International Journal of Pure and Applied Mathematics
Volume 31 No. 1 2006, 47-71

NONOSCILLATION CRITERIA FOR RETARDED FUNCTIONAL DIFFERENTIAL SYSTEMS

Ana M. Pedro
Departamento de Matemática
Faculdade de Ciências e Tecnologia
Universidade Nova de Lisboa
Quinta da Torre
Monte de Caparica, 2825-114, PORTUGAL
e-mail: anap@fct.unl.pt

Abstract: Several criteria are given for having the retarded functional differential systems of the form $\frac{d}{dt}x\left(t\right) = \int_{-1}^{0} x\left(t - r\left(\theta\right)\right) dq\left(\theta\right)$ nonoscillatory, depending upon the smoothness of the delay function $r\left(\theta\right)$ and the properties of the matrix q.

AMS Subject Classification: 34K11
Key Words: retarded functional differential systems, oscillations, non-oscillations

1. Introduction

The purpose of this work is to investigate the oscillatory behavior of the solutions of the functional differential system

$$\frac{d}{dt}x\left(t\right) = \int_{-1}^{0} x\left(t - r\left(\theta\right)\right) d\left[q\left(\theta\right)\right] , \tag{1}$$

where $x(t) \in \mathbb{R}^d$, $r\left(\theta\right)$ is a positive real continuous function on $[-1, 0]$, positive on $[-1, 0[$, and $q\left(\theta\right)$ is a real $d \times d$ matrix valued function of bounded variation on $[-1, 0]$, which in the case having $r\left(0\right) = 0$ whill be assumed atomic at zero, that is, such that

Received: June 27, 2006

$$\lim_{\alpha \to 0^+} \int_{-\alpha}^{0} \|d[q(\theta)]\| = 0,$$

where for a given norm, $\|\cdot\|$, in the space $\mathbb{R}^{d\times d}$, of all real d-by-d matrices, by

$$\int_a^b |d[q(\theta)]|$$

we mean the total variation of q on an interval $[a,b] \subset [-1,0]$. Notice that when $r(\theta)$ is positive on $[-1,0]$, no atomicity assumption on the function q is necessary. The particular case when $d=1$ (scalar case) was considered in a recent work [7]. Some of the results of that paper concerning the oscillatory behavior of the solutions of equations of the form (1) are here extended to the case of a system of equations.

The system (1) for $r(\theta) = -r\theta$ $(r>0)$ and $\theta \in [-1,0]$ reduces to the class of retarded functional systems

$$\frac{d}{dt}x(t) = \int_{-r}^{0} x(t+\theta)\, d[q(\theta)] \ , \tag{2}$$

where $q(\theta) = q(\theta/r)$ is assumed to be atomic at zero. This is the most common general linear retarded functional system appearing in the literature [2]. My preference on system (1) regards the possibility of understanding more clearly the role of the delays on the oscillatory behavior of functional retarded systems.

We will analize the existence or nonexistence of oscillations in terms of the smoothnes of the delay functions $r(\theta)$. Namely, we will consider the cases, when $r(\theta)$ is in the set C^+ of all positive continuous functions in $[-1,0]$, or in D^+, the set of all positive differentiable functions in $[-1,0]$.

It will be also considered the relevant class of differential difference systems

$$\frac{d}{dt}x(t) = \sum_{j=1}^{m} A_j x(t-r_j) \ , \tag{3}$$

where the A_j are nonzero real matrices in $\mathbb{R}^{d\times d}$ and each r_j is a positive real number $(j=1,...,m)$. As it is well-known, this equation can be obtained from (1), under the assumption that $q(\theta)$ is a step function with a number m of jump points. More concretely it can be obtained from (1) with $q(\theta)$ given explicitely, for example, by

$$q(\theta) = \sum_{j=1}^{m} H(\theta-\theta_j)\, A_j, \tag{4}$$

where, for $-1 < \theta_1 < ... < \theta_m < 0$, by H we mean the Heaviside function and the delays, r_j, are obtained through any function $r(\theta) \in C^+$ which satisfies $r(\theta_j) = r_j$ for $j = 1, ..., m$.

A metric is introduced in C^+ through the norm $\|r\| = \max\{r(\theta) : -1 \leq \theta \leq 0\}$ $(r \in C^+)$. The value $m(r) = \min\{r(\theta) : -1 \leq \theta \leq 0\}$ will also have some relevance in the sequel.

By a solution of (1) we mean a continuous function $x : [-\|r\|, \infty[$, which is differentiable on $[0, +\infty[$ in manner that (1) be satisfied for every $t \geq 0$. A solution is said oscillatory whenever it has an infinite number of zeros; otherwise it will be said nonoscillatory. When all solutions are oscillatory the system (1) is called oscillatory. If (1) has at last one nonoscillatory solution the equation will be said nonoscillatory.

Let us denote by BV_d the Banach space of all real functions of bounded variation, $\phi : [-1, 0] \to \mathbb{R}^{d\times d}$ (in the case $d = 1$ we will simply write BV). Moreover we will denote by $\int_{-1}^{0} |d\phi(\theta)|$ the total variation of ϕ on $[-1, 0]$.

For any $\phi \in BV_d$ we can formulate the right and left hand limit matrices at any point $\theta \in [-1, 0]$, $\phi(\theta^+)$, $\phi(\theta^-)$, as well as the right and left hand oscillation matrices

$$H_\phi^+(\theta) = \phi\left(\theta^+\right) - \phi(\theta) \quad \text{and} \quad H_\phi^-(\theta) = \phi(\theta) - \phi\left(\theta^-\right). \tag{5}$$

The definition of nonnegative matrices enables us to consider monotonic d-by-d matrix valued functions. For this purpose we recall that a d-by-d real matrix $C = [c_{jk}]$ $(j, k = 1, ..., d)$ is said to be nonnegative (positive) whenever $c_{jk} \geq 0$ (respectively, $c_{jk} > 0$) for every $j, k = 1, ..., d$. These properties will be expressed as usual, through the notations $C \geq 0$ and $C > 0$, respectively. More generally given two d-by-d real matrices, C and D, will say that $C \leq D$ $(C < D)$ if $D - C \geq 0$ (respectively, $D - C > 0$).

We will say that a function $\phi : [-1, 0] \to \mathbb{R}^{d\times d}$ is increasing (decreasing) on $J \subset [-1, 0]$, if ϕ is non constant on J and for every $\theta_1, \theta_2 \in J$ such that $\theta_1 < \theta_2$, one has $\phi(\theta_1) \leq \phi(\theta_2)$ (respectively, $\phi(\theta_2) \leq \phi(\theta_1)$. Following [1] , a given $\theta \in [-1, 0]$ is said a point of increase (respectively, a point of decrease) of ϕ, if for every $\varepsilon > 0$, sufficiently small, ϕ is increasing (decreasing) in $[\theta - \varepsilon, \theta + \varepsilon]$ $([-\varepsilon, 0]$ if $\theta = 0, [-1, -1 + \varepsilon]$ if $\theta = -1)$. If there exists a $\varepsilon > 0$ such that ϕ is constant in $[\theta - \varepsilon, \theta + \varepsilon]$ $([-\varepsilon, 0]$ if $\theta = 0$, $[-1, -1 + \varepsilon]$ if $\theta = -1)$, θ will be said a point of constancy of ϕ.

As it is well known, any function $\phi \in BV$ can be decomposed as the difference of two nondecreasing functions α and $\beta : \phi = \alpha - \beta$. This decomposition is not unique and a particular decomposition of ϕ is given by

$$\phi = \phi_p - \phi_n, \tag{6}$$

where by ϕ_p and ϕ_n we denote, respectively, the positive and negative variation of ϕ, which are defined as follows. For each $\theta \in [-1,0]$, let $\mathcal{P}$ be the set of all partitions $P = \{-1 = \theta_0, \theta_1, ..., \theta_k = \theta\}$ of the interval $[-1,\theta]$ and to each $P \in \mathcal{P}$ associate the sets

$$A(P) = \{j : \phi(\theta_j) - \phi(\theta_{j-1}) > 0\} \text{ and } B(P) = \{j : \phi(\theta_j) - \phi(\theta_{j-1}) < 0\}.$$

Then ϕ_p and ϕ_n are given, respectively, by

$$\phi_p(\theta) = \sup\left\{\sum_{j\in A(P)} (\phi(\theta_j) - \phi(\theta_{j-1})) : P \in \mathcal{P}\right\}$$

and

$$\phi_n(\theta) = \sup\left\{\sum_{j\in B(P)} |\phi(\theta_j) - \phi(\theta_{j-1})| : P \in \mathcal{P}\right\}$$

(whenever $A(P)$ or $B(P)$ are empty, we make $\phi_p(\theta) = 0, \phi_n(\theta) = 0$). One easily sees that both ϕ_p and ϕ_n are nondecreasing functions such that $\phi(\theta) = \phi_p(\theta) - \phi_n(\theta)$, for every $\theta \in [-1,0]$.

These facts can be extended to functions $\Phi \in BV_d$. In fact, since for each $\theta \in [-1,0]$ we have $\Phi(\theta) = [\phi_{jk}(\theta)]$ $(j,k = 1,...,d)$, where $\phi_{jk} \in BV$, for every $j,k = 1,...,d$, Then decomposing each function ϕ_{jk} $(j,k = 1,...,d)$ as the difference of two nondecreasing functions a_{jk} and b_{jk}, where the d-by-d matrix valued functions $A(\theta) = [a_{jk}(\theta)]$, $B(\theta) = [b_{jk}(\theta)]$, are both nondecreasing functions in BV_d, and for every $\theta \in [-1,0]$, we have

$$\phi(\theta) = A(\theta) - B(\theta). \tag{7}$$

If for every $j,k = 1,..,d$, we decompose each function ϕ_{jk} according (6) then we obtain

$$\Phi(\theta) = \Phi_p(\theta) - \Phi_n(\theta),$$

with $\Phi_n(\theta) = \left[(\phi_{jk})_n(\theta)\right]$ and $\Phi_p(\theta) = \left[(\phi_{jk})_p(\theta)\right]$, where $(\phi_{jk})_p$ and $(\phi_{jk})_n$ are, respectively, the positive and negative variation of ϕ_{jk} $(j,k = 1,...,d)$.

According to [4], the analysis of the oscillatory behavior of solutions of the system (1) can be based upon the existence or absence of real zeros of the characteristic equation

$$\det\left[\lambda I - \int_{-1}^{0} \exp(-\lambda r(\theta))\, d[q(\theta)]\right] = 0, \tag{8}$$

where by I we mean the d-by-d identity matrix. In fact, in this framework, one can conclude that (1) is oscillatory if and only if (8) has no real roots. Therefore nonoscillatory solutions will exist, whenever (8) has at least a real root.

2. Cooperative Systems

Nonnegative matrices can play some role in the study of the oscillatory behavior of the system (1). According the Perron-Frobenious Theorem, a nonnegative matrix $C \in \mathbb{R}^{d\times d}$ has several important spectral properties (see [3] and [8]). As a matter of fact, denoting by $\sigma(C)$ the spectrum of C and by $\rho(C)$ the spectral radius of C, one has that $\rho(C) \in \sigma(C)$. Moreover, $\rho(C) > 0$ if $C > 0$ and $0 \leq C \leq D \Rightarrow \rho(C) \leq \rho(D)$.

For a matrix $C \in \mathbb{R}^{d\times d}$, considering the upper bound

$$s(C) = \max\{\mathrm{Re} z : z \in \sigma(C)\},$$

of the set $\mathrm{Re}\sigma(C) = \{\mathrm{Re} z : z \in \sigma(C)\}$, then $s(C) = \rho(C) \in \sigma(C)$ whenever $C \geq 0$. Through that same theorem, one can conclude that $s(C) \in \sigma(C)$, if $C = [c_{ij}]$ $(i, j = 1, ..., d)$, is essentially nonnegative, that is, if the off-diagonal entries of C (c_{ij} for $i \neq J$) are nonnegative real numbers.

Therefore if the matrix

$$\int_{-1}^{0} \exp(-\lambda r(\theta))\, d[q(\theta)] \tag{9}$$

is essentially nonnegative, for every real λ, the spectral set

$$\sigma\left(\int_{-1}^{0} \exp(-\lambda r(\theta))\, d[q(\theta)]\right)$$

is dominated by the value

$$s(\lambda) = s\left(\int_{-1}^{0} \exp(-\lambda r(\theta))\, d[q(\theta)]\right),$$

that is,

$$s(\lambda) \in \sigma\left(\int_{-1}^{0} \exp(-\lambda r(\theta))\, d[q(\theta)]\right), \quad \forall \lambda \in \mathbb{R}. \tag{10}$$

On this purpose, we notice that the assumption (10) is fulfilled when for each $\theta \in [-1, 0]$, $q(\theta)$ is a symmetric real matrix. Moreover for every real λ, the matrix (9) is nonnegative when the function q is nondecreasing on $[-1,0]$. When all the off-diagonal functions $q_{ij}(\theta)$ $(i \neq j)$ of the matrix q are nondecreasing the system (1) is said cooperative. In this case the matrix (9) is essentialy nonnegative and therefore the assumption (10) is also satisfied. In this section we will obtain some results that are applicable to systems with this property.

Lemma 1. *Under the assumption (10), if* $\lim_{\lambda\to-\infty} s(\lambda) = +\infty$ *then (1) is nonoscillatory.*

Proof. As shown in the proof of [1, Theorem 1] we have $\lim_{\lambda\to\infty} s(\lambda) = 0$. The continuity of $s(\lambda)$ implies that $s(\lambda_0) = \lambda_0$ for some real λ_0. Therefore by (10) we have

$$\det\left[\lambda_0 I - \int_{-1}^{0} \exp\left(-\lambda_0 r(\theta)\right) d\left[q(\theta)\right]\right] = 0,$$

and so (1) is nonoscillatory. □

Taking a decomposition of $\phi \in BV_d$ according to (7)

$$q(\theta) = A(\theta) - B(\theta),$$

when $A, B \in BV_d$ are nondecreasing functions and we obtain

$$\int_{-1}^{0} \exp\left(-\lambda r(\theta)\right) d\left[q(\theta)\right] = \int_{-1}^{0} \exp\left(-\lambda r(\theta)\right) d\left[A(\theta)\right] - \int_{-1}^{0} \exp\left(-\lambda r(\theta)\right) d\left[B(\theta)\right]. \quad (11)$$

The decomposition (11) will be used in statements 2-5 to prove the existence of a nonoscillatory solution of system (1). Note that these theorems are analogs of the results obtained in [1] for retarded functional systems.

Given a matrix function $B(\theta)$ we shall denote by $|B|$ the matrix $|B| = \int_{-1}^{0} |d[B_{jk}(\theta)]|$, $j = 1,..,d$, $k = 1,...,d$.

Theorem 2. *Let* $\theta_0 \in [-1,0]$ *be such that* $r(\theta_0) = \|r\|$. *If either*

$$H_A^+(\theta_0) > |B|, \quad \text{or} \quad H_A^-(\theta_0) > |B|,$$

where H_A^+ *and* H_A^- *are defined by (5), then (1) is nonoscillatory.*

Proof. Let us assume, for example, that $H_A^+(\theta_0) > |B|$.

For $\varepsilon > 0$ small enough, we have

$$\int_{-1}^{0} \exp\left(-\lambda r(\theta)\right) d\left[A(\theta)\right] \geq \int_{-1}^{\theta_0+\varepsilon} \exp\left(-\lambda r(\theta)\right) d\left[A(\theta)\right].$$

Since A is nondecreasing, by application of a mean value property of the functions of bounded variation, we can conclude for every real $\lambda < 0$,

$$\int_{\theta_0}^{\theta_0+\varepsilon} \exp\left(-\lambda r\left(\theta\right)\right) d\left[A\left(\theta\right)\right] \geq \exp\left(-\lambda r\left(\theta_0+\delta\varepsilon\right)\right)\left(A\left(\theta_0+\varepsilon\right)-A\left(\theta_0\right)\right)$$

for some $\delta \in]-1, 0[$, depending upon r, θ_0, ε and A. Then, making $\varepsilon \to 0^+$, we have

$$\int_{-1}^{0} \exp\left(-\lambda r\left(\theta\right)\right) d\left[A\left(\theta\right)\right] \geq \exp\left(-\lambda \left\|r\right\|\right) H_A^+\left(\theta_0\right).$$

On the other hand, for every real $\lambda < 0$ we have

$$\int_{-1}^{0} \exp\left(-\lambda r\left(\theta\right)\right) d\left[B\left(\theta\right)\right] \leq \exp\left(-\lambda \left\|r\right\|\right) \left|B\right|.$$

Therefore, by (11) and for every real $\lambda < 0$, we have that

$$\int_{-1}^{0} \exp\left(-\lambda r\left(\theta\right)\right) d\left[q\left(\theta\right)\right] \geq \exp\left(-\lambda \left\|r\right\|\right)\left(H_A^+\left(\theta_0\right)-\left|B\right|\right).$$

As the matrix $H_A^+\left(\theta_0\right)-\left|B\right|$ is positive, this means in particular that also

$$\int_{-1}^{0} \exp\left(-\lambda r\left(\theta\right)\right) d\left[q\left(\theta\right)\right] > 0,$$

for every $\lambda < 0$. Thus the assumption (10) is fulfilled and

$$s\left(\lambda\right) = \rho\left(\int_{-1}^{0} \exp\left(-\lambda r\left(\theta\right)\right) d\left[q\left(\theta\right)\right]\right)$$

for every $\lambda < 0$ and $\lim_{\lambda\to-\infty} s\left(\lambda\right) = +\infty$. Hence, by Lemma 1, (1) is nonoscillatory. □

Theorem 3. *Let $\theta_0 \in [-1, 0]$ be such that $r\left(\theta_0\right) = \left\|r\right\|$ and $r\left(\theta\right) < \left\|r\right\|$ for every $\theta \neq \theta_0$. If θ_0 is a point of increase of $q\left(\theta\right)$, then the system (1) is nonoscillatory.*

Proof. Let $\theta_0 = -1$. Taking into consideration the decomposition of $q\left(\theta\right)$, given by (7), and since θ_0 is a point of increase of $q\left(\theta\right)$, for some $\varepsilon > 0$ the matrix $B\left(\theta\right)$ is constant on $[-1, -1+\varepsilon]$. By consequence, on this interval

$$A\left(\theta\right) = q\left(\theta\right) - D,$$

for some real d-by-d real matrix D. Therefore

$$\begin{aligned}\int_{-1}^{0} \exp\left(-\lambda r\left(\theta\right)\right) d\left[q\left(\theta\right)\right] \\ &= \int_{-1}^{0} \exp\left(-\lambda r\left(\theta\right)\right) d\left[A\left(\theta\right)\right] - \int_{-1+\varepsilon}^{0} \exp\left(-\lambda r\left(\theta\right)\right) d\left[B\left(\theta\right)\right] \\ &\geq \int_{-1}^{-1+\varepsilon} \exp\left(-\lambda r\left(\theta\right)\right) d\left[A\left(\theta\right)\right] - \int_{-1+\varepsilon}^{0} \exp\left(-\lambda r\left(\theta\right)\right) d\left[B\left(\theta\right)\right].\end{aligned}$$

Take $0 < \delta < \varepsilon$ such that

$$m_0 = \min\left\{r\left(\theta\right) : \theta \in \left[-1, -1+\delta\right]\right\} > M = \max\left\{r\left(\theta\right) : \theta \in \left[-1+\varepsilon, 0\right]\right\}.$$

For every real $\lambda < 0$, the following matrix relations hold

$$\begin{aligned}0 \leq \int_{-1+\varepsilon}^{0} \exp\left(-\lambda r\left(\theta\right)\right) d\left[B\left(\theta\right)\right] \leq \exp\left(-\lambda M\right)\left|B\right| \\ \int_{-1}^{-1+\varepsilon} \exp\left(-\lambda r\left(\theta\right)\right) d\left[A\left(\theta\right)\right] \geq \exp\left(-\lambda m_0\right)\left(A\left(-1+\varepsilon\right) - A\left(-1\right)\right).\end{aligned}$$

Thus for every real $\lambda < 0$,

$$\begin{aligned}\int_{-1}^{0} \exp\left(-\lambda r\left(\theta\right)\right) d\left[q\left(\theta\right)\right] \\ \geq \exp\left(-\lambda m_0\right)\left(A\left(-1+\varepsilon\right) - A\left(-1\right)\right) - \exp\left(-\lambda M\right)\left|B\right|,\end{aligned}$$

which imply that

$$\begin{aligned}\int_{-1}^{0} \exp\left(-\lambda r\left(\theta\right)\right) d\left[q\left(\theta\right)\right] \\ \geq \exp\left(-\lambda m_0\right)\left[A\left(-1+\varepsilon\right) - A\left(-1\right) - \exp\left(\lambda\left(m_0 - M\right)\right)\left|B\right|\right].\end{aligned}$$

Since the nonnegative matrix $\exp\left(\lambda\left(m_0 - M\right)\right)\left|B\right|$ tends to the null matrix, as $\lambda \to -\infty$. Then we can conclude that there exists a real number $n > 0$ sufficiently large, such that, for every $\lambda < -n$, we have

$$\int_{-1}^{0} \exp\left(-\lambda r\left(\theta\right)\right) d\left[q\left(\theta\right)\right] > 0.$$

Therefore (10) is satisfied in a way that

$$s\left(\lambda\right) = \rho\left(\int_{-1}^{0} \exp\left(-\lambda r\left(\theta\right)\right) d\left[q\left(\theta\right)\right]\right).$$

Then we have $\lim_{\lambda\to-\infty} s(\lambda) = +\infty$ and by Lemma 1, system (1) is nonoscillatory. □

Example 4. Let $r(\theta) = -\theta(\theta+1)$. Then $\theta_0 = -\frac{1}{2} \Rightarrow \|r\| = r(\theta_0) = \frac{1}{4}$. Suppose that

$$q(\theta) = \begin{bmatrix} 1 & \left(\theta+\frac{2}{3}\right)^2 \\ \cos\theta & 1 \end{bmatrix}.$$

In this case the functions $q_{12}(\theta)$ and $q_{21}(\theta)$ are increasing for $\theta = -\frac{1}{2}$ and therefore $q(\theta)$ has a point of increase at $\theta = \theta_0$. Then, by Theorem 3, system (1) is nonoscillatory.

Corollary 5. *If* $r_k = \max\{r_j : j = 1, ..., m\}$ *and* $A_k > 0$ *then (3) is nonoscillatory.*

Proof. If $A_k > 0$ then θ_k is a point of increase of $q(\theta) = \sum_{j=1}^{m} H(\theta-\theta_j) A_j$. Therefore choosing $r(\theta)$ continuous and positive on $[-1, 0]$ in manner that $r(\theta_k) = \|r\|$ and $r(\theta) < \|r\|$ for every $\theta \neq \theta_k$, one can conclude by Theorem 3 that (3) is nonoscillatory. □

Theorem 6. *If* $q(\theta)$ *is increasing on* $[-1, 0]$ *and nonconstant on* $[-1, 0[$, *then (1) is nonoscillatory.*

Proof. The proof is similar to the proof of [1, Theorem 6] □

3. Nonoscillations for Continuous Delays

In this paper we shall use some properties of the matrix measures. With this purpose let us introduce the following notations. Given a function $\phi \in BV_d$, on $[\alpha, \beta]$ we can define ϕ^-, ϕ^+, by the following equalities

$$\begin{aligned} \phi^+(\alpha,\beta) &= \phi(\beta) - \phi(\alpha), \\ \phi^-(\alpha,\beta) &= \phi(\alpha) - \phi(\beta) = -\phi^+(\alpha,\beta). \end{aligned} \tag{12}$$

As it is well known matrix measures are a relevant tool for the oscillation theory of delay systems. For a matter of completeness we recall briefly, its definition and the properties which will be used in the sequel.

For each induced norm, $\|\cdot\|$, in $\mathbb{R}^{d\times d}$, we associate a matrix measure $\mu : \mathbb{R}^{d\times d} \to \mathbb{R}$, which is defined for any $C \in \mathbb{R}^{d\times d}$ as

$$\mu(C) = \lim_{y\to 0^+} \frac{\|I + yC\| - 1}{y},$$

where by I we mean the identity matrix.

Some well known matrix measures of a matrix $C = [c_{jk}] \in \mathbb{R}^{d\times d}$, are

$$\mu_1(C) = \max\left\{ c_{kk} + \sum_{j\neq k} |c_{jk}| : k = 1, ..., d \right\},$$

$$\mu_\infty(C) = \max\left\{ c_{jj} + \sum_{k\neq j} |c_{jk}| : j = 1, ..., d \right\},$$

which correspond, respectively, to the induced norms in $\mathbb{R}^{d\times d}$ given by:

$$\|C\|_1 = \max\left\{ \sum_{j=1}^{d} |c_{jk}| : k = 1, ..., d \right\},$$

$$\|C\|_\infty = \max\left\{ \sum_{k=1}^{d} |c_{jk}| : j = 1, ..., d \right\}.$$

Independenly of the considered induced norms in $\mathbb{R}^{d\times d}$, a matrix measure has always the following properties:

(i) $s(C) \leq \mu(C) \leq \|C\|$.

(ii) $\mu(C_1) - \mu(-C_2) \leq \mu(C_1 + C_2) \leq \mu(C_1) + \mu(C_2)$ $\left(C_1, C_2 \in \mathbb{R}^{d\times d}\right)$.

(iii) $\mu(\alpha C) = \alpha\mu(C)$, for every $\alpha \geq 0$.

(iv) $-\mu(-C) \leq \text{Re } \lambda_i(C) \leq \mu(C)$ for all $i = 1, ..., d$, where Re $\lambda_i(C)$ denotes the real part of the eigenvalue $\lambda_i(C)$ of C.

If $\phi \in BV_d$, the continuity of μ on $\mathbb{R}^{d\times d}$ implies that $\mu \circ \phi \in BV_d$, in consequence, with $[a, b] \subset [-1, 0]$, the following inequalities hold (see [5]):

(v) If $\varphi \in C([a, b]; \mathbb{R})$ is decreasing and negative, then

$$\mu\left(\int_{-1}^{0} \varphi(\theta)\, d[\phi(\theta)]\right) \leq \int_{-1}^{0} \varphi(\theta)\, d(\mu(\phi(\theta) - \phi(0))) = \int_{-1}^{0} \varphi(\theta)\, d\left(\mu\left(\phi^-(\theta, 0)\right)\right).$$

(vi) If $\varphi \in C([a,b];\mathbb{R})$ is increasing and negative, then

$$\mu\left(\int_{-1}^{0}\varphi(\theta)\,d[\phi(\theta)]\right) \le -\int_{-1}^{0}\varphi(\theta)\,d(\mu(\phi(-1)-\phi(\theta))) = -\int_{-1}^{0}\varphi(\theta)\,d\left(\mu\left(\phi^{-}(-1,\theta)\right)\right).$$

Lemma 7. *Let $C \in \mathbb{R}^{d\times d}$.*
a) If $\det C < 0 \Rightarrow \mu(-C) > 0$.
b) If $\det C > 0$ and d is odd $\Rightarrow \mu(C) > 0$.

Proof. a) We have $\det C = z_1 \times z_2 \times ... \times z_d$, where z_i are the eigenvalues of C, $i = 1, ..., d$.

Since $\det C < 0$, then $\exists z_k \in \sigma(C) : z_k \in \mathbb{R}$ and $z_k < 0$. Then using the property (iv) of matrix measures we have $-\mu(-C) \le \operatorname{Re} z_k$ and therefore $-\mu(-C) < 0$, i.e. $\mu(-C) > 0$.

b) $\det C > 0 \wedge d$ odd $\Rightarrow \det(-C) = (-1)^d \det C = -\det(C) < 0$. Then using a) we have $\mu(C) > 0$. □

Along the rest of this work we will use the function

$$F(\lambda) = \lambda I - \int_{-1}^{0}\exp(-\lambda r(\theta))\,d[q(\theta)].$$

Lemma 8. *Let d be odd. If there is $\lambda \in \mathbb{R}$ such that $\mu(F(\lambda)) \le 0$, then system (1) is nonoscillatory.*

Proof. Assume the system (1) is oscillatory, then $\det F(\lambda) \ne 0$ for $\forall\lambda \in \mathbb{R}$. On the other hand we know that $\det F(\lambda) \to +\infty$ when $\lambda \to +\infty$. Hence $\det F(\lambda) > 0$ for $\forall\lambda \in \mathbb{R}$. Then, according to Lemma 7, $\mu(F(\lambda)) > 0$. This is a contradiction with the fact $\mu(F(\lambda)) \le 0$. Therefore system (1) is nonoscillatory.
□

Remark 9. If $\mu(q^{-}(-1,0)) \le 0$, then, using (12) we have

$$\mu(F(0)) = \mu\left(q^{-}(-1,0)\right) \le 0.$$

Hence, in such situation system (1) is nonoscillatory.

Lemma 8 will be used along the rest of this article to obtain nonoscillatory results in different situations. Therefore, all these results are valid only for systems of odd order d.

Theorem 10. *Let $r(\theta)$ be decreasing on $[-1,0]$. If $\mu\left(q^{+}(-1,\theta)\right)$ has a point of increase at -1, then the system (1) is nonoscillatory.*

Proof. As it is well-known the function $\mu\left(q^{+}(-1,\theta)\right)$ is a function of bounded variation (see [5]) and we can consider the decomposition of $\mu\left(q^{+}(-1,\theta)\right)$ given by (6):

$$\mu\left(q^{+}(-1,\theta)\right)=\left[\mu\left(q^{+}(-1,\theta)\right)\right]_{p}-\left[\mu\left(q^{+}(-1,\theta)\right)\right]_{n}\quad(\theta\in[-1,0]).$$

Therefore for some sufficiently small $\varepsilon>0$, the function $\left[\mu\left(q^{+}(-1,\theta)\right)\right]_{p}$ is increasing and $\left[\mu\left(q^{+}(-1,\theta)\right)\right]_{n}$ is constant on $[-1,-1+\varepsilon]$. Then by properties (ii) and (vi) of matrix measures, for $\lambda<0$ we obtain

$$\begin{aligned}
\mu(F(\lambda)) &= \mu\left(\lambda I-\int_{-1}^{0}\exp(-\lambda r(\theta))\,d[q(\theta)]\right)\\
&\le \lambda+\mu\left(\int_{-1}^{0}\left(-\exp(-\lambda r(\theta))\right)d[q(\theta)]\right)\\
&\le \lambda+\int_{-1}^{0}\exp(-\lambda r(\theta))\,d\mu\left(q^{-}(-1,\theta)\right)\\
&\le \lambda+\int_{-1}^{0}\exp(-\lambda r(\theta))\,d\left[\mu\left(q^{-}(-1,\theta)\right)\right]_{p}\\
&\quad -\int_{-1+\varepsilon}^{0}\exp(-\lambda r(\theta))\,d\left[\mu\left(q^{-}(-1,\theta)\right)\right]_{n}.
\end{aligned}$$

Therefore

$$\begin{aligned}
-\mu(F(\lambda)) &\ge -\lambda-\int_{-1}^{0}\exp(-\lambda r(\theta))\,d\left[\mu\left(q^{-}(-1,\theta)\right)\right]_{p}\\
&\quad +\int_{-1+\varepsilon}^{0}\exp(-\lambda r(\theta))\,d\left[\mu\left(q^{-}(-1,\theta)\right)\right]_{n}.
\end{aligned}$$

Hence, by property (iv) of matrix measures, we have

$$\begin{aligned}
-\mu(F(\lambda)) &\ge -\lambda+\int_{-1}^{0}\exp(-\lambda r(\theta))\,d\left[\mu\left(q^{+}(-1,\theta)\right)\right]_{p}\\
&\quad -\int_{-1+\varepsilon}^{0}\exp(-\lambda r(\theta))\,d\left[\mu\left(q^{+}(-1,\theta)\right)\right]_{n}.
\end{aligned}$$

Take $0<\delta<\varepsilon$ in manner that

$$m_0=\min\{r(\theta):-1\le\theta\le-1+\delta\}$$

be such that

$$m_0 > M = \max\{r(\theta) : -1+\varepsilon \le \theta \le 0\}.$$

One easily can see that for every real $\lambda < 0$,

$$\int_{-1}^{0} \exp(-\lambda r(\theta))\, d\left[\mu\left(q^+(-1,\theta)\right)\right]_p \ge \int_{-1}^{-1+\delta} \exp(-\lambda r(\theta))\, d\left[\mu\left(q^+(-1,\theta)\right)\right]_p$$
$$\ge \exp(-\lambda m_0)\left[\mu\left(q^+(-1,-1+\delta)\right)\right]_p,$$

and

$$\int_{-1+\varepsilon}^{0} \exp(-\lambda r(\theta))\, d\left[\mu\left(q^+(-1,\theta)\right)\right]_n$$
$$\le \exp(-\lambda M)\left[\left[\mu\left(q^+(-1,0)\right)\right]_n - \left[\mu\left(q^+(-1,-1+\varepsilon)\right)\right]_n\right].$$

Thus, for every real $\lambda < 0$, we have

$$\begin{aligned} -\mu(F(\lambda)) \quad \ge \quad & -\lambda + \exp(-\lambda m_0)\left[\mu\left(q^+(-1,-1+\delta)\right)\right]_p \\ & -\exp(-\lambda M)\left[\left[\mu\left(q^+(-1,0)\right)\right]_n - \left[\mu\left(q^+(-1,-1+\varepsilon)\right)\right]_n\right], \end{aligned}$$

that is,

$$\begin{aligned} -\mu(F(\lambda)) \ge -\lambda + \; & \exp(-\lambda m_0) \times [\left[\mu\left(q^+(-1,-1+\delta)\right)\right]_p \\ & -\exp(\lambda(m_0 - M))\left[\left[\mu\left(q^+(-1,0)\right)\right]_n - \left[\mu\left(q^+(-1,-1+\varepsilon)\right)\right]_n\right]], \end{aligned}$$

Since $m_0 - M > 0$, $\exp(\lambda(m_0 - M)) \to 0$ as $\lambda \to -\infty$. Therefore, $-\mu(F(\lambda)) \to +\infty$, as $\lambda \to -\infty$, which concludes the proof. □

Using similar arguments, we can prove th following result for the case of increasing delay function.

Theorem 11. *Let $r(\theta)$ be increasing on $[-1,0]$. If $\mu(q^-(\theta,0))$ has a point of increase at 0, then the system (1) is nonoscillatory.*

Example 12. Let $r(\theta) = -\theta$ and

$$q(\theta) = \begin{bmatrix} -\theta(\theta+1) & -1-\theta & 0 \\ -1-\theta & -\theta(\theta+1) & 0 \\ 0 & 0 & 1+\theta \end{bmatrix}.$$

We have r decreasing and $\max_{-1\le\theta\le 0} r(\theta) = r(-1) = 1$. It is easy to verify that -1 is not a point of increase of $q(\theta)$, since the off-diagonal elements of q

are decreasing functions of θ. Therefore, the conditions of theorem 3 are not satisfied in this case. However, we can prove that the system (1), with this choise of q and r, is nonoscillatory. We have

$$\begin{aligned}\mu_\infty\left(q^+(-1,\theta)\right) &= \mu_\infty\left(q(\theta)-q(-1)\right)\\ &= \mu_\infty\begin{bmatrix}-\theta(\theta+1) & -1-\theta & 0\\ -1-\theta & -\theta(\theta+1) & 0\\ 0 & 0 & 1+\theta\end{bmatrix} = -\theta^2+1\,.\end{aligned}$$

Therefore, -1 is a point of increase of $\mu_\infty\left(q^+(-1,\theta)\right)$ and the conditions of Theorem 10 are satisfied in this case.

Assuming that $-1\le a\le b\le 0$, let $C^+(a,b)$ be the family of all functions in C^+, which are increasing on $[-1,a]$, constant on $[a,b]$ and decreasing on $[b,0]$. In case of having $a=b=\theta_0$ with $\theta_0\in[-1,0]$ we obtain the family $C^+(\theta_0)$ of all functions in C^+ which are increasing on $[-1,\theta_0]$ and decreasing on $[\theta_0,0]$.

Theorem 13. *Let $r\in C^+(a,b)$. If:*

$$\begin{array}{l}\mu\left(q^-(\theta,0)\right) \text{ is increasing for every } \theta\in[-1,a]\ ,\\ \mu\left(q^-(-1,\theta)\right) \text{ is decreasing for every } \theta\in[b,0],\end{array}\tag{13}$$

and

$$\mu\left(q^-(a,b)\right)\le 0,\tag{14}$$

then (1) is nonoscillatory for every delay function in $C^+(a,b)$.

Proof. With $r\in C^+(a,b)$, using the property (ii) of matrix mesures, we obtain

$$\begin{aligned}\mu(F(\lambda)) \le \lambda &+ \mu\left(\int_{-1}^{a}\left(-\exp(-\lambda r(\theta))\right)d\left[q(\theta)\right]\right)\\ &+\exp(-\lambda r(a))\,\mu\left(q^-(a,b)\right)+\mu\left(\int_{b}^{0}\left(-\exp(-\lambda r(\theta))\right)d\left[q(\theta)\right]\right).\end{aligned}$$

Moreover, if $\lambda<0$ and by properties v) and vi) of matrix measures, we have

$$\begin{aligned}\mu(F(\lambda)) \le \lambda &- \int_{-1}^{a}\exp(-\lambda r(\theta))\,d\mu\left(q^-(\theta,0)\right)\\ &+\exp(-\lambda r(a))\,\mu\left(q^-(a,b)\right)+\int_{b}^{0}\exp(-\lambda r(\theta))\,d\mu\left(q^-(-1,\theta)\right)\,.\end{aligned}$$

Then, by (14) and (13), $\mu(F(\lambda))<0$ and by Lemma 8, system (1) is nonoscillatory. □

For the case where $a = b = \theta_0$ we have the following corollary.

Corollary 14. *Let be $r \in C^{+}(\theta_0)$. If:*

$$\mu\left(q^{-}(\theta,0)\right) \text{ is increasing for every } \theta \in [-1,\theta_0],$$
$$\mu\left(q^{-}(-1,\theta)\right) \text{ is decreasing for every } \theta \in [\theta_0,0],$$

then (1) is nonoscillatory for every delay function in $C^{+}(\theta_0)$.

Considering the particular case when $\theta_0 = 0$, we obtain the following result.

Corollary 15. *Let r be increasing on $[-1,0]$. If $\mu\left(q^{-}(\theta,0)\right)$ is increasing for every $\theta \in [-1,0]$, then (1) is nonoscillatory.*

Analogously, for $\theta_0 = -1$ we obtain the following result.

Corollary 16. *Let r be decreasing on $[-1,0]$. If $\mu\left(q^{-}(-1,\theta)\right)$ is decreasing for every $\theta \in [-1,0]$, then (1) is nonoscillatory.*

As noted in the introduction the system (3) can be considered as a particular case of the system (1), when the function $q(\theta)$ has the form (4). Taking this into account, analogs of Corollaris 15 and 16 can be obtained for system (3). With this purpose, following [5], let us define

$$\alpha(A_1) = \mu(-A_1),$$
$$\alpha(A_j) = \mu\left(\sum_{k=1}^{j}(-A_k)\right) - \mu\left(\sum_{k=1}^{j-1}(-A_k)\right), \quad j = 2,...,m, \tag{15}$$
$$\beta(A_m) = \mu(-A_m),$$
$$\beta(A_j) = \mu\left(\sum_{k=j}^{m}(-A_k)\right) - \mu\left(\sum_{k=j+1}^{m}(-A_k)\right), \quad j = 1,...,m-1. \tag{16}$$

In these notations, $\mu\left(q^{-}(\theta,0)\right)$ is increasing if and only if $\beta(A_j) \leq 0$, $j = 1,...,m-1$, and $\mu\left(q^{-}(-1,\theta)\right)$ is decreasing if and only if $\alpha(A_j) \leq 0, \forall j = 2,...,m$. Therefore, with respect to the system (3), from Corollary 15 and Corollary 16 we can state the following results

Corollary 17. *Let $r_1 < r_2 < ... < r_m$. If $\beta(A_j) \leq 0, \forall j = 1,...,m-1$ then (3) is nonoscillatory.*

Corollary 18. *Let $r_1 > r_2 > ... > r_m$. If $\alpha(A_j) \leq 0, \forall j = 2,...,m$ then (3) is nonoscillatory.*

Example 19. Let us consider the system

$$\frac{d}{dt}x(t) = A_1 x(t-1) + A_2 x(t-2) + A_3 x(t-3), \tag{17}$$

where

$$A_1 = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; A_2 = \begin{bmatrix} 1 & -1/2 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; A_3 = \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since we do not have $A_3 > 0$, we cannot apply Corollary 5 to this system. Let us check the conditions of Corollary 17. Actually,

$$\beta(A_3) = \mu_\infty \begin{bmatrix} -1 & 1/2 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \max\left\{-\frac{1}{2}, -1, -1\right\} = -\frac{1}{2} < 0\,,$$

$$\beta(A_2) = \mu_\infty \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} - \mu_\infty \begin{bmatrix} -1 & 1/2 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \max\{-1, -1, -2\} + \frac{1}{2} = -\frac{1}{2} < 0\,,$$

$$\beta(A_1) = \mu_\infty \begin{bmatrix} -3 & -1 & 0 \\ -1 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix} - \mu_\infty \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$= \max\{-2, -2, -3\} + 1 = -2 < 0\,.$$

Hence, by corollary 17, the system (3) is nonoscillatory.

Remark 20. Consider Example 19, but change the order of the retards:

$$\overline{r}_1 = 3 > \overline{r}_2 = 2 > \overline{r}_3 = 1.$$

For the new system we have

$$\bar{A}_1 = A_3,\; \bar{A}_2 = A_2,\; \bar{A}_3 = A_1.$$

Then it is easy to check that $\alpha\left(\bar{A}_1\right) = \beta(A_3) = -\frac{1}{2} < 0$, $\alpha\left(\bar{A}_2\right) = \beta(A_2) = -\frac{1}{2} < 0$ and $\alpha\left(\bar{A}_3\right) = \beta(A_1) = -2 < 0$. Therefore, this system also satisfies the conditions of Corollary 18.

Theorem 21. *Let $r \in C^+(a,b)$. If:*

$$\mu\left(q^-(\theta,0)\right) \text{ is decreasing for every } \theta \in [-1,a]\,,$$
$$\mu\left(q^-(-1,\theta)\right) \text{ is increasing for every } \theta \in [b,0]\,,$$

and

$$0 < \mu\left(q^-(b,0)\right) + \mu\left(q^-(-1,a)\right) + \mu\left(q^-(a,b)\right) < \frac{1}{er(a)}. \tag{18}$$

Then (1) is nonoscillatory for every delay function in $C^+(a,b)$.

Proof. With $r \in C^+(a,b)$ we have

$$\mu(F(\lambda)) \leq \lambda + \mu\left(\int_{-1}^{a}(-\exp(-\lambda r(\theta)))\, d[q(\theta)]\right) + \exp(-\lambda r(a))\,\mu\left(q^-(a,b)\right) + \mu\left(\int_{b}^{0}(-\exp(-\lambda r(\theta)))\, d[q(\theta)]\right).$$

By properties $v)$ and $vi)$ of matrix measures and for $\lambda < 0$, we have

$$\mu(F(\lambda)) \leq \lambda - \int_{-1}^{a}\exp(-\lambda r(\theta))\, d\mu\left(q^-(\theta,0)\right) + \exp(-\lambda r(a))\,\mu\left(q^-(a,b)\right) + \int_{b}^{0}\exp(-\lambda r(\theta))\, d\mu\left(q^-(-1,\theta)\right). \quad (19)$$

If $\lambda < 0$, we have $\exp(-\lambda r(\theta)) \leq \exp(-\lambda\|r\|) = \exp(-\lambda r(a))$, which implies that

$$\begin{aligned}
&\mu(F(\lambda)) \leq \lambda \\
&+ \exp(-\lambda r(a))\left[-\int_{-1}^{a} d\mu\left(q^-(\theta,0)\right) + \mu\left(q^-(a,b)\right) + \int_{b}^{0} d\mu\left(q^-(-1,\theta)\right)\right] \\
&\leq \lambda + \exp(-\lambda r(a))\left[\mu\left(q^-(-1,0)\right) - \mu\left(q^-(a,0)\right)\right. \\
&\left. + \mu\left(q^-(-1,0)\right) - \mu\left(q^-(-1,b)\right) + \mu\left(q^-(a,b)\right)\right] \\
&\leq \lambda + \exp(-\lambda r(a)) \times \left[\mu\left(q^-(-1,0)\right) - \mu\left(-q^+(a,0)\right) + \mu\left(q^-(-1,0)\right)\right. \\
&\left. - \mu\left(-q^+(-1,b)\right) + \mu\left(q^-(a,b)\right)\right] \\
&\leq \lambda + \exp(-\lambda r(a))\left[\mu\left(q^-(b,0)\right) + \mu\left(q^-(-1,a)\right) + \mu\left(q^-(a,b)\right)\right].
\end{aligned}$$

Thus

$$\mu(F(\lambda)) \leq f(\lambda) = \lambda + \exp(-\lambda r(a))\left[\mu\left(q^-(b,0)\right) + \mu\left(q^-(-1,a)\right) + \mu\left(q^-(a,b)\right)\right].$$

Since $f(\lambda) \to +\infty$, as $\lambda \to \pm\infty$, then $f(\lambda)$ has as absolute minimum the value $f(\lambda_0)$, for

$$\lambda_0 = \frac{1}{r(a)}\log\left[r(a)\left(\mu\left(q^-(b,0)\right) + \mu\left(q^-(-1,a)\right) + \mu\left(q^-(a,b)\right)\right)\right].$$

But, by (18),

$$f(\lambda_0) = \frac{1}{r(a)}\left[\log\left(r(a)\left(\mu\left(q^-(b,0)\right) + \mu\left(q^-(-1,a)\right)\right.\right.\right.$$
$$\left.\left.\left. + \mu\left(q^-(a,b)\right)\right)\right) + 1\right] \leq 0,$$

which means that $\mu(F(\lambda_0)) \leq 0$. Hence (1) is nonoscillatory. □

For the case where $a = b = \theta_0$ we have the following corollary.

Corollary 22. *Let $r \in C^+(\theta_0)$. If:*

$$\begin{array}{l} \mu(q^-(\theta,0)) \text{ is decreasing for every } \theta \in [-1,\theta_0] \\ \mu(q^-(-1,\theta)) \text{ is increasing for every } \theta \in [\theta_0,0] \end{array}$$

and

$$0 < \mu\left(q^-(\theta_0,0)\right) + \mu\left(q^-(-1,\theta_0)\right) < \frac{1}{er(\theta_0)}.$$

Then (1) is nonoscillatory for every delay function in $C^+(\theta_0)$.

For the case where $\theta_0 = 0$ we have the following corollary.

Corollary 23. *Let r be increasing on $[-1,0]$. If $\mu(q^-(\theta,0))$ is decreasing for every $\theta \in [-1,0]$ and $0 < \mu(q^-(-1,0)) < \frac{1}{er(0)}$, then (1) is nonoscillatory.*

For the case where $\theta_0 = -1$ we have the following corollary.

Corollary 24. *Let r be decreasing on $[-1,0]$. If $\mu(q^-(-1,\theta))$ is increasing for every $\theta \in [-1,0]$ and $0 < \mu(q^-(-1,0)) < \frac{1}{er(-1)}$,, then (1) is nonoscillatory.*

With respect to the system (3), using the definitions of $\beta(A_j)$ and $\alpha(A_j)$ given by (15) and (16) we obtain the following results.

Corollary 25. *Let $r_1 < r_2 < ... < r_m$. If $\beta(A_j) \geq 0$ for $j = 1, ..., m$ and $0 < \mu\left(-\sum_{j=1}^{m} A_j\right) < \frac{1}{er_m}$ then (3) is nonoscillatory.*

Corollary 26. *Let $r_1 > r_2 > ... > r_m$. If $\alpha(A_j) \geq 0$ for $j = 1, ..., m$ and $0 < \mu\left(-\sum_{j=1}^{m} A_j\right) < \frac{1}{er_1}$, then (3) is nonoscillatory.*

Example 27. Let us consider the system

$$\frac{d}{dt}x(t) = A_1 x\left(t - \frac{1}{6}\right) + A_2 x\left(t - \frac{1}{5}\right) + A_3 x\left(t - \frac{1}{4}\right), \tag{20}$$

where

$$A_1 = \begin{bmatrix} -5/3 & 1 & 0 \\ 1/3 & 5/3 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \; A_2 = \begin{bmatrix} 2/3 & 0 & 0 \\ 2/3 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 1/3 & -2/3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Note that in this case the conditions of Corollary 5 are not satisfied since $\max_{1\le j\le 3} r_j = r_3$ and A_3 its not a positive matrix. However, it is possible to show that this system is nonoscillatory. Actually,

$$\mu_\infty \left[-A_1 - A_2 - A_3\right] = 1 < \frac{1}{er_3} < \frac{4}{e},$$

$$\beta\left(A_3\right) = \mu_\infty \begin{bmatrix} -1/3 & -2/3 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \max\{1/3, -1, -1\} = 1/3 > 0,$$

$$\beta\left(A_2\right) = \mu_\infty \begin{bmatrix} -1 & 2/3 & 0 \\ -2/3 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} - \mu_\infty \begin{bmatrix} -1/3 & -2/3 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \frac{2}{3} - \frac{1}{3} = \frac{1}{3} > 0,$$

$$\beta\left(A_1\right) = \mu_\infty \begin{bmatrix} 2/3 & -1/3 & 0 \\ -1 & -5/3 & 0 \\ 0 & 0 & -1 \end{bmatrix} - \mu_\infty \begin{bmatrix} -1 & 2/3 & 0 \\ -2/3 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$= 1 - \frac{2}{3} = \frac{1}{3} > 0.$$

Therefore, by Corollary 25, the system (20) is nonoscillatory.

Remark 28. Consider again Example 27, but change the order of the retards:

$$\overline{r}_1 = \frac{1}{4} > \overline{r}_2 = \frac{1}{5} > \overline{r}_3 = \frac{1}{6}.$$

For the new system we have

$$\bar{A}_1 = A_3,\ \bar{A}_2 = A_2,\ \bar{A}_3 = A_1.$$

Then it is easy to check that $\alpha\left(\bar{A}_1\right) = \beta\left(A_3\right) = \frac{1}{3} > 0$, $\alpha\left(\bar{A}_2\right) = \beta\left(A_2\right) = \frac{1}{3} > 0$ and $\alpha\left(\bar{A}_3\right) = \beta\left(A_1\right) = \frac{1}{3} > 0$. Moreover, $\mu_\infty\left[-\bar{A}_1 - \bar{A}_2 - \bar{A}_3\right] < \frac{1}{e\overline{r}_1}$. Therefore this system also satisfies the conditions of Corollary 26.

4. Nonoscillations for Differentiable Delays

With $-1 \le a \le b \le 0$, let $D^+(a,b)$ be the family of all functions in D^+ which are increasing on $[-1,a]$, constant on $[a,b]$ and decreasing on $[b,0]$. In case of having $a = b = \theta_0$ with $\theta_0 \in [-1, 0]$ we obtain the family $D^+(\theta_0)$ of all differentiable and positive functions which are increasing on $[-1, \theta_0]$ and decreasing on $[\theta_0, 0]$. If $\theta_0 = -1$, $D^+(-1)$ is the class of all positive differentiable and decreasing functions on $[-1, 0]$ wich we will denote by D_d^+. For $\theta_0 = 0$ we obtain the family D_i^+ of all positive differentiable and increasing funtions on $[-1, 0]$. For these families of delays we start by stating the following nonoscillatory situation.

Theorem 29. *Let $r \in D^+(a, b)$ and $q \in BV_d$ such that:*

$$\begin{aligned} &\mu\left(q^-(\theta, 0)\right) \geq 0 \text{ for every } \theta \in [-1, a], \\ &\mu\left(q^-(-1, \theta)\right) \geq 0 \text{ for every } \theta \in [b, 0], \\ &0 < \mu\left(q^-(-1, a)\right) + \mu\left(q^-(b, 0)\right) + \mu\left(q^-(a, b)\right) < \tfrac{1}{er(a)}. \end{aligned} \tag{21}$$

If

$$\begin{aligned} &\int_{-1}^{a} \mu\left(q^-(\theta, 0)\right) dr(\theta) - \int_{b}^{0} \mu\left(q^-(-1, \theta)\right) dr(\theta) \\ &\quad \leq \left(1 + \frac{1}{\log\left(r(a)\left(\mu\left(q^-(-1, a)\right) + \mu\left(q^-(b, 0)\right) + \mu\left(q^-(a, b)\right)\right)\right)}\right) \\ &\qquad \times r(a)\left(\mu\left(q^-(-1, a)\right) + \mu\left(q^-(b, 0)\right) + \mu\left(q^-(a, b)\right)\right), \end{aligned} \tag{22}$$

then (1) is nonoscillatory.

Proof. Recalling (19), for $\lambda < 0$, we have

$$\begin{aligned} \mu(F(\lambda)) \leq \lambda &- \int_{-1}^{a} \exp(-\lambda r(\theta))\, d\mu\left(q^-(\theta, 0)\right) \\ &+ \exp(-\lambda r(a))\, \mu\left(q^-(a, b)\right) + \int_{b}^{0} \exp(-\lambda r(\theta))\, d\mu\left(q^-(-1, \theta)\right). \end{aligned}$$

Integrating by parts each one of the above integrals and taking into account that $\exp(-\lambda r(a)) \geq \exp(-\lambda r(-1))$ and $\exp(-\lambda r(a)) \geq \exp(-\lambda r(0))$, we have

$$\begin{aligned} \mu(F(\lambda)) \leq \lambda &+ \exp(-\lambda r(a))\left(\mu\left(q^-(-1, a)\right) + \mu\left(q^-(b, 0)\right)\right. \\ &\left. + \mu\left(q^-(a, b)\right)\right) - \lambda\left[\int_{-1}^{a} \exp(-\lambda r(\theta))\, \mu\left(q^-(\theta, 0)\right) dr(\theta)\right. \\ &\left. - \int_{b}^{0} \exp(-\lambda r(\theta))\, \mu\left(q^-(-1, \theta)\right) dr(\theta)\right]. \end{aligned}$$

Moreover, since $\lambda < 0$,

$$\mu(F(\lambda)) \leq \lambda + \exp(-\lambda r(a)) \left(\mu\left(q^-(-1,a)\right) + \mu\left(q^-(b,0)\right) + \mu\left(q^-(a,b)\right)\right) - \lambda \exp(-\lambda r(a)) \times \left[\int_{-1}^{a} \mu\left(q^-(\theta,0)\right) dr(\theta) - \int_{b}^{0} \mu\left(q^-(-1,\theta)\right) dr(\theta)\right]. \quad (23)$$

The function

$$\varphi(\lambda) = \lambda + \exp(-\lambda r(a)) \left(\mu\left(q^-(-1,a)\right) + \mu\left(q^-(b,0)\right) + \mu\left(q^-(a,b)\right)\right),$$

has the absolute minimum

$$\varphi(\lambda_0) = \frac{1}{r(a)} \left[\log\left(r(a)\left(\mu\left(q^-(-1,a)\right) + \mu\left(q^-(b,0)\right) + \mu\left(q^-(a,b)\right)\right)\right) + 1\right],$$

attained at

$$\lambda_0 = \frac{1}{r(a)} \log\left(r(a)\left(\mu\left(q^-(-1,a)\right) + \mu\left(q^-(b,0)\right) + \mu\left(q^-(a,b)\right)\right)\right).$$

Since, from (21)

$$0 < r(a)\left(\mu\left(q^-(-1,a)\right) + \mu\left(q^-(b,0)\right) + \mu\left(q^-(a,b)\right)\right) < e^{-1},$$

one has $\lambda_0 < 0$. Therefore by (23) and (22) we obtain

$$\mu(F(\lambda_0)) \leq \varphi(\lambda_0) - \lambda_0 \exp(-\lambda_0 r(a)) \times \left(1 + \frac{1}{\log\left[r(a)\left(\mu(q^-(-1,a)) + \mu(q^-(b,0)) + \mu(q^-(a,b))\right)\right]}\right) \times \left(r(a)\left(\mu\left(q^-(-1,a)\right) + \mu\left(q^-(b,0)\right) + \mu\left(q^-(a,b)\right)\right)\right).$$

Taking into account that

$$\lambda_0 \exp(-\lambda_0 r(a)) = \left(\mu\left(q^-(-1,a)\right) + \mu\left(q^+(-1,a)\right)\right) \times \log\left(r(a)\left(\mu\left(q^-(-1,a)\right) + \mu\left(q^-(b,0)\right) + \mu\left(q^-(a,b)\right)\right)\right),$$

we have then $\mu(F(\lambda_0)) \leq 0$ and (1) is nonoscillatory. □

For the case where $a = b = \theta_0$ we have the following corollary.

Corollary 30. *Let $r \in D^{+}(\theta_0)$ and $q \in BV_d$ such that:*

$$\mu\left(q^{-}(\theta,0)\right) \geq 0 \text{ for every } \theta \in [-1,\theta_0],$$
$$\mu\left(q^{-}(-1,\theta)\right) \geq 0 \text{ for every } \theta \in [\theta_0,0],$$
$$0 < \left(\mu\left(q^{-}(-1,\theta_0)\right) + \mu\left(q^{-}(\theta_0,0)\right)\right) < \frac{1}{er(a)}.$$

If

$$\int_{-1}^{\theta_0} \mu\left(q^{-}(\theta,0)\right) dr(\theta) - \int_{\theta_0}^{0} \mu\left(q^{-}(-1,\theta)\right) dr(\theta)$$
$$\leq \left(1 + \frac{1}{\log\left(r(a)\left(\mu\left(q^{-}(-1,\theta_0)\right) + \mu\left(q^{-}(\theta_0,0)\right)\right)\right)}\right)$$
$$\times\, r(a)\left(\mu\left(q^{-}(-1,\theta_0)\right) + \mu\left(q^{-}(\theta_0,0)\right)\right),$$

then (1) is nonoscillatory.

In particular, for the case $\theta_0 = 0$ we obtain the following result.

Corollary 31. *Let $r(\theta) \in D_i^{+}$. If*

$$\mu\left(q^{-}(\theta,0)\right) \geq 0 \text{ for every } \theta \in [-1,0], \quad 0 < \mu\left(q^{-}(-1,0)\right) < \frac{1}{er(0)}.$$

and

$$\int_{-1}^{0} \mu\left(q^{-}(\theta,0)\right) dr(\theta)$$
$$\leq \left(1 + \frac{1}{\log\left(r(0)\,\mu\left(q^{-}(-1,0)\right)\right)}\right) r(0)\,\mu\left(q^{-}(-1,0)\right),$$

then (1) is nonoscillatory.

On the other hand, for the case $\theta_0 = -1$ we obtain the following result.

Corollary 32. *Let $r(\theta) \in D_d^{+}$. If*

$$\mu\left(q^{-}(-1,\theta)\right) \geq 0 \text{ for every } \theta \in [-1,0], \quad 0 < \mu\left(q^{-}(-1,0)\right) < \frac{1}{er(-1)}.$$

and

$$-\int_{-1}^{0} \mu\left(q^{-}(-1,\theta)\right) dr(\theta)$$
$$\leq \left(1 + \frac{1}{\log\left(r(-1)\,\mu\left(q^{-}(-1,0)\right)\right)}\right) r(-1)\,\mu\left(q^{-}(-1,0)\right).$$

Then (1) is nonoscillatory.

Considering in (1), $q(\theta)$ given by (4), for $-1 < \theta_1 < \theta_2 < ... < \theta_m < 0$, and $r(\theta)$ differentiable, increasing and positive on $[-1, 0]$, one obtains the system (3) with $r_j = r(\theta_j)$ for $j = 1, ..., m$, such that $r_1 < r_2 < ... < r_m$. In this situation we can apply the Corollary 31 and consequently obtain the following corollary.

Corollary 33. *Let $r_1 < r_2 < ... < r_m$. If*

$$\mu\left(-\sum_{k=j}^{m} A_k\right) \geq 0, \quad j = 2, ..., m; \quad 0 < \mu\left(-\sum_{k=1}^{m} A_k\right) < \frac{1}{er_m}, \tag{24}$$

and

$$\sum_{j=2}^{m} \mu\left(-\sum_{k=j}^{m} A_k\right) \log \frac{r_j}{r_{j-1}} \leq \left(1 + \frac{1}{\log\left(r_m \mu\left(-\sum_{k=1}^{m} A_k\right)\right)}\right) r_m \mu\left(-\sum_{k=1}^{m} A_k\right), \tag{25}$$

then (3) is nonoscillatory.

Analogously, by application of Corollary 33, we can obtain the following result for system (3).

Corollary 34. *Let be $r_1 > r_2 > ... > r_m$. If*

$$0 < \mu\left(-\sum_{k=1}^{m} A_k\right) < \frac{1}{er_1}; \quad \mu\left(-\sum_{k=1}^{j} A_k\right) \geq 0, \quad j = 1, ..., m-1, \tag{26}$$

and

$$\sum_{j=1}^{m-1} \mu\left(-\sum_{k=1}^{j} A_k\right) \log \frac{r_{J+1}}{r_j} \leq \left(1 + \frac{1}{\log\left(r_1 \mu\left(-\sum_{k=1}^{m} A_k\right)\right)}\right) r_1 \mu\left(-\sum_{k=1}^{m} A_k\right), \tag{27}$$

then (3) is nonoscillatory.

Example 35. Let us consider the system

$$\frac{d}{dt}x(t) = A_1 x\left(t - \frac{1}{6}\right) + A_2 x\left(t - \frac{1}{5}\right) + A_3 x\left(t - \frac{2}{7}\right), \tag{28}$$

where

$$\begin{aligned} A_1 &= \begin{bmatrix} -6/5 & 1/25 & 1/25 \\ -3/50 & 2/25 & 3/25 \\ -1/50 & 0 & 1 \end{bmatrix}, \; A_2 = \begin{bmatrix} 1/25 & 3/25 & -1/25 \\ 1/25 & 2/25 & 0 \\ 0 & 1/25 & 3/25 \end{bmatrix}, \\ A_3 &= \begin{bmatrix} 0 & -1/25 & 3/25 \\ 1/50 & 2/25 & 1/25 \\ 1/50 & 0 & 2/25 \end{bmatrix}. \end{aligned}$$

Here again the condition $A_3 > 0$ is not satisfied and therefore we cannot apply Corollary 5. However, as we shall see, the system is also nonoscillatory. On the other hand, Corollary 24 is also not applicable to this case, because here we have $\beta(A_2) = \mu_1(-A_2 - A_3) - \mu_1(-A_3) = \frac{1}{25} - \frac{2}{25} < 0$. However, as we shall see, the system is also nonoscillatory. Actually,

$$\mu_1(-A_3) = \mu_1 \begin{bmatrix} 0 & 1/25 & -3/25 \\ -1/50 & -2/25 & -1/25 \\ -1/50 & 0 & -2/25 \end{bmatrix} = \max\left\{\frac{1}{25}, \frac{-1}{25}, \frac{2}{25}\right\} = \frac{2}{25} \geq 0,$$

$$\begin{aligned} \mu_1(-A_2 - A_3) &= \mu_1 \begin{bmatrix} -1/25 & -2/25 & -2/25 \\ -3/50 & -4/25 & -1/25 \\ -1/50 & -1/25 & -1/25 \end{bmatrix} \\ &= \max\left\{\frac{1}{25}, \frac{-1}{25}, \frac{-2}{25}\right\} = \frac{1}{25} \geq 0, \end{aligned}$$

$$0 < \mu_1(-A_1 - A_2 - A_3) = \mu_1 \begin{bmatrix} 29/25 & -3/25 & -3/25 \\ 0 & -6/25 & -4/25 \\ 0 & -1/25 & -6/5 \end{bmatrix}$$

$$= \max\{29/25, -2/25, -23/25\} = 29/25 < \frac{7}{2e},$$

So, (24) is satisfied. The same holds to (25), since

$$\mu_1(-A_2 - A_3)\log\frac{r_2}{r_1} + \mu_1(-A_3)\log\frac{r_3}{r_2}$$

$$= \frac{1}{25}\left(\log\frac{6}{5} + \log\frac{10}{7}\right) \approx 0,0215 \leq \frac{12}{35}\left(1 + \frac{1}{\log{(12/35)}}\right) \approx 0,022\,.$$

Then, by Corollary 33, the system (28) is nonoscillatory.

Acknowledgments

I would like to thank Professor José M. Ferreira for his support and helpful suggestions during the preparation of this paper.

References

[1] J.M. Ferreira, S. Pinelas, Oscillatory retarded functional systems, *J. Math. Anal. Appl.*, **285** (2003), 506-527.

[2] J.K. Hale, S.M.V. Lunel, *Introduction to Functional Differential Equations*, Springer (1993).

[3] R.A. Horn, C.R. Johnson, *Matrix Analysis*, Cambridge Univ. Press (1990).

[4] T. Krisztyn, Oscillations in linear functional differential systems, *Differential Equations Dynam. Systems*, **2** (1994), 99-112.

[5] J. Kirchner, U. Stroinski, Explicit oscillation criteria for systems of neutral differential equations with distributed delay, *Differential Equations Dynam. Systems*, **3** (1995), 101-120.

[6] Q. Kong, Oscillation for systems of functional differential equations, *J. Mat. Anal. Appl.*, **198** (1996), 608-619

[7] Ana Pedro, Oscillation and nonoscillation criteria for retarded functional differential equations, *Proceedings of Equadiff*, **11**, Bratislava (2005).

[8] R.S. Varga, *Matrix Iterative Analysis*, Prentice Hall (1962).