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GENERALIZED BINOMIAL COEFFICIENTS ASSOCIATED WITH COMPLEX ZONAL POLYNOMIALS

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Abstract: The generalized binomial coefficient $\binom{\lambda}{\kappa}$ associated with complex zonal polynomials $\tilde{C}_\kappa(X)$ is defined as the coefficient of $\tilde{C}_\kappa(X)/\tilde{C}_\kappa(I_m)$ in the complex binomial expansion

$$\frac{\tilde{C}_\lambda(I_m+X)}{\tilde{C}_\lambda(I_m)}=\sum_{k=0}^{l}\sum_{\kappa}\binom{\lambda}{\kappa}\frac{\tilde{C}_\kappa(X)}{\tilde{C}_\kappa(I_m)}.$$

In this article we have derived several results which facilitate computations of these coefficients.

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1. Introduction

In multivariate statistical analysis based on the multivariate normal distribution, many of the distributional results of random matrices are often derived in terms of functions of matrix arguments. Constantine [2] gave the power series representation of hypergeometric functions of matrix arguments in series involving zonal polynomials. The theory of zonal polynomials was developed in a series of papers by A.T. James and A.G. Constantine. For applications and

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properties of zonal polynomials and functions of matrix arguments the reader is referred to Muirhead [11] and Gupta and Nagar [5].

The statistical analysis based on the complex multivariate normal distribution has been developed as counterpart of the classical statistical analysis based on the real multivariate normal distribution. To deal with the distributional problems of the complex random matrices necessary tools such as the zonal polynomials, generalized hypergeometric functions, Laguerre polynomials, generalized Hayakawa polynomials etc. of Hermitian matrix argument have also been defined and studied. For example, see, Chikuse [1], Farrell [3, 4], Hayakawa [6, 7], James [8], Khatri [9], Smith and Gao [12] and Takemura [13].

Let X be an $m\times m$ Hermitian matrix and let $\lambda=(l_1,\ldots,l_m)$ $l_1\geq\cdots\geq l_m\geq 0$ and $\kappa=(k_1,\ldots,k_m)$, $k_1\geq\cdots\geq k_m\geq 0$ be partitions of the integers l and k, respectively ($k\leq l$). The generalized binomial coefficient $\binom{\lambda}{\kappa}$ associated with complex zonal polynomials $\tilde{C}_\kappa(X)$ is defined as the coefficient of $\tilde{C}_\kappa(X)/\tilde{C}_\kappa(I_m)$ in the complex binomial expansion

$$\frac{\tilde{C}_\lambda(I_m+X)}{\tilde{C}_\lambda(I_m)}=\sum_{k=0}^{l}\sum_{\kappa}\binom{\lambda}{\kappa}\frac{\tilde{C}_\kappa(X)}{\tilde{C}_\kappa(I_m)}. \tag{1}$$

These coefficients occur in the non-central distribution of Hotelling's generalized T^2 statistic and in certain moments derived from the non-central means distribution (Muirhead [10]). In this article, we will give expressions for the generalized binomial coefficient $\binom{\lambda}{\kappa}$.

2. Generalized Binomial Coefficients

Let $x_1,\ldots,x_m$ be the latent roots of the Hermitian matrix X, $E=\sum_{i=1}^m x_i\frac{\partial}{\partial x_i}$ and $\epsilon=\sum_{i=1}^m\frac{\partial}{\partial x_i}$. Then, Chikuse [1] showed that

$$E\tilde{C}_\kappa(X)=k\tilde{C}_\kappa(X), \tag{2}$$

$$\epsilon\left[\frac{\tilde{C}_\kappa(X)}{\tilde{C}_\kappa(I_m)}\right]=\sum_{i=1}^{m}\binom{\kappa}{\kappa^{(i)}}\frac{\tilde{C}_{\kappa^{(i)}}(X)}{\tilde{C}_{\kappa^{(i)}}(I_m)}, \tag{3}$$

where corresponding to the partition $\kappa=(k_1,k_2,\ldots)$ of k, the partitions κ_i and $\kappa^{(i)}$ are defined as $\kappa_i=(k_1,\ldots,k_{i-1},k_i+1,k_{i+1},\ldots), i=1,2,\ldots,$ and $\kappa^{(i)}=(k_1,\ldots,k_{i-1},k_i-1,k_{i+1},\ldots), i=1,2,\ldots,$ respectively. Note that the partitions κ_i and $\kappa^{(i)}$ are well defined whenever they are admissible, i.e., so long as the parts are in non increasing order. Further, using above and several other results, Chikuse [1] has shown that for an $m \times m$ Hermitian matrix X,

$$\sum_{i=1}^{m} \binom{\kappa_i}{\kappa} \tilde{C}_{\kappa_i}(X) = (k+1)(\operatorname{tr} X)\tilde{C}_{\kappa}(X) \tag{4}$$

and

$$\frac{1}{k+1} \sum_{i=1}^{m} \binom{\kappa_i}{\kappa} [k_i - (i-1)]\tilde{C}_{\kappa_i}(X) = \gamma_X \tilde{C}_{\kappa}(X)\,, \tag{5}$$

where $\gamma_X = \sum_{i=1}^{m} x_i^2 \frac{\partial}{\partial x_i}$. Substituting $X = I_m$ in (4) and (5) and simplifying the resulting expression, we obtain

$$\sum_{i=1}^{m} \binom{\kappa_i}{\kappa} \tilde{C}_{\kappa_i}(I_m) = m(k+1)\tilde{C}_{\kappa}(I_m) \tag{6}$$

and

$$\sum_{i} \binom{\kappa_i}{\kappa} [k_i - (i-1)]\tilde{C}_{\kappa_i}(I_m) = k(k+1)\tilde{C}_{\kappa}(I_m). \tag{7}$$

From Khatri [9, Lemma 1], the zonal polynomial $\tilde{C}_{\kappa}(I_m)$ is given as

$$\tilde{C}_{\kappa}(I_m) = \frac{[\chi_{[\kappa]}(1)]^2 [m]_{\kappa}}{k!}\,, \tag{8}$$

where $\chi_{[\kappa]}(1)$ is the dimension of the representation $[\kappa]$ of the symmetric group given by

$$\chi_{[\kappa]}(1) = \frac{k! \prod_{i<j}^{m} (k_i - k_j - i + j)}{\prod_{i=1}^{m} (k_i + m - i)!}, \tag{9}$$

and the complex multivariate hypergeometric coefficient $[a]_{\kappa}$ is defined as

$$[a]_{\kappa} = \prod_{i=1}^{m} (a - i + 1)_{k_i}, \tag{10}$$

with $(a)_0 = 1$ and $(a)_k = a(a+1)\cdots(a+k-1) = (a)_{k-1}(a+k-1)$. Using (8) and (10), the relationship between $C_{\kappa_i}(I_m)$ and $C_{\kappa}(I_m)$ is obtained as

$$\tilde{C}_{\kappa_i}(I_m) = \frac{m + k_i - i + 1}{k+1} \left[\frac{\chi_{[\kappa_i]}(1)}{\chi_{[\kappa]}(1)} \right]^2 \tilde{C}_{\kappa}(I_m). \tag{11}$$

Now, substitution of (11) in (6) and (7) yields

$$\sum_i \binom{\kappa_i}{\kappa}(m+k_i-i+1)[\chi_{[\kappa_i]}(1)]^2 = m(k+1)^2[\chi_{[\kappa]}(1)]^2 \tag{12}$$

and

$$\sum_i \binom{\kappa_i}{\kappa}(k_i-i+1)(m+k_i-i+1)[\chi_{[\kappa_i]}(1)]^2 = k(k+1)^2[\chi_{[\kappa]}(1)]^2. \tag{13}$$

Note that in (12) and (13) there is no longer any restriction on the number of nonzero parts of the partition κ and the summation is taken over all i such that κ_i is admissible. Now, since $\tilde{C}_\kappa(X) \equiv 0$ for any partition κ of k into more than m nonzero parts, it is clear from (1) that $\binom{\lambda}{\kappa}$ can be defined for partitions into any number of parts, and hence can be regarded as independent of m, so equating the constant terms in (12) or the coefficients of m in (13) both give

$$\sum_i \binom{\kappa_i}{\kappa}(k_i-i+1)[\chi_{[\kappa_i]}(1)]^2 = 0. \tag{14}$$

Further, equating the coefficients of m on both the sides of (12) gives

$$\sum_i \binom{\kappa_i}{\kappa}[\chi_{[\kappa_i]}(1)]^2 = (k+1)^2[\chi_{[\kappa]}(1)]^2, \tag{15}$$

while equating the constant terms in (13) results

$$\sum_i \binom{\kappa_i}{\kappa}(k_i-i+1)^2[\chi_{[\kappa_i]}(1)]^2 = k(k+1)^2[\chi_{[\kappa]}(1)]^2. \tag{16}$$

If the partition κ is such that it gives rise to at most three admissible partitions κ_i, then solving the simultaneous equations (14), (15) and (16) gives formulas for the $\binom{\kappa_i}{\kappa}$. First we compute the generalized binomial coefficient $\binom{\kappa_i}{\kappa}$ for a partition having two nonzero parts.

Theorem 2.1. *Let* $\kappa = (k-j, j)$, $k > 2j$, $j \neq 0$, *so that* $\kappa_1 = (k-j+1, j)$, $\kappa_2 = (k-j, j+1)$ *and* $\kappa_3 = (k-j, j, 1)$. *Then,* $\binom{\kappa_1}{\kappa} = (k-j+2)(k-2j+1)/(k-2j+2)$, $\binom{\kappa_2}{\kappa} = (j+1)(k-2j+1)/(k-2j)$ *and* $\binom{\kappa_3}{\kappa} = (j+1)(k-j+2)/j(k-j+1)$.

Proof. Note that when $k = 2j$, the partition κ_2 is not admissible and thus $\binom{\kappa_2}{\kappa} = 0$. If $j = 0$, then κ_3 is not admissible and thus $\binom{\kappa_3}{\kappa} = 0$. For $k > 2j$,

$j \neq 0$, the partitions κ_1, κ_2, κ_3 are admissible and from (14), (15) and (16) we have the system of equations

$$
\begin{aligned}
&(k-j)a_1y_1 + (j-1)a_2y_2 + (-2)a_3y_3 = 0\,,\\
&a_1y_1 + a_2y_2 + a_3y_3 = (k+1)^2\,,\\
&(k-j)^2a_1y_1 + (j-1)^2a_2y_2 + (-2)^2a_3y_3 = k(k+1)^2\,,
\end{aligned}
$$

where $y_1 = \binom{\kappa_1}{\kappa}$, $y_2 = \binom{\kappa_2}{\kappa}$ and $y_3 = \binom{\kappa_3}{\kappa}$. Further, using (9), a_1, a_2 and a_3 are obtained as $a_1 = [(k+1)(k-2j+2)/(k-j+2)(k-2j+1)]^2$, $a_2 = [(k+1)(k-2j)/(k-2j+1)(j+1)]^2$ and $a_3 = [(k+1)(k-j+1)j/(k-j+2)(j+1)]^2$, respectively. Solving the above system we arrive at the required result. □

Next, we compute the generalized binomial coefficient $\binom{\kappa_i}{\kappa}$ for a partition having $j+1$ nonzero parts.

Theorem 2.2. *Let $\kappa = (k-j, 1^j)$, $k-j > 1$, so that $\kappa_1 = (k-j+1, 1^j)$, $\kappa_2 = (k-j, 2, 1^{j-1})$ and $\kappa_{j+2} = (k-j, 1^{j+1})$. Then $\binom{\kappa_1}{\kappa} = (k+1)(k-j)/k$, $\binom{\kappa_2}{\kappa} = (j+1)(k-j)/j(k-j-1)$ and $\binom{\kappa_{j+2}}{\kappa} = (j+1)(k+1)/k$.*

Proof. Note that when $j = 1$ the above theorem reduces to the Theorem 2.1. When $k-j = 1$, the partition κ_2 is not admissible and hence $\binom{\kappa_2}{\kappa} = 0$. For for $k-j > 1$, the partition κ_1, κ_2, κ_{j+2} are admissible and we have the system of equations

$$
\begin{aligned}
&(k-j)b_1y_1 + (-j-1)b_3y_3 = 0\,,\\
&b_1y_1 + b_2y_2 + b_3y_3 = (k+1)^2\,,\\
&(k-j)^2b_1y_1 + (-j-1)^2b_3y_3 = k(k+1)^2\,,
\end{aligned}
$$

where $y_1 = \binom{\kappa_1}{\kappa}$, $y_2 = \binom{\kappa_2}{\kappa}$, $y_3 = \binom{\kappa_{j+2}}{\kappa}$, $b_1 = [k/(k-j)]^2$, $b_2 = [(k-j-1)(k+1)j/(k-j)\ (j+1)]^2$ and $b_3 = [k/(j+1)]^2$. Solving the above system we get the desired result. □

Next, we give certain results which are used in generalizing the above two theorems.

Theorem 2.3. *If κ and λ are partitions of k and l, respectively ($k < l$), then*

$$\sum_i \binom{\lambda}{\kappa_i}\binom{\kappa_i}{\kappa} = (l-k)\binom{\lambda}{\kappa}. \tag{17}$$

Proof. Replacing X and κ by $I_m + R$ and λ in (2), we have

$$\sum_{i=1}^{m}(1+r_i)\frac{\partial}{\partial r_i}\tilde{C}_\lambda(I_m+R) = l\tilde{C}_\lambda(I_m+R), \tag{18}$$

where $r_1, \ldots, r_m$ are the latent roots of the matrix R. Substituting the complex binomial expansion of $\tilde{C}_\lambda(I_m+R)$ given by (1) in (18) and using (3), we obtain

$$\sum_{k=0}^{l}\sum_{\kappa}\binom{\lambda}{\kappa}\sum_{i=1}^{m}\binom{\kappa}{\kappa^{(i)}}\frac{\tilde{C}_{\kappa^{(i)}}(R)}{\tilde{C}_{\kappa^{(i)}}(I_m)} + \sum_{k=0}^{l}\sum_{\kappa}\binom{\lambda}{\kappa}k\frac{\tilde{C}_\kappa(R)}{\tilde{C}_\kappa(I_m)} = l\sum_{k=0}^{l}\sum_{\kappa}\binom{\lambda}{\kappa}\frac{\tilde{C}_\kappa(R)}{\tilde{C}_\kappa(I_m)},$$

where the summation is taken over all i such that $\kappa^{(i)}$ is admissible. Finally, equating the coefficients of $\tilde{C}_\kappa(R)/\tilde{C}_\kappa(I_m)$ in the above equation, we obtain the desired result. □

Some important consequences of the above theorem are given below.

Corollary 2.3.1. *Let $\kappa = (k-j, j)$ be a partition of k, $\sigma = (k-j+s, j)$ and $\rho = (k-j, j+s)$ be partitions of $k+s$. Then, for any integer $s \geq 1$, $\binom{\sigma}{\kappa} = \binom{\sigma}{\kappa_1}\binom{\kappa_1}{\kappa}/s$ and $\binom{\rho}{\kappa} = \binom{\rho}{\kappa_2}\binom{\kappa_2}{\kappa}/s$.*

Proof. To prove the first part we use the fact that if $\sigma = (s_1, s_2, \ldots)$ and $\kappa = (k_1, k_2, \ldots)$ are partitions of s and $k(\leq s)$, respectively, then $\binom{\sigma}{\kappa} = 0$ if $s_i < k_i$ for any $i = 1, 2, \ldots$. Thus, applying this property to (17) it is easy to see $\sum_{i\neq 1}\binom{\sigma}{\kappa_i}\binom{\kappa_i}{\kappa} = 0$. Hence the required result follows. The second part can be proved similarly. □

Corollary 2.3.2. *Let $\kappa = (k-j, 1^j)$ be a partition of k, $\sigma = (k-j+s, 1^j)$ and $\rho = (k-j, 1^{j+s})$ be partitions of $k+s$. Then, for any integer $s \geq 1$, $\binom{\sigma}{\kappa} = \binom{\sigma}{\kappa_1}\binom{\kappa_1}{\kappa}/s$ and $\binom{\rho}{\kappa} = \binom{\rho}{\kappa_{j+2}}\binom{\kappa_{j+2}}{\kappa}/s$.*

Now, we can generalize Theorem 2.1 and Theorem 2.2 as follows.

Theorem 2.4. *Let $\kappa = (k-j, j)$, and $\sigma = (k-j+s, j)$ be partitions of k and $k+s$, respectively. Then, for any integer $s \geq 1$,*

$$\binom{\sigma}{\kappa} = \frac{1}{s!}\prod_{i=1}^{s}\frac{(k-j+1+i)(k-2j+i)}{k-2j+1+i}.$$

Proof. Using Corollary 2.3.1, it is straighforward to see that $\binom{\sigma}{\kappa} = (s!)^{-1} \prod_{i=1}^{s} \binom{(k-j+i,j)}{(k-j+i-1,j)}$. Further, Theorem 2.1 gives $\binom{(k-j+i,j)}{(k-j+i-1,j)} = (k-j+i+1)(k-2j+i)/(k-2j+i+1)$. □

Theorem 2.5. *Let $\kappa = (k-j, j)$ and $\rho = (k-j, j+s)$ be partitions of k and $k+s$, respectively. Then, for any integer $s \geq 1$,*

$$\binom{\rho}{\kappa} = \frac{1}{s!} \prod_{i=1}^{s} \frac{(j+i)(k-2j-i+2)}{(k-2j-i+1)}.$$

Theorem 2.6. *Let $\kappa = (k-j, 1^j)$ be a partition of k, $\sigma = (k-j+s, 1^j)$ and $\rho = (k-j, 1^{j+s})$ be partitions of $k+s$. Then, for any integer $s \geq 1$,*

$$\binom{\sigma}{\kappa} = \frac{1}{s!} \prod_{i=1}^{s} \frac{(k+i)(k+i-j-1)}{(k+i-1)} \quad \text{and} \quad \binom{\rho}{\kappa} = \frac{1}{s!} \prod_{i=1}^{s} \frac{(j+i)(k+i)}{(k+i-1)}.$$

Finally, it is clear that the system of equations (14), (15) and (16) can be used to get different sets of three general coefficients $\binom{\kappa_i}{\kappa}$. Further, a system having four equations can also be constructed and different sets of four general coefficients $\binom{\kappa_i}{\kappa}$ can be obtained. This can be achieved by considering (14), (15) and (16) together with a fourth equation which can be obtained by putting $X = I_m$ and substituting for $C_{\kappa_i}(I_m)$ from (11) in the equation (3.11) of Chikuse [1] and equating coefficients of m and the constant terms of the resulting equation.

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