Ind., **551**, Hermann, Paris (1937).