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COMPUTING BLOW-UP SOLUTIONS

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Abstract: We show the existence of power series type of solutions of reaction diffusion equations with quadratic nonlinearities. Local existence of these solutions follows from the uniform convergence of the series. We also show that a positive source leads to a finite blow-up time of a solution.

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1. Introduction

We would like to construct power series solutions for the reaction diffusion equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + au + bu^2 + f(x,t)\,, \quad x \in \Omega \subset \mathbb{R}^m,\ t \geq 0\,, \\ u(x,0) = \psi(x)\,, \\ u(\partial\Omega, t) = 0\,, \end{cases} \tag{1}$$

where $f(x,t) = \sum_{n\geq 0} f_n(x)e^{nt}$ is a convergent series, $a,\ b \in \mathbb{R},\ m = 1,2,3$ and Ω is a bounded convex domain with a smooth boundary $\partial\Omega \in C^2$.

The existence, qualitative behavior and asymptotic growth of solutions of nonlinear reaction diffusion equations is classical, see for example [2] and the references therein. However the computational side of the problem remains difficult to address due to the rapid growth of the solution. For example finite difference schemes are not appropriate since they may not detect the single blow-up point, see [1]. Instead of looking for solutions of a general type, we

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look for power series solutions of the type

$$u(x,t) = \sum_{n\geq 0} \varphi_n(x) e^{nt} \,, \tag{2}$$

where φ_n would be determined recursively by solving linear equations only, through a sequential algorithm. Clearly a power series solution would provide numerical analysis with computable test solutions and shed more light into the analytical properties of the solution, see [5]. Thus by truncating the series, one can compute symbolically and approximate solutions (1) to any given precision.

Observe that if $\varphi_n(x) \geq 0$, then the divergence of the series in (2) leads to blow-up. Also once the φ_n are obtained, the profile of admissible initial conditions ψ is given by $\psi(x) = \sum_{n\geq 0} \varphi_n(x)$. The main focus of the algorithm is on the existence and construction of the sequence $\{\varphi_n\}_{n\geq 0}$. Convergence for the series in (2) for small $t > 0$, follows from estimates obtained by generating functions since the φ_n satisfy a recurrence relation.

2. Notation

Let Δ denote the Laplacian operator generated by the expression $\Delta = \sum\limits_{i=0}^{m} \frac{\partial^2}{\partial x_i^2}$ and acting in the Sobolev space $W_{2,0}^2(\Omega) := \overset{\circ}{W_2^1}(\Omega) \cap W_2^2(\Omega)$, where W_2^p represents functions in $L^2(\Omega)$ with their first p derivatives. Functions in $\overset{\circ}{W_2^1}(\Omega)$ have zero trace on the boundary and denote the norms $||f||_{2,0}^{(2)} = \left(\int_\Omega \sum\limits_{k=0}^{2} \left| \frac{\partial^k f}{\partial x_1^{k1} .. \partial x_m^{km}} \right|^2 dx \right)^{\frac{1}{2}}$, $||f||_2 = \sqrt{\int_\Omega |f(x)|^2 \, dx}$ and $|f|_0 = \sup\limits_{x\in\Omega} |f(x)|$. Let σ_Δ denote the spectrum of the Laplacian Δ, as an operator acting in $L^2(\Omega)$ under Dirichlet boundary conditions. Recall that σ_Δ is discrete and we say that (a,T) is a blow-up point if $u(x,t) \to \infty$ as $(x,t) \to (a,T)$. In case $|u(x,T)| < \infty$ for $x \neq a$, then we say that (a,T) is a single point blow-up.

If (2) is a solution of (1) then φ_n satisfy a recurrence relation. We formally have

$$\sum_{n\geq 0} \{\Delta\varphi_n(x) - n\varphi_n(x) + a\varphi_n(x)\} e^{nt} + b \sum_{n\geq 0} e^{nt} \sum_{k=0}^{n} \varphi_{n-k}(x)\varphi_k(x) + \sum_{n\geq 0} f_n(x) e^{nt} = 0 \,, \tag{3}$$

$$\sum_{n\geq 0}\left((\Delta - n + a)\,\varphi_n(x) + b\sum_{k=0}^{n}\varphi_{n-k}(x)\varphi_k(x) + f_n(x)\right)e^{nt} = 0\,.$$

Since for a fixed x, the above power series in t vanishes we must have

$$(\Delta - n + a)\,\varphi_n(x) + b\sum_{k=0}^{n}\varphi_{n-k}(x)\varphi_k(x) + f_n(x) = 0 \qquad \text{for } n \geq 0.$$

We now describe the sequential algorithm. We begin by solving the first equation, i.e. $n = 0$,

$$(\Delta + a)\,\varphi_0(x) + b\varphi_0^2(x) + f_0(x) = 0\,, \tag{4}$$

which is nonlinear. For simplicity we could choose the trivial solution $\varphi_0(x) = 0$, if $f_0 = 0$ or any non trivial easily computable solution.

The remaining functions are given by the subsequent recurrence relations

$$[\Delta + 2b\varphi_0(x) + a - 1]\,\varphi_1(x) \;=\; -f_1(x)\,, \tag{5}$$

$$[\Delta + 2b\varphi_0(x) + a - n]\,\varphi_n(x) \;=\; R_n(x)\,, \quad n = 2, 3, \ldots\,, \tag{6}$$

where

$$\varphi_n(x) = 0\,,\ x \in \partial\Omega \quad \text{and} \quad R_n(x) = -b\sum_{k=1}^{n-1}\varphi_{n-k}(x)\varphi_k(x) - f_n(x)\,,$$

which shows that φ_n depends on the previous terms $\varphi_1, \ldots, \varphi_{n-1}$ only.

Observe that all equations defining the φ_n are linear, but to avoid the trivial solution in (5), we need to choose one of the following alternatives for equation (5).

Starting conditions for $\varphi_1 \neq 0$: One of the following conditions should hold

$$\begin{array}{ll}[A] & 1 - a \text{ is an eigenvalue of } \Delta + 2b\varphi_0(x) \text{ and } f_1 = 0\,. \\ [B] & 1 - a \text{ is not an eigenvalue of } \Delta + 2b\varphi_0(x) \text{ and } 0 \neq f_1 \in L^2(\Omega)\,.\end{array} \tag{7}$$

If none of the starting conditions is met for φ_1, then the alternative should hold for the next equations (6) until we obtain $\varphi_k \neq 0$. From (6) we have the following proposition.

Proposition 1. *Assume that for all $n \geq 1$, $n - a$ is not an eigenvalue of $\Delta + 2b\varphi_0$ and $f_n = 0$, then the only solution of (1) of the form (2) is the trivial steady state, $u(x,t) = \varphi_0(x)$.*

Here are two simple examples on how the sequence φ_n can be generated. Convergence proofs are in the next section.

Example 1. Consider the simple case when $a = 0$, $b = 1$ and $f_1 \in L^2(\Omega)$

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + u^2 + e^t f_1(x)\,, \quad x \in \Omega \subset \mathbb{R}^m, \quad t \geq 0\,, \\ u(x,0) = \psi(x)\,, \\ u(x,t) = 0\,, \quad x \in \partial\Omega. \end{cases} \tag{8}$$

For the sake of simplicity, take $\varphi_0 = 0$ as a solution for the first equation

$$\Delta\varphi_0(x) + \varphi_0^2(x) = 0$$

and the recurrence (6) provides for the next equations

$$(-\Delta + 1)\,\varphi_1(x) = f_1(x)\,, \qquad (-\Delta + 2)\,\varphi_2(x) = \varphi_1^2(x)\,,$$
$$(-\Delta + 3)\,\varphi_3(x) = 2\varphi_1(x)\varphi_2(x)\,, \quad (-\Delta + 4)\,\varphi_4(x) = 2\varphi_1(x)\varphi_3(x) + \varphi_2^2(x)\,,$$

and in general for $n \geq 2$

$$(-\Delta + n)\,\varphi_n(x) = \sum_{k=1}^{n-1} \varphi_{n-k}(x)\varphi_k(x). \tag{9}$$

Obviously $1 \notin \sigma_\Delta$, since all eigenvalues of Δ are negative, and $\varphi_1(x) = (-\Delta + 1)^{-1} f_1(x) \neq 0$ is well defined as a classical solution. This is enough to trigger a non trivial sequence $\{\varphi_n\}_{n\geq 1}$ since the inverse $(-\Delta + n)^{-1}$ exists for $n \geq 2$.

In the next example we use condition [A], i.e. an eigenvalue instead of $f_1(x)$ to start the sequence φ_n.

Example 2. Consider

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + au + u^2\,, \quad x \in \mathbb{R}^m, \quad t \geq 0\,, \\ u(x,0) = \psi(x)\,, \end{cases}$$

where a is a positive constant to be chosen later. The recursive equations (4), (5) and (6) are again

$$\begin{aligned} (\Delta + a)\,\varphi_0(x) + \varphi_0^2(x) &= 0\,, \\ [\Delta + 2\varphi_0(x) + a - 1]\,\varphi_1(x) &= 0\,, \\ [\Delta + 2\varphi_0(x) + a - n]\,\varphi_n(x) &= R_n(x)\,, \quad n = 2, 3, \ldots. \end{aligned}$$

We need to choose $n-a$ as one of the eigenvalue of $\Delta+2\varphi_0(x)$, so that $\varphi_n(x) \neq 0$. For example if $\varphi_0 = 0$, then for $\varphi_1 \neq 0$ we need $a := 1 + \lambda_k$, where λ_k is an eigenvalue of $-\Delta$. The sequence of equations reduces to

$$[\Delta + a - 1]\,\varphi_1(x) = 0\,,$$

$$[\Delta + a - n]\,\varphi_n(x) \;=\; \sum_{k=1}^{n-1} \varphi_{n-k}(x)\varphi_k(x)\,, \quad n \geq 2.$$

3. Convergence

For the sake of simplicity, consider equation (1) with $b = 1$, $f(x,t) = f_1(x)e^t$ and $\varphi_0 = 0$ but $\varphi_1 \neq 0$. The starting function φ_1 could be obtained under condition [A] or [B] while the remaining terms of the sequence φ_n are generated by, see (6),

$$[\Delta + a - n]\,\varphi_n(x) = R_n(x)\,, \quad n \geq 2\,, \tag{10}$$

where we recall

$$R_n(x) = -\sum_{k=1}^{n-1} \varphi_{n-k}(x)\varphi_k(x). \tag{11}$$

Since the operator $(\Delta + a - n)$ is bounded $W^2_{2,0}(\Omega) \to L_2(\Omega)$, we have, see [4],

$$\|\varphi_n\|^{(2)}_{2,0} \leq c(n)\,\|R_n\|_2\,. \tag{12}$$

Thus $\varphi_1 \in C(\Omega)$ would lead to $R_2(x) = -\varphi_1^2(x) \in C(\Omega)$, which by (12) implies that, $\varphi_2 \in W^2_{2,0}(\Omega) \hookrightarrow C(\Omega)$, by the Sobolev embedding for $\Omega \subset \mathbb{R}^m$, with $m = 1, 2, 3$. If we assume that $\varphi_k \in W^2_{2,0}(\Omega) \hookrightarrow C(\Omega)$, for $k = 1, ..., n-1$, then by (11) $R_n \in C(\Omega) \subset L^2(\Omega)$ and (12) yields again that

$$\varphi_n \in W^2_{2,0}(\Omega) \hookrightarrow C(\Omega)\,. \tag{13}$$

To proceed further we need the following lemma.

Lemma 2. *Assume that $m = 1, 2, 3$ and φ_n are solution of (10). If $0 \neq \varphi_1 \in C(\Omega)$ then $\varphi_n \in C(\Omega)$ for $n \geq 2$ and*

$$\sup_{x\in\Omega} |\varphi_n(x)| \leq \gamma \sum_{k=1}^{n-1} \sup_{x\in\Omega} |\varphi_{n-k}(x)| \;\sup_{x\in\Omega} |\varphi_k(x)|\;, \tag{14}$$

where γ is a constant that depends on Ω only.

Proof. The fact that all $\varphi_n \in C(\Omega)$ has been shown in (13). We now show that the constant $c(n)$ in (12) is bounded uniformly for n, i.e. $c(n) \leq M$. To this end we consider the Dirichlet problem $(\Delta + a - n)\, u = f$, where $f \in L^2(\Omega)$ and $u \in W^2_{2,0}(\Omega)$. Denote by F the unique solution

$$\begin{cases} \Delta F = f\,, \\ F(x) = 0 \quad x \in \partial\Omega\,, \end{cases}$$

which leads to

$$\begin{cases} (\Delta + a - n)\, u = \Delta F\,, \\ u = 0\,, \quad x \in \partial\Omega\,. \end{cases}$$

Let ϕ_n denote the eigenfunctions of Δ under Dirichlet boundary conditions, i.e. $\Delta\phi_n = \lambda_n\phi_n$ and $\phi_n(x) = 0 \quad$ if $x \in \partial\Omega$. Recall that $\{\phi_n\}_{n\geq 0}$ forms an orthonormal basis in $W^2_{2,0}(\Omega)$ whose inner product is denoted by $(.,.)_2$, see [4]. Thus for $u = \sum_{n\geq 0} \alpha_n\phi_n$ we have

$$\begin{aligned} (\lambda_n + a - n)\,\alpha_n &= (\Delta F,\ \phi_n)_2\ , \\ \alpha_n &= \frac{(F,\ \Delta\phi_n)_2}{\lambda_n + a - n} = \frac{\lambda_n}{\lambda_n + a - n}\,(F,\ \phi_n)_2\ . \end{aligned}$$

Since $\lambda_n \to -\infty$, let us denote by $M := \sup_{n\geq 0}\left|\frac{\lambda_n}{\lambda_n+a-n}\right|$, which yields

$$|\alpha_n| \leq M\,|(F,\ \phi_n)_2|\,.$$

The norm then is bounded by

$$\begin{aligned} \sum_{n\geq 0}|\alpha_n|^2 &\leq M^2\sum_{n\geq 0}|(F,\ \phi_n)_2|^2\ , \\ ||u||^{(2)}_{2,0} &\leq M\,||F||^{(2)}_{2,0} \leq c\,||\Delta F||_2 \leq c\,||f||_2\,. \end{aligned}$$

Thus from (10) we have

$$\begin{aligned} ||\varphi_n||^{(2)}_{2,0} &\leq c\,||R_n||_2 \leq c\,m(\Omega)\,|R_n|_0\ , \\ ||\varphi_n||^{(2)}_{2,0} &\leq c\,m(\Omega)\sum_{k=1}^{n-1}|\varphi_{n-k}|_0\,|\varphi_k|_0\ , \end{aligned} \tag{15}$$

where $m(\Omega) = \int_\Omega 1dx$. By the Sobolev embedding $W^2_{2,0}(\Omega) \hookrightarrow C(\Omega)$, we have

$$|\varphi_n|_0 \leq \kappa\,||\varphi_n||^{(2)}_{2,0}$$

which by (15) leads to

$$|\varphi_n|_0 \leq \kappa c\, m\,(\Omega) \sum_{k=1}^{n-1} |\varphi_{n-k}|_0\, |\varphi_k|_0 \,.$$

If we denote by $\gamma := \kappa c\, m\,(\Omega)$ we have (14). ☐

We now prove convergence of the series of the solution in (2).

Proposition 3. *Assume that* $0 < |\varphi_1|_0 < \frac{1}{4\gamma}$ *and the* φ_n *are defined by (10) then*

$$w(x,t) = \sum_{n\geq 1} \varphi_n(x) e^{nt} \tag{16}$$

converges uniformly for $(x,t) \in \Omega \times [0,\, \text{-}\ln(4\gamma\, |\varphi_1|_0))$ *and is a solution to (8).*

Proof. If we set $b_n := \gamma\, |\varphi_n|_0$ in (14) then the rescaled numerical sequence b_n satisfies $b_n \leq \sum_{k=1}^{n-1} b_k b_{n-k}$. An upper bound for the b_n is given by the increasing sequence a_n defined by

$$a_n = \sum_{k=1}^{n-1} a_{n-k} a_k \,, \quad a_0 = 0\,, \quad a_1 = b_1 > 0\,.$$

The sequence a_n can be computed explicitly by observing that the generating function $g(x) = \sum_{n\geq 1} a_n x^n$, satisfies the quadratic equation

$$g(x)^2 = \sum_{n\geq 2} \left(\sum_{k\geq 1}^{n-1} a_{n-k} a_k \right) x^n = \sum_{n\geq 2} a_n x^n = g(x) - a_1 x\,.$$

The fact that $a_n \geq 0$ and increasing leads us to choose the solution $g(x) = \frac{1}{2} - \frac{1}{2}\sqrt{1-4a_1 x}$ and so $a_n = g^{(n)}(0)/n!$. Thus it is enough to examine the k-th derivative of $g(x)$, which leads to

$$g^{(k)}(x) = \left(-\frac{1}{2}\right) \cdot \left(\frac{1}{2}\right) \left(\frac{1}{2} - 1\right) .. \left(\frac{1}{2} - k + 1\right) (1-4a_1 x)^{\left(\frac{1}{2}-k\right)} (-4a_1)^k$$

$$= \left(-\frac{1}{2}\right) \cdot \left(\frac{1}{2}\right) \left(\frac{-1}{2}\right) \left(\frac{-3}{2}\right) \left(\frac{-5}{2}\right) .. \left(\frac{3-2k}{2}\right) (1-4a_1 x)^{\left(\frac{1}{2}-k\right)} (-4a_1)^k \,.$$

By the obvious estimate $\left|\frac{3-2k}{2}\right| < k-1$ we get

$$\left|g^{(k)}(0)\right| \leq \left(\frac{1}{2}\right) \left(\frac{1}{2}\right) (1)\,(2) .. (k-1)\,(4a_1)^k \leq k! 4^k a_1^k.$$

Thus an upper bound for the sequence a_k: $a_k = \frac{g^{(k)}(0)}{k!} \leq 4^k a_1^k$. This estimate should come at no surprise and is best possible. Clearly the radius of convergence of the Taylor series of g about $x = 0$ is the distance from $x = 0$ to the first singularity of $\sqrt{1 - 4a_1 x}$, which is $x = \frac{1}{4a_1}$.

Thus going back to $b_n \leq a_n$ yields

$$\gamma \left|\varphi_n\right|_0 \leq 4^n a_1^n , \quad \left|\varphi_n\right|_0 \leq \frac{1}{\gamma} 4^n a_1^n.$$

The formal solution

$$w(x,t) = \sum_{n\geq 1} \varphi_n(x) e^{nt} \tag{17}$$

thus converges uniformly for $x \in \Omega$ when $4\left|a_1\right| e^t < 1$, i.e. $t \in [0, \text{-ln}(4a_1))$, and $4a_1 < 1$, which follows from Weierstrass Theorem

$$|w(x,t)| \leq \sum_{n\geq 1} \left|\varphi_n(x)\right|_0 e^{nt} \leq \sum_{n\geq 1} \frac{1}{\gamma} 4^n a_1^n e^{nt} \leq \frac{1}{\gamma} \sum_{n\geq 1} \left(4a_1 e^t\right)^n .$$

So we need to have $b_1 = \gamma \left|\varphi_1\right|_0 \leq a_1 < \frac{1}{4}$ for the series (16) to converge.

We now show that the formal solution w in (16) is in fact a classical solution to (1). For the sake of simplicity we shall prove it for the particular case of (8). Denote the partial sums of the power series in t by $u_k(x,t) = \sum_{n=1}^{k} \varphi_n(x) e^{nt}$. For $x \in \Omega$ and $t \in (0, -\ln(4a_1))$, u_k converges uniformly to w, which we denote by

$$u_k(x,t) \to w(x,t) \tag{18}$$

and so does $\frac{\partial}{\partial t} u_k \to \frac{\partial}{\partial t} w$, since it is a power series in t. We also have $u_k^2 \to w^2$ and so as $k \to \infty$

$$\frac{\partial}{\partial t} u_k - u_k^2 \to \frac{\partial}{\partial t} w - w^2 \quad x \in \Omega \text{ and } t \in [0, -\ln(4a_1)). \tag{19}$$

Since φ_n satisfy the recurrence relation (9), the partial sums also satisfy

$$\begin{aligned} -\Delta u_k = \sum_{n=1}^{k} -\Delta \varphi_n(x) e^{nt} &= \sum_{n=2}^{k} e^{nt} \left\{ -n\varphi_n(x) + \sum_{k=1}^{n-1} \varphi_{n-k}(x) \varphi_k(x) \right\} \\ &+ \left[-\varphi_1(x) + f_1(x)\right] e^t = -\frac{\partial}{\partial t} u_k + u_k^2 + e^t f_1(x). \end{aligned}$$

Thus from (19) we obtain as $k \to \infty$

$$-\Delta u_k \to -\frac{\partial}{\partial t} w + w^2 + e^t f_1(x).$$

Let us denote by g the family of $t-$solutions of the problem

$$\begin{cases} -\Delta g(x,t) = -\frac{\partial}{\partial t} w + w^2 + e^t f_1(x)\,, \\ g(x,t) = 0\,, \quad x \in \partial\Omega, \quad 0 < t < -\ln{(4a_1)}\ . \end{cases} \tag{20}$$

Thus we have $\Delta u_k \to \Delta g(x,t)$ and at the same time

$$\sup_{x \in \Omega} |u_k(x,t) - g(x,t)| \leq C \sup_{x \in \Omega} |\Delta u_k - \Delta g(x,t)|\ ,$$

where C is independent of t, see [3], we deduce that for each $t \in (0, -\ln{(4a_1)})$ we have

$$u_k(.,t) \to g(.,t) \quad \text{in } \Omega. \tag{21}$$

On the other hand since the limit is unique, (18) and (21) imply that $w(x,t) = g(x,t)$ and thus (20) yields

$$\begin{cases} \frac{\partial}{\partial t} w(x,t) = \Delta w(x,t) + w^2(x,t) + e^t f_1(x)\,, \\ w(x,t) = 0\,, \quad x \in \partial\Omega, \quad 0 < t < -\ln{(4a_1)}\ . \end{cases}$$

The initial condition is given by $w(x,0) = \sum_{n \geq 1} \varphi_n(x)$, which converges if $|\varphi_1|_0 < \frac{1}{4\gamma}$. □

The proof can be extended to equation of type (1) when $f(x,t)$ is defined by a finite series, since the main argument uses the sequence φ_n for $n \geq 2$ and the first φ_1 can be generated by one of the starting condition (7).

4. Blow-up

For simplicity we shall consider the equation in (8)

$$\begin{cases} u_t = \Delta u + u^2 + f_1(x) e^t\,, \quad x \in \Omega\,, \\ u(x,t) = 0\,, \quad x \in \partial\Omega\,, \\ u(x,0) = \psi(x)\,, \end{cases} \tag{22}$$

where we now assume that $f_1(x) > 0$. This allows the solution to grow and eventually blow-up in finite time. As in Example 1, we take $\varphi_0(x) = 0$ and this leads to the recurrence equations (9).

Lemma 4. *Assume that* $f_1(x) > 0$, *then the sequence* $\{\varphi_n\}_{n\geq 1}$ *defined by* (9) *is positive* $\varphi_n(x) > 0$ *for* $x \in \Omega - \partial\Omega$.

Proof. To use an induction argument, we first prove that $\varphi_1(x) > 0$. From

$$\begin{cases} (\Delta - 1)\,\varphi_1(x) = -f_1(x) < 0\,, \\ \varphi_1(x) = 0\,, \quad x \in \partial\Omega\,, \end{cases} \tag{23}$$

the maximum principle implies that $\varphi_1(x) > 0$ for $x \in \Omega$. The same also holds for the next iterates

$$\begin{aligned} (\Delta - 2)\,\varphi_2(x) &= -\varphi_1^2(x) < 0\,, \quad (\Delta - 3)\,\varphi_3(x) = -2\varphi_1(x)\varphi_2(x) < 0\,, \\ (\Delta - 4)\,\varphi_4(x) &= -2\varphi_1(x)\varphi_3(x) - \varphi_2^2(x) < 0\,. \end{aligned}$$

If we now assume that $\varphi_1, ..., \varphi_{n-1}$ are positive then $(\Delta - n)\,\varphi_n(x) = -R_n(x) = -\sum_{k=1}^{n-1} \varphi_{n-k}(x)\varphi_k(x) < 0$ also implies that φ_n is also a positive solution and thus for $n \geq 2$

$$\varphi_n(x) > 0\,, \quad \text{for } x \in \Omega\,.$$

To show that the solution blows up we use a standard argument. Denote by ϕ_1 the first eigenfunction of the Laplacian under Dirichlet condition, $-\Delta\phi_1 = \lambda_1\phi_1$, which is known to be positive $\phi_1 > 0$. This leads to

$$\begin{aligned} \int_\Omega u_t\phi_1 dx &= \int_\Omega \Delta u\; \phi_1 dx + \int_\Omega u^2\phi_1 dx + e^t \int_\Omega f_1(x)\phi_1 dx\,, \\ h'(t) &= -\lambda_1 h(t) + \int_\Omega u^2\phi_1 dx + e^t c\,, \end{aligned}$$

where $h(t) := \int_\Omega u(x,t)\phi_1(x)dx$, and the constant $c := \int_\Omega f_1(x)\phi_1(x)dx$. Using Jensen's inequality and the normalization $\int_\Omega \phi_1(x)dx = 1$, we have $\int_\Omega u^2\phi_1 dx \geq \left(\int_\Omega u\phi_1 dx\right)^2$ and so

$$h'(t) \geq -\lambda_1 h(t) + h^2(t) + e^t c\,, \quad h(0) > 0 \text{ and } h(t) > 0\,. \tag{24}$$

Since the solution of the Bernoulli equation $\widetilde{h}'(t) = -\lambda_1\widetilde{h}(t) + \widetilde{h}^2(t)$ is a lower solution of (24), i.e. $h \geq \widetilde{h}$, which blows up in finite time, if $\widetilde{h}(0) = \delta + \lambda_1$ where $\delta > 0$, we deduce that $h(t)$ and so $u(x,t) \to \infty$ as $t \to T \leq \frac{1}{\lambda_1} \ln\left(1 + \frac{\lambda_1}{\delta}\right)$.

In other words we need $h(0) = \int_\Omega \phi_1(x)u(x,0)dx = \int_\Omega \sum\limits_{n\geq 1} \varphi_n(x)\phi_1(x)dx = \sum\limits_{n\geq 1} \int_\Omega \varphi_n(x)\phi_1(x)dx > \lambda_1$, i.e.

$$\sum_{n\geq 1} \int_\Omega \varphi_n(x)\phi_1(x)dx - \lambda_1 = \delta > 0$$

in order for $h(t) \to \infty$ in finite time. □

Proposition 5. *Assume that* $f_1 > 0$ *and* $|\varphi_1|_0 < \frac{1}{4\gamma}$ *then the solution* $\sum_{n\geq 1}\varphi_n(x)e^{nt}$ *is positive and blows up in finite time.*

Proof. By Proposition 3 and Lemma 4 we have the existence of a local positive solution. Assume that the solution remains bounded for all $t > 0$. In this case

$$w(x,t) = \sum_{n\geq 1}\varphi_n(x)e^{nt} > \varphi_1(x)e^t$$

and if t_0 is large enough, we have

$$\sum_{n\geq 1}\int_\Omega e^{nt}\varphi_n(x)\phi_1(x)dx > e^{t_0}\int_\Omega \varphi_1(x)\phi_1(x)dx > \lambda_1\,.$$

By taking $w(x,t_0)$ as a new initial condition, the solution will blow-up at a later time

$$T \leq t_0 + \frac{1}{\lambda_1}\ln\left(1+\frac{\lambda_1}{\delta}\right). \qquad \square$$

Example 3. A simple example of a blow-up solution, consider the equation (22) with, $m = 1$ and $f_1(x) = \frac{1}{4\gamma}\sin(x)$, where γ is the positive constant mentioned in Proposition 3,

$$u_t = u_{xx} + u^2 + \frac{1}{4\gamma}\sin(x)e^t \quad 0 \leq x \leq \pi \text{ and } t > 0\,, \tag{25}$$

if we take $\varphi_0(x) = 0$ then (23) becomes

$$-\varphi_1''(x) - \varphi_1(x) = \frac{1}{4\gamma}\sin(x)\,, \quad \varphi_1(x) = \frac{1}{8\gamma}\sin(x)\,,$$

$$\varphi_2(x) = \frac{1}{64\gamma^2}\left(\frac{\left(1-e^{-\pi\sqrt{2}}\right)\sinh\left(x\sqrt{2}\right)}{12\sinh\left(\pi\sqrt{2}\right)} - \frac{1}{4} + \frac{1}{12}\cos\left(2x\right)\right)\,.$$

Here φ_3 and φ_4 can be obtained from

$$\begin{cases} \varphi_3'' - 3\varphi_3 = -2\varphi_1\varphi_2\,, \\ \varphi_3(0) = \varphi_3(\pi) = 0\,, \end{cases} \qquad \begin{cases} \varphi_4'' - 4\varphi_4 = -2\varphi_1\varphi_3 - \varphi_2^2\,, \\ \varphi_4(0) = \varphi_4(\pi) = 0\,, \end{cases}$$

and the solution to (25), which is given by the series

$$u(x,t) = \sin(x)e^t + \varphi_2(x)e^{2t} + \varphi_3(x)e^{3t} + ...,$$

blows up in finite time by Proposition 5 since $|\varphi_1|_0 < \frac{1}{4\gamma}$.

Future work will investigate the radius of convergence of the series in time $\sum_{n\geq 0}\varphi_n(x)e^{nt}$. If it is not constant, as a function of x, then the extention beyond blow-up is possible.

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