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FIXED POINT THEOREMS AND STABILITY FOR SEVERAL CLASSES OF MULTI-VALUED MAPPINGS

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Abstract: In this paper several fixed point theorems and a stability theorem for some new classes of multi-valued contractive type and nonlinear mappings are given. The results presented in this paper improve and generalize some results in literatures.

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1. Introduction

Let (X, d) be a metric space and $CL(X)$ (respectively, $CB(X)$, $C(X)$) denote the family of all nonempty closed (nonempty closed bounded, nonempty compact) subsets of X. For $c \in X$, $A, B \in CL(X)$ and $T : X \to CL(X)$, define

$$F(T) = \{x \in X : x \in Tx\},\ d(c, B) = \inf_{y \in B} d(c, y),$$
$$H_-(A, B) = \sup_{x \in A} d(x, B),\ H_+(A, B) = H_-(B, A),$$
$$H(A, B) = \max\{H_-(A, B), H_+(B, A)\}.$$

Obviously,

$$d(x, y) = H_-(\{x\}, \{y\}) = H_+(\{x\}, \{y\}),$$
$$H_-(\{x\}, A) = d(x, A), \quad \forall x, y \in X,\ A \in CL(X).$$

In 1969, Nadler [10] first studied the existence and stability of fixed points for the multi-valued contraction mapping and proved the following nice result.

Theorem 1.1. (see [10]) *Let (X, d) be a complete metric space and $T : X \to CB(X)$ be a multi-valued contraction mapping, that is, there exists a constant $r \in [0, 1)$ satisfying*

$$H(Tx, Ty) \le rd(x, y), \quad \forall x, y \in X. \tag{1.1}$$

Then T has a fixed point in X.

Afterwards, many researchers gave various generalizations of Nadler's results, for example, see [1-11] and the references cited therein. In particular, Lim [3] extended Nadler's Stability Theorem from $CB(X)$ to $CL(X)$ and Wang [11] obtained a few extensions of Nadler and Lim's results from the class of multi-valued contraction mappings (1.1) to the larger class of multi-valued mappings (1.2) below: there exist there exist constants $\alpha, \beta \in [0, 1)$ with $\alpha + 2\beta < 1$ satisfying

$$H_-(Tx, Ty) \le \alpha d(x, y) + \beta[d(x, Tx) + d(y, Ty)], \quad \forall x, y \in X. \tag{1.2}$$

On the other hand, Liu [4] proved some periodic point results for the following single-valued mappings:

$$d(x, y) > a[d(x, T^m x) + d(y, T^m y)] + bd(T^m x, T^m y),$$
$$\forall x, y \in \overline{O(x_0, T^m)} \text{ with } x \ne y, \tag{1.3}$$

where a and b are constants with $2a+b>1$, $a\le 1$, $b>0$, m is a positive integer and $O(x_0,T^m)=\{T^{nm}x_0\}_{n\ge 0}$.

Motivated and inspired by the work [1-4, 10, 11], in this paper, we introduce a new class of multi-valued contractive type mappings and a few classes of multi-valued nonlinear mappings in complete metric spaces and show several fixed point theorems and a stability theorem for the multi-valued contractive type mappings and multi-valued nonlinear mappings. The results presented in this paper extend and improve some known results in [1, 3, 10, 11].

2. Preliminaries and Lemmas

Lemma 2.1. (see [11]) *Let (X,d) be a metric space and $A,B\in CL(X)$. Then:*

(a) $H_-(A,B)\ge 0$;

(b) $H_-(A,B)=0$ *if and only if* $A\subseteq B$;

(c) $H_-(A,B)\le H_-(A,C)+H_-(C,B)$ *for any* $A,B,C\in CL(X)$.

Remark 2.1. If H_- is replaced by H_+, then Lemma 2.1 holds also.

Lemma 2.2. *Let (X,d) be a metric space and $A,B\in CL(X)$. Then:*

(a) *for any $\varepsilon>0$ and $a\in A$, there exists $b\in B$ such that*

$$d(a,b)<H_-(A,B)+\varepsilon;$$

(b) *for any $\varepsilon>0$, $r>1$ and $a\in A$, there exists $b\in B$ such that*

$$d(a,b)\le rH_-(A,B).$$

Proof. (a) Let $\varepsilon>0$ and $a\in A$ be arbitrary. It follows that there exists some $b\in B$ with $d(a,b)<d(a,B)+\varepsilon$. Note that $d(a,B)\le H_-(A,B)$. Thus $d(a,b)<H_-(A,B)+\varepsilon$.

(b) Let $\varepsilon>0, r>1$ and $a\in A$. Suppose that $H_-(A,B)=0$. Lemma 2.1 ensures that $A\subseteq B$. Consequently, for $b=a\in A\subseteq B$, we deduce that $d(a,b)=0=rH_-(A,B)$. Suppose that $H_-(A,B)>0$. If (b) does not hold, it follows that there exist $r_0>1$ and $a_0\in A$ with $d(a_0,b)>r_0H_-(A,B)$, $\forall b\in B$. It is clear that

$$d(a_0,B)\ge r_0H_-(A,B)\ge r_0d(a_0,B), \tag{2.1}$$

which implies that $(1-r_0)d(a_0,B)\ge 0$, that is, $d(a_0,B)=0$. It follows from (2.1) that $H_-(A,B)=0$, which is a contradiction. This completes the proof. □

Remark 2.2. Lemma 2.2 (a) is a generalization of Lemma 1 in [1].

Lemma 2.3. *Let (X, d) be a metric space. Then:*
(a) $|d(x, A) - d(x, B)| \le H(A, B)$, $\forall x, y \in X, A, B \in CL(X)$;
(b) $|d(x, A) - d(y, A)| \le d(x, y)$, $\forall x, y \in X, A \in CL(X)$.

Proof. (a) Let $A, B \in CL(X), x, y \in X, z \in B$. It follows from Lemma 2.1 that

$$d(x, A) \le d(x, z) + d(z, A) \le d(x, z) + H_{-}(B, A),$$

which implies that

$$d(x, A) \le d(x, B) + H_{-}(B, A).$$

Similarly, $d(x, B) \le d(x, A) + H_{-}(A, B)$. Consequently, we get that

$$|d(x, A) - d(x, B)| \le \max\{H_{-}(A, B), H_{-}(B, A)\} = H(A, B).$$

(b) Let $A \in CL(X), x, y \in X, a \in A$. Obviously,

$$d(x, a) \le d(x, y) + d(y, a),$$

which implies that

$$d(x, A) \le d(x, y) + d(y, A).$$

Similarly, $d(y, A) \le d(x, y) + d(x, A)$. Therefore, $|d(x, A) - d(y, A)| \le d(x, y)$. This completes the proof. □

3. Fixed Point Theorems

In this section, we first prove a fixed point theorem for multi-valued contractive type mapping (3.1), which extends the corresponding results of Nadler [10] and Wang [11]. Furthermore, we give an example to show that our result indeed generalizes Theorem 2.4 of [11]. Second, we prove a fixed point theorem for a few classes of multi-valued nonlinear mappings (3.5)-(3.8) in complete metric spaces.

Theorem 3.1. *Let (X, d) be a complete metric space and $T : X \to CL(X)$ be a multi-valued mapping satisfying*

$$H_{-}(Tx, Ty) \le \alpha \max\Big\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)], \frac{[1 + d(y, Ty)]d(y, Tx)}{1 + d(x, Tx)}\Big\}, \quad \forall x, y \in X, \tag{3.1}$$

where α is a constant in $[0, 1)$. Then T has a fixed point in X.

Proof. Let $\varepsilon > 0$ be arbitrary. We claim that there exists a sequence $\{x_n\}_{n\geq 0}$ satisfying $x_{n+1} \in Tx_n$ and

$$d(x_n, x_{n+1}) \leq \alpha^n d(x_0, x_1) + n\alpha^n \varepsilon \quad n \geq 0. \tag{3.2}$$

Pick an arbitrary point $x_0 \in X$. Choose a point $x_1 \in Tx_0$. Clearly, (3.2) holds for $n = 0$. Suppose that (3.2) holds for some $n \geq 0$. Lemma 2.1 implies that there exists some point $x_{n+2} \in Tx_{n+1}$ such that

$$\begin{aligned}
&d(x_{n+1}, x_{n+2})\\
&\leq H_-(Tx_n, Tx_{n+1}) + \alpha(1-\alpha)\varepsilon\\
&\leq \alpha \max\{d(x_n, x_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1}),\\
&\qquad \frac{1}{2}[d(x_n, Tx_{n+1}) + d(x_{n+1}, Tx_n)],\\
&\qquad \frac{[1 + d(x_{n+1}, Tx_{n+1})]d(x_{n+1}, Tx_n)}{1 + d(x_n, Tx_n)}\Big\} + \alpha(1-\alpha)\varepsilon\\
&\leq \alpha \max\Big\{d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \frac{1}{2}d(x_n, x_{n+2}), 0\Big\}\\
&\quad + \alpha(1-\alpha)\varepsilon\\
&= \alpha \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\} + \alpha(1-\alpha)\varepsilon.
\end{aligned} \tag{3.3}$$

Suppose that $d(x_n, x_{n+1}) \leq d(x_{n+1}, x_{n+2})$. It follows from (3.3) that

$$\begin{aligned}
d(x_{n+1}, x_{n+2}) &\leq \alpha d(x_{n+1}, x_{n+2}) + \alpha(1-\alpha)\varepsilon,\\
&\leq \alpha d(x_{n+1}, x_{n+2}) + \alpha(1-\alpha)d(x_n, x_{n+1}) + \alpha(1-\alpha)\varepsilon,
\end{aligned}$$

which yields that

$$d(x_{n+1}, x_{n+2}) \leq \alpha d(x_n, x_{n+1}) + \alpha\varepsilon. \tag{3.4}$$

Suppose that $d(x_n, x_{n+1}) > d(x_{n+1}, x_{n+2})$. (3.3) means that

$$\begin{aligned}
d(x_{n+1}, x_{n+2}) &\leq \alpha d(x_n, x_{n+1}) + \alpha(1-\alpha)\varepsilon\\
&\leq \alpha d(x_n, x_{n+1}) + \alpha\varepsilon,
\end{aligned}$$

that is, (3.4) holds. In terms of (3.4), we see that

$$\begin{aligned}
d(x_{n+1}, x_{n+2}) &\leq \alpha[\alpha^n d(x_0, x_1) + n\alpha^n\varepsilon] + \alpha\varepsilon\\
&= \alpha^{n+1} d(x_0, x_1) + (n+1)\alpha^{n+1}\varepsilon.
\end{aligned}$$

Therefore (3.2) holds for $n+1$. Consequently, (3.2) holds for every $n \geq 0$. For any $m \geq 1$ and $p \geq 1$, by (3.2) we infer that

$$\begin{aligned} d(x_m, x_{m+p}) &\leq \sum_{n=m}^{p-1} d(x_n, x_{n+1}) \leq \sum_{n=m}^{p-1} [\alpha^n d(x_0, x_1) + n\alpha\varepsilon] \\ &\leq \frac{\alpha^m}{1-\alpha} d(x_0, x_1) + \varepsilon \sum_{n=m}^{p-1} n\alpha^n. \end{aligned}$$

Notice that $\alpha \in [0, 1)$ and

$$\lim_{m\to\infty} \left[\frac{\alpha^m}{1-\alpha} d(x_0, x_1) + \varepsilon \sum_{n=m}^{\infty} n\alpha^n \right] = 0.$$

Hence $\{x_n\}_{n\geq 0}$ is a Cauchy sequence and there exists some $z \in X$ with $\lim_{n\to\infty} x_n = z$ by completeness of X.

Now we prove that z is the fixed point of T. In fact, by (3.1), Lemma 2.1 and Lemma 2.3, we infer that

$$\begin{aligned} &d(z, Tz) \\ &\leq d(z, x_n) + d(x_n, Tx_n) + H_-(Tx_n, Tz) \\ &\leq d(z, x_n) + d(x_n, x_{n+1}) + \alpha \max\{d(x_n, z), d(x_n, Tx_n), d(z, Tz), \\ &\quad \frac{1}{2}[d(x_n, Tz) + d(z, Tx_n)], \frac{[1 + d(z, Tz)]d(z, Tx_n)}{1 + d(x_n, Tx_n)}\Big\} \\ &\leq d(z, x_n) + d(x_n, x_{n+1}) + \alpha \max\{d(x_n, z), d(x_n, x_{n+1}), d(z, Tz), \\ &\quad \frac{1}{2}[d(x_n, z) + d(z, Tz) + d(z, x_{n+1})], [1 + d(z, Tz)]d(z, x_{n+1})\} \end{aligned}$$

for all $n \geq 0$. Letting $n \to \infty$ in the above inequalities, we conclude that

$$\begin{aligned} d(z, Tz) &\leq \alpha \max\left\{0, 0, d(z, Tz), \frac{1}{2}[0 + d(z, Tz) + 0], 0\right\} \\ &= \alpha d(z, Tz), \end{aligned}$$

which implies that $z \in Tz$. This completes the proof. □

Remark 3.1. Theorem 5 of Nadler [10] and Theorem 2.4 of Wang [11] are special cases of Theorem 3.1. The following examples reveal that Theorem 3.1 extends properly these results of Nadler [10] and Wang [11].

Example 3.1. Let $X = [0, 1]$ with the usual metric and define a multi-valued mapping $T : X \to CL(X)$ by

$$Tx = \begin{cases} \left\{0, \frac{x}{5}\right\} & \text{for } x \in \left[0, \frac{1}{2}\right), \\ \left\{0, \frac{x}{6}\right\} & \text{for } x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

It is clear that the multi-valued mapping T satisfies (3.1) with $\alpha = \frac{7}{10}$. Thus Theorem 3.1 guarantees that T has a fixed point in X. Note that for $x \in [0, \frac{1}{2})$,

$$\begin{aligned} H\Big(Tx, T\frac{1}{2}\Big) &= H\Big(\Big\{0, \frac{x}{5}\Big\}, \Big\{0, \frac{1}{12}\Big\}\Big) \\ &= \max\Big\{ \min\Big\{\frac{x}{5}, \Big|\frac{x}{5} - \frac{1}{12}\Big|\Big\}, \min\Big\{\frac{1}{12}, \Big|\frac{x}{5} - \frac{1}{12}\Big|\Big\}\Big\}. \end{aligned}$$

This implies that $\lim_{x \to (\frac{1}{2})^-} H(Tx, T\frac{1}{2}) = \frac{1}{60} \neq 0$. It follows that T is not continuous at $\frac{1}{2}$. Consequently, T is not a multi-valued contraction mapping and therefore, Theorem 5 in [10] is not applicable.

Example 3.2. Let $X = \{1, 2, 3, 8\}$ with the usual metric. Define a multi-valued mapping $T : X \to CL(X)$ by $T2 = \{8\}, Tx = \{3, 8\}$ for $x \in \{1, 3, 8\}$. It is easy to check the multi-valued mapping T satisfies (3.1) with $\alpha = \frac{6}{7}$. Thus Theorem 3.1 guarantees that T has a fixed point in the complete metric space X. But we do not invoke Theorem 2.4 in [11] to show that T possesses a fixed point in X since

$$\begin{aligned} H_-(T1, T2) = 5 > 4 &\geq (\alpha + 2\beta) \max\Big\{d(2, 1), \frac{1}{2}[d(2, T2) + d(1, T1)]\Big\} \\ &\geq \alpha d(2, 1) + \beta[d(2, T2) + d(1, T1)] \end{aligned}$$

for any nonnegative constants α, β with $\alpha + 2\beta < 1$.

Theorem 3.2. *Let (X, d) be a complete metric space and $T : X \to C(X)$ be a continuous multi-valued mapping such that for any different $x, y \in X$, at least one of the following conditions is fulfilled:*

$$\begin{aligned} &\max\Big\{d(x, y), d(x, Tx), \frac{d(y, Tx)d(x, Ty)}{d(x, y)}, \frac{d(y, Tx)d(x, Tx)}{d(x, y)}\Big\} \\ &> a \min\Big\{d(x, Tx), d(y, Ty), \frac{d^2(x, Tx)}{d(x, y)}, \frac{d(x, Tx)d(y, Ty)}{d(x, y)}\Big\} \qquad (3.5) \\ &\quad + bH_-(Tx, Ty), \end{aligned}$$

where a, b are nonnegative constants with $a + b = 1$;

$$\begin{aligned}&\max\{d^2(x,y), d(y,Tx)d(x,Ty), d(y,Tx)d(x,Tx), d(x,y)d(x,Tx)\}\\ &> a\min\{d^2(x,Tx), d^2(y,Ty), d(x,Tx)H_-(Tx,Ty),\\ &\quad d(y,Ty)H_-(Tx,Ty)\} + bH_-^2(Tx,Ty)\,,\end{aligned} \tag{3.6}$$

where a, b are nonnegative constants with $a + b = 1$;

$$\begin{aligned}&\max\{d^2(x,y), d^2(y,Ty), d(x,y)d(x,Tx)\}\\ &> a\min\{d^2(x,Tx), d(x,y)d(y,Ty), d(y,Tx)d(x,Ty)\} + H_-^2(Tx,Ty)\,,\end{aligned} \tag{3.7}$$

where a is a constant in $(-\infty, +\infty)$;

$$\begin{aligned}H_-(Tx,Ty) < \max\Big\{&d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}[d(x,Ty) + d(y,Tx)],\\ &\frac{[1 + d(y,Ty)]d(y,Tx)}{1 + d(x,Tx)}\Big\}.\end{aligned} \tag{3.8}$$

Assume that there exists a sequence $\{x_n\}_{n\ge0} \subset X$ with

$$d(x_n, x_{n+1}) = d(x_n, Tx_n),\ x_{n+1} \in Tx_n, \quad \forall n \ge 0 \tag{3.9}$$

and the sequence has a cluster point $z \in X$. Then z is a fixed point of T.

Proof. We claim that $x_n \neq x_{n+1}$ for each $n \ge 0$. Otherwise there exists some $k \ge 0$ such that $x_k = x_{k+1}$. Consequently, $x_k = x_{k+1} \in Tx_k$ and $d(x_{k+1}, x_{k+2}) = d(x_{k+1}, Tx_{k+1}) = 0$, which gives that $x_k = x_{k+1} = x_{k+2}$. It is easy to see that $x_n = x_k$ for any $n > k$. Thus $\{x_n\}_{n\ge0}$ does not have a cluster point, which is impossible.

We now assert that

$$d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}), \quad \forall n \ge 0. \tag{3.10}$$

Let n be a nonnegative integer. We have to consider the following cases:

Case 1. Suppose that (3.5) holds for x_n and x_{n+1}. It follows from (3.5) and

(3.9) that

$$\begin{aligned}
&d(x_n, x_{n+1})\\
&= \max\Big\{d(x_n, x_{n+1}), d(x_n, Tx_n), \frac{d(x_{n+1}, Tx_n)d(x_n, Tx_{n+1})}{d(x_n, x_{n+1})},\\
&\quad \frac{d(x_{n+1}, Tx_n)d(x_n, Tx_n)}{d(x_n, x_{n+1})}\Big\}\\
&> a \min\Big\{d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1}), \frac{d^2(x_n, Tx_n)}{d(x_n, x_{n+1})},\\
&\quad \frac{d(x_n, Tx_n)d(x_{n+1}, Tx_{n+1})}{d(x_n, x_{n+1})}\Big\} + bH_-(Tx_n, Tx_{n+1})\\
&\geq a \min\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}\\
&\quad + bd(x_{n+1}, Tx_{n+1})\\
&= a \min\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\} + bd(x_{n+1}, x_{n+2}).
\end{aligned} \tag{3.11}$$

If $d(x_n, x_{n+1}) \leq d(x_{n+1}, x_{n+2})$, it follows from (3.11) that

$$d(x_n, x_{n+1}) > ad(x_n, x_{n+1}) + bd(x_{n+1}, x_{n+2}),$$

which yields that

$$(1-a)d(x_n, x_{n+1}) > bd(x_{n+1}, x_{n+2}) \geq (1-a)d(x_n, x_{n+1}),$$

which is a contradiction. Hence $d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1})$;

Case 2. Suppose that (3.6) holds for x_n and x_{n+1}. By virtue of (3.6) and (3.9) we see that

$$\begin{aligned}
&d^2(x_n, x_{n+1})\\
&= \max\{d^2(x_n, x_{n+1}), d(x_{n+1}, Tx_n)d(x_n, Tx_{n+1}),\\
&\quad d(x_{n+1}, Tx_n)d(x_n, Tx_n), d(x_n, x_{n+1})d(x_n, Tx_n)\}\\
&> a \min\{d^2(x_n, Tx_n), d^2(x_{n+1}, Tx_{n+1}), d(x_n, Tx_n)H_-(Tx_n, Tx_{n+1}),\\
&\quad d(x_{n+1}, Tx_{n+1})H_-(Tx_n, Tx_{n+1})\} + bH_-^2(Tx_n, Tx_{n+1})\\
&\geq a \min\{d^2(x_n, x_{n+1}), d^2(x_{n+1}, x_{n+2}), d(x_n, x_{n+1})d(x_{n+1}, Tx_{n+1}),\\
&\quad d(x_{n+1}, x_{n+2})d(x_{n+1}, Tx_{n+1})\} + bd^2(x_{n+1}, Tx_{n+1})\\
&= a \min\{d^2(x_n, x_{n+1}), d^2(x_{n+1}, x_{n+2})\} + bd^2(x_{n+1}, x_{n+2}).
\end{aligned} \tag{3.12}$$

If $d(x_n, x_{n+1}) \leq d(x_{n+1}, x_{n+2})$, from (3.12) we deduce that

$$d^2(x_n, x_{n+1}) > ad^2(x_n, x_{n+1}) + bd^2(x_{n+1}, x_{n+2})$$

$$\geq (a+b)d^2(x_n, x_{n+1}) = d^2(x_n, x_{n+1}),$$

which is impossible and hence $d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1})$;

Case 3. Suppose that (3.7) holds for x_n and x_{n+1}. Using (3.7) and (3.9), we have

$$\begin{aligned}
&\max\{d^2(x_n, x_{n+1}), d^2(x_{n+1}, x_{n+2})\} \\
&= \max\{d^2(x_n, x_{n+1}), d^2(x_{n+1}, Tx_{n+1}), d(x_n, x_{n+1})d(x_n, Tx_n)\} \\
&> a\min\{d^2(x_n, Tx_n), d(x_n, x_{n+1})d(x_{n+1}, Tx_{n+1}), \\
&\qquad d(x_{n+1}, Tx_n)d(x_n, Tx_{n+1})\} + H_-^2(Tx_n, Tx_{n+1}) \\
&\geq a\min\{d^2(x_n, x_{n+1}), d(x_n, x_{n+1})d(x_{n+1}, x_{n+2}), 0\} + d^2(x_{n+1}, x_{n+2}) \\
&= d^2(x_{n+1}, x_{n+2}),
\end{aligned}$$

which implies that $d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1})$;

Case 4. Suppose that (3.8) holds for x_n and x_{n+1}. It follows from (3.8) and (3.9) that

$$\begin{aligned}
&H_-(Tx_n, Tx_{n+1}) \\
&< \max\Bigg\{d(x_n, x_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1}), \\
&\qquad \frac{1}{2}[d(x_n, Tx_{n+1}) + d(x_{n+1}, Tx_n)], \frac{[1 + d(x_{n+1}, Tx_{n+1})]d(x_{n+1}, Tx_n)}{1 + d(x_n, Tx_n)}\Bigg\} \\
&= \max\left\{d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \frac{1}{2}[d(x_n, x_{n+2}) + 0], 0\right\} \\
&= \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\},
\end{aligned}$$

which implies that

$$\begin{aligned}
d(x_{n+1}, x_{n+2}) = d(x_{n+1}, Tx_{n+1}) &\leq H_-(Tx_n, Tx_{n+1}) \\
&< \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\},
\end{aligned}$$

that is, $d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1})$.

Hence (3.10) holds for $n \geq 0$. It follows that $\{d(x_n, x_{n+1})\}_{n\geq 0}$ is a strictly decreasing sequence and it converges to some $r \geq 0$. Since $\{x_n\}_{n\geq 0}$ has a cluster point z, there exists a subsequence $\{x_{n_i}\}_{i\geq 1}$ of $\{x_n\}_{n\geq 0}$ such that $\lim_{i\to\infty} x_{n_i} = z$. It follows from Lemma 2.3 that

$$\begin{aligned}
&|d(x_{n_i}, Tx_{n_i}) - d(x_{n_i}, Tz)| \leq H(Tx_{n_i}, Tz) \to 0 \quad \text{as } i \to \infty, \\
&|d(x_{n_i}, Tz) - d(z, Tz)| \leq d(x_{n_i}, z) \to 0 \quad \text{as } i \to \infty,
\end{aligned}$$

which imply that

$$d(x_{n_i}, x_{n_i+1}) = d(x_{n_i}, Tx_{n_i}) \to d(z, Fz) = r \quad \text{as } i \to \infty. \tag{3.13}$$

Notice that the continuity of T guarantees that

$$d(x_{n_i+1}, Tz) \leq H_-(Tx_{n_i}, Tz) \leq H(Tx_{n_i}, Tz) \to 0 \quad \text{as } i \to \infty. \tag{3.14}$$

From $Tz \in C(X)$, we can select a sequence $\{z_{n_i}\}_{i\geq 1} \subset Tz$ such that $d(x_{n_i+1}, z_{n_i}) = d(x_{n_i+1}, Tz)$ for each $i \geq 1$. The compactness of Tz ensures that there exists a subsequence $\{z_{n_{i_k}}\}_{k\geq 1}$ of $\{z_{n_i}\}_{i\geq 1}$ with $\lim_{k\to\infty} z_{n_{i_k}} = z_1 \in Tz$. It is to easy see that

$$d(x_{n_{i_k}+1}, z_1) \leq d(x_{n_{i_k}+1}, z_{n_{i_k}}) + d(z_{n_{i_k}}, z_1) \to 0 \quad \text{as } k \to \infty\,. \tag{3.15}$$

From Lemma 2.3, we conclude that

$$|d(x_{n_{i_k}+1}, Tx_{n_{i_k}+1}) - d(x_{n_{i_k}+1}, Tz_1)| \leq H(Tx_{n_{i_k}+1}, Tz_1) \to 0 \quad \text{as } k \to \infty,$$
$$|d(x_{n_{i_k}+1}, Tz_1) - d(z_1, Tz_1)| \leq d(x_{n_{i_k}+1}, z_1) \to 0 \quad \text{as } k \to \infty,$$

which imply that

$$d(x_{n_{i_k}+1}, x_{n_{i_k}+2}) = d(x_{n_{i_k}+1}, Tx_{i_k+1}) \to d(z_1, Tz_1) = r \quad \text{as } k \to \infty. \tag{3.16}$$

In light of (3.11) and (3.13) we derive that

$$d(x_{n_{i_k}}, z_1) \leq d(x_{n_{i_k}}, x_{n_{i_k}+1}) + d(x_{n_{i_k}+1}, z_1) \to r \quad \text{as } k \to \infty,$$

that is, $d(z, z_1) = r$. It follows from (3.13)-(3.16) that

$$r = d(z, z_1) = d(z_1, Tz_1) = d(z, Tz). \tag{3.17}$$

Next we show that $z = z_1$. Suppose that $z \neq z_1$. We have to consider the following cases:

Case 1. Suppose that (3.5) holds for z and z_1. It follows from (3.5) and (3.17) that

$$\begin{aligned} r &= \max\Big\{d(z, z_1), d(z, Tz), \frac{d(z_1, Tz)d(z, Tz_1)}{d(z, z_1)}, \frac{d(z_1, Tz)d(z, Tz)}{d(z, z_1)}\Big\} \\ &> a \min\Big\{d(z, Tz), d(z_1, Tz_1), \frac{d^2(z, Tz)}{d(z, z_1)}, \frac{d(z, Tz)d(z_1, Tz_1)}{d(z, z_1)}\Big\} \\ &\quad + bH_-(Tz, Tz_1) \geq a \min\{r, r, r, r\} + br = r, \end{aligned}$$

which is a contradiction;

Case 2. Suppose that (3.6) holds for z and z_1. From (3.6) and (3.17) we infer immediately that

$$\begin{aligned} r^2 &= \max\{d^2(z,z_1), d(z_1,Tz)d(z,Tz_1), d(z_1,Tz)d(z,Tz), d(z,z_1)d(z,Tz)\} \\ &> a\min\{d^2(z,Tz), d^2(z_1,Tz_1), d(z,Tz)H_-(Tz,Tz_1), \\ &\quad d(z_1,Tz_1)H_-(Tz,Tz_1)\} + bH_-^2(Tz,Tz_1) \\ &\geq a\min\{r^2, r^2, rd(z_1,Tz_1), rd(z_1,Tz_1)\} + bd^2(z_1,Tz_1) = r^2, \end{aligned}$$

which is impossible;

Case 3. Suppose that (3.7) holds for z and z_1. By virtue of (3.7) and (3.17) we obtain easily that

$$\begin{aligned} r^2 &= \max\{d^2(z,z_1), d^2(z_1,Tz_1), d(z,z_1)d(z,Tz)\} \\ &> a\min\{d^2(z,Tz), d(z,z_1)d(z_1,Tz_1), d(z_1,Tz)d(z,Tz_1)\} + H_-^2(Tz,Tz_1) \\ &\geq a\min\{r^2, r^2, 0\} + r^2 = r^2, \end{aligned}$$

which is a contradiction;

Case 4. Suppose that (3.8) holds for z and z_1. In view of (3.8) and (3.17) we get that

$$\begin{aligned} r &= d(z_1,Tz_1) \leq H_-(Tz,Tz_1) \\ &< \max\Big\{d(z,z_1), d(z,Tz), d(z_1,Tz_1), \frac{1}{2}[d(z,Tz_1) + d(z_1,Tz)], \\ &\quad \frac{[1+d(z_1,Tz_1)]d(z_1,Tz)}{1+d(z,Tz)}\Big\} \leq \max\{r,r,r,r,0\} = r, \end{aligned}$$

which is impossible.

Hence $z = z_1$ and $z \in Fz$ by (3.17). This completes the proof. □

Theorem 3.3. *Let (X,d) be a complete metric space and $T : X \to C(X)$ be a continuous multi-valued mapping such that for any different $x, y \in X$, one of the following conditions is satisfied:*

$$d(x,y) > a\min\{d(x,Tx), d(y,Ty)\} + bH_-(Tx,Ty), \tag{3.18}$$

where a, b are nonnegative constants with $a + b = 1$;

$$d(x,y) > a[d(x,Tx) + d(y,Ty)] + bH_-(Tx,Ty), \tag{3.19}$$

where a, b are nonnegative constants with $2a + b = 1$. Assume that there exists a sequence $\{x_n\}_{n\geq 0} \subset X$ such that (3.9) holds and the sequence has a cluster point z in X. Then z is a fixed point of T.

Proof. Note that (3.19) implies that (3.18), which, in turn, implies (3.5). Thus Theorem 3.3 follows from Theorem 3.2. This completes the proof. □

4. Fixed Point Stability

In this section, we shall study the stability of fixed points for the multi-valued contractive type mapping (3.1) in a complete metric space.

Lemma 4.1. *Let (X, d) be a complete metric space. Suppose that T_1 and $T_2 : X \to CL(X)$ satisfy* (3.1) *with a common constant $\alpha \in [0, 1)$. Then*

$$H_-(F(T_1), F(T_2)) \le \frac{1}{1-\alpha} \sup_{x \in X} H_-(T_1 x, T_2 x).$$

Proof. Put $K = \sup_{x \in X} H_-(T_1 x, T_2 x)$. Without loss of generality we may assume that $K < +\infty$. It follows from Theorem 3.1 that $F(T_1) \ne \emptyset$. Let $\varepsilon > 0$ be arbitrary and $x_0 \in F(T_1)$. Lemma 2.2 ensures that there exists a point $x_1 \in T_2 x_0$ with $d(x_0, x_1) < H_-(T_1 x_0, T_2 x_0) + \varepsilon \le K + \varepsilon$. Similarly, there exists $x_2 \in T_2 x_1$ with $d(x_1, x_2) \le H_-(T_2 x_0, T_2 x_1) + \alpha(1-\alpha)\varepsilon$. As in the proof of Theorem 3.1, we conclude that there exists a sequence $\{x_n\}_{n \ge 0} \subset X$ with $x_n \in T_2 x_{n-1}$ and $d(x_n, x_{n+1}) \le \alpha^n d(x_0, x_1) + n\alpha^n \varepsilon, n \ge 0$. Furthermore, we derive that

$$\sum_{n=m}^{p-1} d(x_n, x_{n+1}) \le \frac{\alpha^m}{1-\alpha} d(x_0, x_1) + \varepsilon \sum_{n=m}^{p-1} n\alpha^n, \quad \forall m \ge 1,\, p \ge 1,$$

which implies that $\{x_n\}_{n \ge 0}$ is a Cauchy sequence with $\lim_{n\to\infty} x_n = \overline{x} \in X$ by completeness of X. By Lemmas 2.1 and 2.3, we obtain that

$$\begin{aligned}
d(\overline{x}, T_2\overline{x}) &\le d(\overline{x}, x_n) + d(x_n, T_2 x_n) + H_-(T_2 x_n, T_2 \overline{x}) \\
&\le d(\overline{x}, x_n) + d(x_n, x_{n+1}) + \alpha \max\{d(x_n, \overline{x}), d(x_n, T_2 x_n), \\
&\quad d(\overline{x}, T_2\overline{x}), \frac{1}{2}[d(x_n, T_2\overline{x}) + d(\overline{x}, T_2 x_n)], \frac{[1 + d(\overline{x}, T_2\overline{x})] d(\overline{x}, T_2 x_n)}{1 + d(x_n, T x_n)}\Big\} \\
&\le d(\overline{x}, x_n) + d(x_n, x_{n+1}) + \alpha \max\{d(x_n, \overline{x}), d(x_n, x_{n+1}), \\
&\quad d(\overline{x}, T_2\overline{x}), \frac{1}{2}[d(x_n, \overline{x}) + d(\overline{x}, T_2 x_n) + d(\overline{x}, x_{n+1})], \\
&\quad [1 + d(\overline{x}, T_2\overline{x})] d(\overline{x}, T_2 x_n)\} \to \alpha d(\overline{x}, T_2\overline{x}) \quad \text{as } n \to \infty,
\end{aligned}$$

which implies that $\overline{x} \in T_2\overline{x}$ by $\alpha \in [0, 1)$. It follows that

$$d(x_0,\overline{x}) \le \sum_{n=0}^{\infty} d(x_n,x_{n+1}) \le \frac{d(x_0,x_1)}{1-\alpha} + \varepsilon \sum_{n=0}^{\infty} n\alpha^n$$
$$= \frac{d(x_0,x_1)}{1-\alpha} + \frac{\varepsilon\alpha^2}{1-\alpha} \le \frac{d(x_0,x_1)+\varepsilon}{1-\alpha} \le \frac{K+2\varepsilon}{1-\alpha}. \quad (4.1)$$

Letting $\varepsilon \to 0^+$ in (4.1), we have

$$d(x_0,F(T_2)) \le d(x_0,\overline{x}) \le \frac{K}{1-\alpha},$$

which implies that

$$H_-(F(T_1),F(T_2)) \le \frac{1}{1-\alpha} \sup_{x\in X} H_-(T_1,T_2)$$

by the arbitrariness of x_0. This completes the proof. □

Theorem 4.1. *Let (X,d) be a complete metric space and $\{T_i\}_{i\ge 0}$ be a sequence of multi-valued contractive type mappings from X into $CL(X)$ satisfying (3.1) with a common constant $\alpha \in [0,1)$ for each $i \ge 0$. If $\lim_{i\to\infty} H_-(T_ix,T_0x) = 0$ uniformly for all $x \in X$, then*

$$\lim_{i\to\infty} H_-(F(T_i),F(T_0)) = 0.$$

Proof. For $\varepsilon > 0$, choose N such that

$$\sup_{x\in X} H_-(T_ix,T_0x) < (1-\alpha)\varepsilon, \quad \forall i \ge N,$$

which implies that

$$H_-(F(T_i),F(T_0)) < \varepsilon, \quad \forall i \ge N$$

by Theorem 4.1. This completes the proof. □

Remark 4.1. Theorem 3.2 of Wang [11] is a special case of Theorem 4.2.

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