**International Journal of Pure and Applied Mathematics**

**Volume 31 No. 1 2006, 91-96**

# A NOTE ON λ-STATISTICALLY CAUCHY SEQUENCES

A. Nihal Tuncer[1 §], F. Berna Benli[2]

[1,2]Department of Mathematics
Erciyes University
Kayseri, 38039, TURKEY
[1]e-mail: ntuncer@erciyes.edu.tr
[2]e-mail: akpinarb@erciyes.edu.tr

**Abstract:** In this paper, we introduce the concept of $\lambda$-statistically Cauchy sequence of fuzzy numbers and show that it is equivalent to $\lambda$-statistically convergent sequence of fuzzy numbers.

**AMS Subject Classification:** 40A05, 40C05, 40D05
**Key Words:** fuzzy number, statistical convergence, $\lambda$-statistical convergence

## 1. Introduction

Let $D$ denote the set of all closed bounded intervals $A = [\underline{A}, \overline{A}]$ on the real line $R$. For $A, B \in D$ define

$$A \leq B \Leftrightarrow \underline{A} \leq \underline{B} \quad \text{and} \quad \overline{A} \leq \overline{B}\,, \quad d(A,B) = \max(|\underline{A} - \underline{B}|, |\overline{A} - \overline{B}|)\,.$$

It is easy to see that $d$ defines a metric on $D$ and $(D, d)$ is a complete metric space. Also $\leq$ is a partial order in $D$.

A fuzzy number is a fuzzy subset of the real line $R$ which is bounded, convex and normal. Let $L(R)$ denote the set of all fuzzy numbers which are upper semicontinuous and have compact support. In other words, if $X \in L(R)$, then for any $\alpha \in [0,1]$, $X^\alpha$ is compact, where

Received: July 7, 2006 

§Correspondence author

$$X^\alpha = \begin{cases} t : X(t) \geq \alpha & \text{if } \alpha \in (0,1], \\ t : X(t) > 0 & \text{if } \alpha = 0. \end{cases}$$

Define a map

$$\overline{d} : L(R) \times L(R) \to R$$

by

$$\overline{d}(X,Y) = [\sup_{0\leq\alpha\leq1} d(X^\alpha, Y^\alpha)].$$

It is known that (see [7]) $L(R)$ is a complete metric space with the metric $\overline{d}$ .

For $X, Y \in L(R)$ define $X \leq Y$ if and only if $X^\alpha \leq Y^\alpha$ for any $\alpha \in [0,1]$. We now recall the following definitions which were given in [4].

**Definition 1.1.** A sequence $X = (X_k)$ of fuzzy numbers is a function $X$ from the set $N$ of all positive integers into $L(R)$. The fuzzy number $X_k$ denotes the value of the function at $k \in N$ and is called the $k$-th term of the sequence.

**Definition 1.2.** A sequence $X = (X_k)$ of fuzzy numbers is said to be convergent to the fuzzy number $X_0$ if for every $\varepsilon > 0$ there exists a positive integer $k_0$ such that

$$\overline{d}(X_k, X_0) < \varepsilon \quad \text{for} \quad k > k_0 \ .$$

Let $c$ denote the set of all convergent sequences of fuzzy numbers.

$X = (X_k)$ is said to be a Cauchy sequences if for every $\varepsilon > 0$ there exists $k_0 \in N$ such that

$$\overline{d}(X_k, X_m) < \varepsilon \quad \text{for} \quad k, m > k_0.$$

Let $C$ denote the set of all Cauchy sequences of fuzzy numbers. It is easy to see that $c \subset C$.

## 2. Statistical Convergence

The notation of statistical convergence was introduced by Fast [1] and also independently Schoenberg [9] for real and complex sequences. Also Fridy [2] obtained an equivalent criterion for statistically convergent real sequences similar to the Cauchy criterion of convergence. The concept of statistical convergence for sequences of fuzzy numbers was introduced and studied by Nuray and Savas [6]. On the other hand Fridy and Orhan [3] introduced the lacunary statistically convergent and Lacunary statistically Cauchy sequences of real- or complex-valued sequences and they have proved the following theorem.

**Theorem 2.1.** *The sequence $X$ is $S_\theta$-convergent if and only if $X$ is an $S_\theta$-Cauchy sequence.*

Also the following definitions are known.

The natural density of a set $K$ of positive integers (see [5]) is defined by

$$\delta(K) = [\lim_n \frac{1}{n} |\{k \leq n : k \in K\}],$$

where $|\{k \leq n : k \in K\}|$ denotes the number of elements of $K$ not exceeding $n$ [5]. We shall be particularly concerned with integer sets having natural density zero.

If $X = (X_k)$ is a sequence that satisfies some property $P$ for all $k$ except a set of natural density zero, then we say that $X_k$ satisfies $P$ for "almost all $k$" and abbreviate this by "a.a. $k$".

**Definition 2.2.** (see [6]) A sequence $X = (X_k)$ of fuzzy numbers is said to be statistically convergent to the fuzzy number $X_0$, written as st-lim $X_k = X_0$ if for every $\varepsilon > 0$,

$$[\lim_n \frac{1}{n} |\{k \leq n : \overline{d}(X_k, X_0) \geq \varepsilon\}| = 0],$$

i.e.,

$$\overline{d}(X_k, X_0) < \varepsilon \quad \text{a.a. } k.$$

It is clear that $[\lim_k X_k = X_0]$ implies st-lim $X_k = X_0$.

**Definition 2.3.** (see [3]) A sequence $X = (X_k)$ of fuzzy numbers is a statistically Cauchy sequence if for every $\varepsilon > 0$ there exists a number $m = m(\varepsilon)$ such that

$$[\lim_n \frac{1}{n} |\{k \leq n : \overline{d}(X_k, X_m) \geq \varepsilon\}| = 0],$$

i.e.,

$$\overline{d}(X_k, X_m) < \varepsilon \quad \text{a.a. } k.$$

**Definition 2.4.** (see [8]) A sequence $X = (X_k)$ of fuzzy numbers is said to be $\lambda$-statistically convergent or $s\lambda$-convergent to fuzzy numbers $X_0$, written as $s\lambda - lim X_k = X_0$ if for every $\varepsilon > 0$

$$[\lim_n \frac{1}{\lambda_n} |\{k \in I_n : \overline{d}(X_k, X_0) \geq \varepsilon\}| = 0].$$

Also Savas [8] has proved the following theorem concerning the $\lambda$-statistically convergence.

**Theorem 2.5.** *If a sequence $X = X_k$ is statistically convergent to $X_0$ and $\liminf_n(\frac{\lambda_n}{n}) > 0$ then it is $\lambda$-statistically convergent to $X_0$.*

## 3. The Main Result

In this paper we introduce $\lambda$-statistically Cauchy sequences of fuzzy numbers and we will prove two theorems. First we will give the following definition.

**Definition 3.1.** Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to $\infty$ and $\lambda_{n+1} \le \lambda_n + 1$, $\lambda_1 = 1$. A sequence $X = (X_k)$ of fuzzy numbers is said to be $\lambda$-statistically Cauchy sequence if for every $\varepsilon > 0$ there exists a number $m = m(\varepsilon)$ such that

$$\lim_n \frac{1}{\lambda_n} |\{k \in I_n : \overline{d}(X_k, X_m) \ge \varepsilon\}| = 0\,, \tag{1}$$

where $I_n = [n - \lambda_n + 1, n]$. If $\lambda_n = n$, then $\lambda$-statistically Cauchy sequence is same as statistically Cauchy sequence. If $\lambda_n = (1, 1, ..)$, then $\lambda$-statistically Cauchy sequence is same as ordinary Cauchy sequence.

**Theorem 3.2.** *A sequence $X = (X_k)$ of fuzzy numbers is $\lambda$-statistically convergent if and only if $X$ is an $\lambda$-statistically Cauchy sequence.*

*Proof.* Let s$\lambda$-$\lim X_k = X_0$ and write $K^{(j)} = \{k \in N : \overline{d}(X_k, X_0) < \frac{1}{j}\}$, for each $j \in N$. Hence, for each $j$, $K^{(j)} \supseteq K^{(j+1)}$ and

$$\frac{|K^{(j)} \bigcap I_n|}{\lambda_n} \to 1 \quad \text{as} \quad n \to \infty\,.$$

Choose $m(1)$ such that $n \ge m(1)$ implies $\frac{|K^{(1)} \bigcap I_n|}{\lambda_n} > 0$ i.e., $K^{(1)} \bigcap I_n \ne \emptyset$. Next choose $m(2) > m(1)$ so that $n \ge m(2)$ implies $K^{(2)} \bigcap I_n \ne \emptyset$. Then for each $n$ satisfying $m(1) \le n < m(2)$, choose $k^{'}(n) \in I_n$ such that $k^{'}(n) \in I_n \bigcap K^{(1)}$, i.e., $\overline{d}(X_k, X_0) < 1$. In general, choose $m(p+1) > m(p)$ such that $n > m(p+1)$ implies $I_n \bigcap K^{(p+1)} \ne \emptyset$. Then for all n satisfying $m(p) \le n < m(p+1)$, choose $k^{'}(n) \in I_n \bigcap K^{(p)}$, i.e.,

$$\overline{d}(X_{k^{'}(n)}, X_0) < \frac{1}{p}\,. \tag{2}$$

Hence, we get $k^{'}(n) \in I_n$ for every $n$, and (2) implies that $[\lim_n X_{k^{'}(n)} = X_0]$. Furthermore, for every $\varepsilon > 0$, we have that

$$[\frac{1}{\lambda_n} |\{k \in I_n : \overline{d}(X_k, X_{k^{'}(n)}) \ge \varepsilon\}| \le [\frac{1}{\lambda_n} |\{k \in I_n : \overline{d}(X_k, X_0) \ge \frac{\varepsilon}{2}\}|$$

$$+[\frac{1}{\lambda_n} |\{k \in I_n : \overline{d}(X_{k^{'}(n)}, X_0) \ge \frac{\varepsilon}{2}\}|]$$

Using the assumptions that s$\lambda$-lim $X_k = X_0$ and $\lim_n X_{k'(n)} = X_0$, we infer (1), hence $X$ is an $\lambda$-statistically Cauchy sequence.

Conversely, suppose that $X$ is an $\lambda$-statistically Cauchy sequence. For every $\varepsilon > 0$, we have

$$|\{k \in I_n : \overline{d}(X_k, X_0) \geq \varepsilon\}| \leq |\{k \in I_n : \overline{d}(X_k, X_{k'(n)}) \geq \frac{\varepsilon}{2}\}|$$

$$+|\{k'(n) \in I_n : \overline{d}(X_{k'(n)}, X_0) \geq \frac{\varepsilon}{2}\}|]$$

from which it follows that s$\lambda$-lim $X_k = X_0$. This completes the proof of the theorem. ☐

**Theorem 3.3.** *If a sequence $X = X_k$ is statistically Cauchy sequence and $\liminf_n(\frac{\lambda_n}{n}) > 0$, then it is $\lambda$-statistically Cauchy sequence.*

*Proof.* For given $\varepsilon > 0$, we have $\exists m(\varepsilon) \in N$ such that

$$\{k \leq n : \overline{d}(X_k, X_m) \geq \varepsilon\} \supset \{k \in I_n : \overline{d}(X_k, X_m) \geq \varepsilon\}$$

Therefore,

$$\frac{1}{n}|\{k \leq n : \overline{d}(X_k, X_m) \geq \varepsilon\}| \geq \frac{1}{n}|\{k \in I_n : \overline{d}(X_k, X_m) \geq \varepsilon\}|$$

$$\geq \frac{\lambda_n}{n}\frac{1}{\lambda_n}|\{k \in I_n : \overline{d}(X_k, X_m) \geq \varepsilon\}| .$$

Taking limit as $n \to \infty$ and using $\liminf_n(\frac{\lambda_n}{n}) > 0$, we get that $X$ is $\lambda$-statistically Cauchy sequence. ☐

## References


[1] H. Fast, Sur la convergence statistique, *Colloq. Math.* (1951), 241-244.

[2] J.A. Fridy, On statistical convergence, *Analysis*, **5** (1985), 301-313.

[3] J.A. Fridy, C. Orhan, Lacunary statistically summability, *J. Math. Anal. Appl.*, **173** (1993), 497-504.

[4] M. Matloka, Sequences of fuzzy numbers, *Busefal*, **28** (1986), 28-37.

[5] I. Niven, H.S. Zuckerman, *An Introduction to the Theory of Numbers*, Forth Edition, John Wiley and Sons, New York (1980).

[6] F. Nuray, E. Savas, Statistically convergence of sequences of fuzzy numbers, *Math. Slovaca*, **45** (1995), 269-273.

[7] M.L. Puri, D.A. Ralescu, Fuzzy random variables, *J. Math. Anal. Appl.*, **114** (1986), 409-422.

[8] E. Savas, On Strongly $\lambda$-summable sequences of fuzzy numbers, *Inf. Sci.*, **125** (2000), 181-186.

[9] I.J. Schoenberg, The integrebility of certain functions and related summability methods, *Amer. Math. Monthly*, **66** (1959), 362-375.