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ON THE HAUSDORFF INTUITIONISTIC FUZZY METRIC ON COMPACT SETS

Hakan Efe[1 §], Cemil Yildiz[2]

[1,2]Department of Mathematics
Faculty of Science and Arts
Gazi University
Teknikokullar, Ankara, 06500, TURKEY
[1]e-mail: hakanefe@gazi.edu.tr
[2]e-mail: cyildiz@gazi.edu.tr

Abstract: In this paper, we give important properties as completeness, completion and precompactness of Hausdorff intuitionistic fuzzy metric spaces which is given by Gregori et al [13] and establish the precise relationship between the Hausdorff metric of a metric space (X, d) and the Hausdorff intuitionistic fuzzy metric of the standart intuitionistic fuzzy metric of d, and give two examples.

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1. Introduction

Since the introduction of the concept of fuzzy set by Zadeh [22] in 1965, many authors have introduced the concept of fuzzy metric space in different ways (see [7], [11], [14]). George and Veeramani (see [8]-[10], [12]) modified the concept of fuzzy metric space introduced by Kromosil and Michalek [14] and defined a Hausdorff topology on this fuzzy metric space. Lopez and Romaguera [15] purposed a method for constructing a Hausdorff fuzzy metric on the set of the nonempty compact subsets of a given fuzzy metric space. They discussed

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§Correspondence author

several important properties as completeness, completion and precompactness. On the other hand, Atanassov (see [3], [4]) introduced and studied the concept of intuitionistic fuzzy sets and later there has been much progress in the study of intuitionistic fuzzy sets by many authors (see [2], [6], [13], [19], [20]). Also, Alaca et al [1] defined the completions of intuitionistic fuzzy metric spaces. A complete intuitionistic fuzzy metric space Y is said to be an intuitionistic fuzzy completion of a given intuitionistic fuzzy metric space X if X is isometric to a dense subspace of Y. They gave an example of an intuitionistic fuzzy metric space that does not admit any intuitionistic fuzzy metric completion. However, they proved that every standard intuitionistic fuzzy metric space has an (up to isometry) unique intuitionistic fuzzy metric completion. They also showed that for each intuitionistic fuzzy metric space there is an (up to uniform isomorphism) unique complete intuitionistic fuzzy metric space that contains a dense subspace uniformly isomorphic to it.

In this paper, in Section 3 we give important properties as completeness, completion and precompactness of Hausdorff intuitionistic fuzzy metric spaces which are given by Gregori et al [13] and establish the precise relationship between the Hausdorff metric of a metric space (X, d) and the Hausdorff intuitionistic fuzzy metric of the standart intuitionistic fuzzy metric of d and give two examples.

2. Preliminaries

Now, we give the fundamental properties of intuitionistic fuzzy metric space.

Definition 1. (see [21]) A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is continuous t-norm if $*$ satisfies the following conditions: (i) $*$ is commutative and associative; (ii) $*$ is continuous; (iii) $a*1 = a$ for all $a \in [0,1]$; (iv) $a*b \leq c*d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0,1]$.

Definition 2. (see [21]) A binary operation $\Diamond : [0,1] \times [0,1] \to [0,1]$ is continuous t-conorm if $\Diamond$ satisfies the following conditions: (i) $\Diamond$ is commutative and associative; (ii) $\Diamond$ is continuous; (iii) $a\Diamond 0 = a$ for all $a \in [0,1]$; (iv) $a\Diamond b \leq c\Diamond d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0,1]$.

Definition 3. (see [19]) A 5-tuple $(X, M, N, *, \Diamond)$ is said to be an intuitionistic fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm, $\Diamond$ is a continuous t-conorm and M, N are fuzzy sets on $X \times X \times (0, \infty)$ satisfying the following conditions, for all $x, y, z \in X$, $s, t > 0$:

(i) $M(x, y, t) + N(x, y, t) \leq 1$;

(ii) $M(x, y, t) > 0$;
(iii) $M(x, y, t) = 1$ if and only if $x = y$;
(iv) $M(x, y, t) = M(y, x, t)$;
(v) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
(vi) $M(x, y, .) : (0, \infty) \to (0, 1]$ is continuous;
(vii) $N(x, y, t) > 1$;
(viii) $N(x, y, t) = 0$ if and only if $x = y$;
(ix) $N(x, y, t) = N(y, x, t)$;
(x) $N(x, y, t) \Diamond N(y, z, s) \geq N(x, z, t + s)$;
(xi) $N(x, y, .) : (0, \infty) \to [0, 1)$ is continuous.

Then (M, N) is called an intuitionistic fuzzy metric on X. The functions $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.

Remark 1. (i) Every fuzzy metric space $(X, M, *)$ is an intuitionistic fuzzy metric space of the form $(X, M, 1-M, *, \Diamond)$ such that t-norm $*$ and t-conorm $\Diamond$ are assosiated, see [16], i.e., $a \Diamond b = 1-((1-a)*(1-b))$ for any $a, b \in [0, 1]$. (ii) In intuitionistic fuzzy metric space X, $M(x, y, .)$ is non-decreasing and $N(x, y, .)$ is non-increasing for all $x, y \in X$.

Example 1. (see [19]) (Induced Intuitionisticfuzzy Metric) Let (X, d) be a metric space. Denote $a * b = a \cdot b$ and $a \Diamond b = \min\{1, a + b\}$ for all $a, b \in [0, 1]$ and let M_d and N_d be fuzzy sets on $X \times X \times (0, \infty)$ defined as follows:

$$M_d(x, y, t) = \frac{ht^n}{ht^n + md(x, y)}, \quad N_d(x, y, t) = \frac{d(x, y)}{kt^n + md(x, y)}.$$

Remark 2. Note the above example holds even with the t-norm $a * b = \min\{a, b\}$ and the t-conorm $a \Diamond b = \max\{a, b\}$ and hence (M, N) is an intuitionistic fuzzy metric with respect to any continuous t-norm and continuous t-conorm. In the above example by taking $h = k = m = n = 1$ we get

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}, \quad N_d(x, y, t) = \frac{d(x, y)}{t + d(x, y)}.$$

We call this intuitionistic fuzzy metric induced by a metric d the standart intuitionistic fuzzy metric.

Example 2. (see [19]) Let $X = \mathbb{N}$. Define $a * b = \max\{0, a + b - 1\}$ and $a \Diamond b = a + b - ab$ for all $a, b \in [0, 1]$ and let M and N be fuzzy sets on $X^2 \times (0, \infty)$ as follows:

$$M(x, y, t) = \begin{cases} \frac{x}{y} & \text{if} \quad x \leq y, \\ \frac{y}{x} & \text{if} \quad y \leq x, \end{cases} \qquad N(x, y, t) = \begin{cases} \frac{y-x}{y} & \text{if} \quad x \leq y, \\ \frac{x-y}{x} & \text{if} \quad y \leq x, \end{cases}$$

for all $x, y \in X$ and $t > 0$. Then $(X, M, N, *, \Diamond)$ is an intuitionistic fuzzy metric space.

Remark 3. Note that, in the above example, t-norm $*$ and t-conorm $\Diamond$ are not associated. And there exists no metric d on X satisfying

$$M(x,y,t) = \frac{t}{t+d(x,y)}, \quad N(x,y,t) = \frac{d(x,y)}{t+d(x,y)},$$

where $M(x,y,t)$ and $N(x,y,t)$ are as defined in above example. Also note the above functions (M, N) is not an intuitionistic fuzzy metric with the t-norm and t-conorm defined as $a * b = \min\{a, b\}$ and $a \Diamond b = \max\{a, b\}$.

Definition 4. (see [19]) Let $(X, M, N, *, \Diamond)$ is an intuitionistic fuzzy metric space and $r \in (0, 1)$, $t > 0$ and $x \in X$. The set $B_{(M,N)}(x, r, t) = \{y \in X : M(x, y, t) > 1 - r,\ N(x, y, t) < r\}$ is called open ball with center x and radius r with respect to t.

Theorem 1. (see [19]) *Every open ball $B(x, r, t)$ is open set.*

Remark 4. (i) Let $(X, M, N, *, \Diamond)$ is an intuitionistic fuzzy metric space. Define $\tau_{(M,N)} = \{A \subset X$: for each $x \in A$, there exist $t > 0$ and $r \in (0, 1)$ such that $B_{(M,N)}(x, r, t) \subset A\}$. Then $\tau_{(M,N)}$ is a topology on X.

(ii) Every intuitionistic fuzzy metric (M, N) on X generates a topology $\tau_{(M,N)}$ on X which has a base the family of open sets of the form $\{B_{(M,N)}(x, r, t) : x \in X,\ r \in (0, 1),\ t > 0\}$.

(iii) Since $\{B_{(M,N)}(x, \frac{1}{n}, \frac{1}{n}) : n = 1, 2, ...\}$ is a local base at x, the topology $\tau_{(M,N)}$ is first countable.

Theorem 2. (see [19]) *Every intuitionistic fuzzy metric space is Hausdorff.*

Remark 5. Let (X, d) be a metric space. Let

$$M_d(x,y,t) = \frac{t}{t+d(x,y)}, \quad N_d(x,y,t) = \frac{d(x,y)}{kt+d(x,y)}, \quad k \in R^+,$$

be the intuitionistic fuzzy metric defined on X. Then the topology τ_d induced by the metric d and the topology $\tau_{(M,N)}$ induced by the intuitionistic fuzzy metric (M, N) are one and the same.

Theorem 3. (see [19]) *Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space and $\tau_{(M,N)}$ be the topology on X induced by the fuzzy metric. Then for a sequence $\{x_n\}$ in X, $x_n \to x$ if and only if $M(x_n, x, t) \to 1$ and $N(x_n, x, t) \to 0$ as $n \to \infty$.*

Definition 5. (see [19]) Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space. Then: (i) A sequence $\{x_n\}$ in X is said to be Cauchy if for each $\varepsilon > 0$ and each $t > 0$, there exist $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ and $N(x_n, x_m, t) < \varepsilon$ for all $n, m \geq n_0$, (ii) $(X, M, N, *, \diamondsuit)$ is called complete if every Cauchy sequence convergent with respect to $\tau_{(M,N)}$.

3. Main Results

Definition 6. (see [5) Denote by $CB_0(X)$ the set of nonempty closed and bounded subset of a metric space (X, d). It is well known that the function H_d defined on $CB_0(X) \times CB_0(X)$ by

$$H_d(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\right\},$$

for all $A, B \in CB_0(X)$, is a metric on $CB_0(X)$ called the Hausdorff metric of d.

Definition 7. (see [20) Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space, $a \in A$ and $\emptyset \neq B \subset X$. We define

$$M(a, B, t) = \sup\{M(a, b, t) : b \in B\},\ N(a, B, t) = \inf\{N(a, b, t) : b \in B\},$$

for all $t > 0$. Note that $M(a, B, t)$ and $N(a, B, t)$ are a degree of closeness and non closeness of a to B at t respectively.

Remark 6. Given an intuitionistic fuzzy metric space, we shall denote by $P_0(X)$, $F_0(X)$ and $K_0(X)$, the set of nonempty subsets, the set of nonempty finite subsets and the set of nonempty compact subsets of $(X, \tau_{(M,N)})$, respectively.

The following construction was introduced by V. Gregori et al [13].

Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space. We define functions H_M and H_N on $K_0(X) \times K_0(X) \times (0, \infty)$ by

$$\begin{aligned} H_M(A, B, t) &= \min\left\{\inf_{a\in A} M(a, B, t), \inf_{b\in B} M(A, b, t)\right\}, \\ H_N(A, B, t) &= \max\left\{\sup_{a\in A} N(a, B, t), \sup_{b\in B} N(A, b, t)\right\}, \end{aligned}$$

for all $A, B \in K_0(X)$ and $t > 0$. Then, $(H_M, H_N, *, \diamondsuit)$ is Hausdorff intuitionistic fuzzy metric on $K_0(X)$.

Now, we will give other important properties of Hausdorff intuitionistic fuzzy metric. This results suggest that the Hausdorff intuitionistic fuzzy metric introduced above, provides a suitable notation in the context of intuitionistic version of metric fuzziness on hyperspaces.

Proposition 1. *Let (X, d) be a metric space. Then, the Hausdorff intuitionistic fuzzy metric $(H_{M_d}, H_{N_d}, *, \Diamond)$ of the standard intuitionistic fuzzy metric $(M_d, N_d, *, \Diamond)$ coincides with the standard intuitionistic fuzzy metric $(M_{H_d}, N_{H_d}, *, \Diamond)$ of the Hausdorff metric H_d on $K_0(X)$.*

Proof. Let $A, B \in K_0(X)$ and $t > 0$. Therefore,

$$M_d(a, B, t) = \frac{t}{t + d(a, B)} \text{ and } N_d(a, B, t) = \frac{d(a, B)}{t + d(a, B)}$$

for all $a \in A$. By using this fact it is routine to see that

$$\inf_{a \in A} M_d(a, B, t) = \frac{t}{t + \sup_{a \in A} d(a, B)}$$

and

$$\sup_{a \in A} N_d(a, B, t) = \frac{\sup_{a \in A} d(a, B)}{t + \sup_{a \in A} d(a, B)}.$$

Similarly, we obtain

$$\inf_{b \in B} M_d(A, b, t) = \frac{t}{t + \sup_{b \in B} d(A, b)}$$

and

$$\sup_{b \in B} N_d(A, b, t) = \frac{\sup_{b \in B} d(A, b)}{t + \sup_{b \in B} d(A, b)}.$$

Therefore $H_{M_d}(A, B, t) = M_{H_d}(A, B, t)$ and $H_{N_d}(A, B, t) = N_{H_d}(A, B, t)$. □

Now we give two examples to validate Proposition 1.

Example 3. Let d be the Euclidean metric on $\mathbb{R}$, and let $A = [a_1, a_2]$ and $B = [b_1, b_2]$ be two compact intervals. Then

$$H_d(A, B) = \max\{|a_1 - b_1|, |a_2 - b_2|\};$$

so, by Proposition 1,

$$H_{M_d}(A, B, t) \quad = \quad \frac{t}{t + \max\{|a_1 - b_1|, |a_2 - b_2|\}},$$

$$H_{N_d}(A,B,t) \quad = \quad \frac{\max\{|a_1-b_1|,|a_2-b_2|\}}{t+\max\{|a_1-b_1|,|a_2-b_2|\}}\,,$$

for all $t>0$.

Example 4. Let d be the discrete metric on a (nonempty) set X with $|X|\geq 2$. Let A and B be two nonempty finite subsets of X, with $A\neq B$. Then $H_d(A,B)=1$; so by Proposition 1,

$$H_{M_d}(A,B,t)=\frac{t}{t+1} \text{ and } H_{N_d}(A,B,t)=\frac{1}{t+1}$$

for all $t>0$.

Let us recall that if $(X,\breve{U})$ is a uniform space, then the Hausdorff-Baurbaki uniformity $H_{\breve{U}}$ (of $\breve{U}$) on $P_0(X)$, has a base the family of sets of the form

$$H_U=\{(A,B)\in P_0(X)\times P_0(X): B\subseteq U(A) \text{ and } A\subseteq U(B)\},$$

where $U\in\breve{U}$.

The restiriction of $H_{\breve{U}}$ to $K_0(X)\times K_0(X)$, will be also denoted by $H_{\breve{U}}$.

On the other hand, if $(X,M,N,*,\Diamond)$ is an intuitionistic fuzzy metric space, then $\{U_n : n\in\mathbb{N}\}$ is a countable base for uniformity $\breve{U}_{(M,N)}$ on X whose induced topology coincides with $\tau_{(M,N)}$, where

$$U_n=\left\{(x,y)\in X\times X : M\left(x,y,\frac{1}{n}\right)>1-\frac{1}{n},\, N\left(x,y,\frac{1}{n}\right)<\frac{1}{n}\right\},$$

for all $n\in\mathbb{N}$. In particular, $\breve{U}_{(H_M,H_N)}$ is exactly the unifomity induced by the Hausdorff intuitionistic fuzzy metric of (M,N).

Theorem 4. *Let $(X,M,N,*,\Diamond)$ be an intuitionistic fuzzy metric space. Then, the Hausdorff-Baurbaki uniformity $H_{\breve{U}_{(M,N)}}$, coincides with the uniformity $\breve{U}_{(H_M,H_N)}$ on $K_0(X)$.*

Proof. For each $n\in\mathbb{N}$,

$$\begin{aligned}
\{(A,B) \quad &\in \quad K_0(X)\times K_0(X): B\subseteq U_{n+1}(A) \text{ and } A\subseteq U_{n+1}(B)\}\\
&\subseteq \quad \{(A,B)\in K_0(X)\times K_0(X): H_M\left(A,B,\frac{1}{n+1}\right)\\
&\geq \quad 1-\frac{1}{n+1} \text{ and } H_N\left(A,B,\frac{1}{n+1}\right)\leq\frac{1}{n+1}\}\\
&\subseteq \quad \{(A,B)\in K_0(X)\times K_0(X): H_M\left(A,B,\frac{1}{n}\right)
\end{aligned}$$

$$
\begin{aligned}
&> \quad 1-\frac{1}{n} \text{ and } H_N\left(A,B,\frac{1}{n}\right)<\frac{1}{n}\}\\
&\subseteq \quad \{(A,B)\in K_0(X)\times K_0(X): B\subseteq U_n(A) \text{ and } A\subseteq U_n(B)\}.
\end{aligned}
$$

We conclude that $H_{\breve{U}_{(M,N)}}=\breve{U}_{(H_M,H_N)}$ on $K_0(X)$. □

Definition 8. (see [20) Let $(X,M,N,*,\Diamond)$ be an intuitionistic fuzzy metric space and $A\subset X$. We say A is precompact if for each $0<r<1$ and $t>0$ there exists a finite subset S of A such that

$$A\subseteq \cup_{x\in S}B_{(M,N)}(x,r,t).$$

Remark 7. (i) It is well known (see [18]) that if $(X,\breve{U})$ is uniform space, then $(K_0(X),H_{\breve{U}})$ is complete (resp. precompact) if and only if $(X,\breve{U})$ is complete (resp. precompact). (ii) It is easy to see that an intuitionistic fuzzy metric space $(X,M,N,*,\Diamond)$ is complete if and only if $(X,\breve{U}_{(M,N)})$ is a complete uniform space. (iii) Since an intuitionistic fuzzy metric space $(X,M,N,*,\Diamond)$ is precompact if and only if every sequence in X has a Cauchy subsequence [20], and a sequence in $(X,M,N,*,\Diamond)$ is Cauchy if and only if it is Cauchy in the uniform space $(X,\breve{U}_{(M,N)})$, we deduce that $(X,M,N,*,\Diamond)$ is precompact if and only if $(X,\breve{U}_{(M,N)})$ is precompact.

Theorem 5. *Let $(X,M,N,*,\Diamond)$ be an intuitionistic fuzzy metric space. Then $(K_0(X),H_M,H_N,*,\Diamond)$ is complete if and only if $(X,M,N,*,\Diamond)$ is complete.*

Proof. By Remark 7(ii), $(K_0(X),H_M,H_N,*,\Diamond)$ is complete if and only if $(K_0(X),\breve{U}_{(H_M,H_N)})$ is a complete uniform space. Since, by Theorem 4, $\breve{U}_{(H_M,H_N)} = H_{\breve{U}_{(M,N)}}$ on $K_0(X)$, it follows from Remark 7(i) that $(K_0(X),\breve{U}_{(H_M,H_N)})$ is complete if and only if $(X,\breve{U}_{(M,N)})$ is complete. Therefore, $(K_0(X),H_M,H_N,*,\Diamond)$ is complete if and only if $(X,M,N,*,\Diamond)$ is complete. □

By replacing "complete" by "precompact" and "Remark 7(ii)" by "Remark 7(iii)", in the proof of Theorem 5, we show the following result.

Theorem 6. *Let $(X,M,N,*,\Diamond)$ be an intuitionistic fuzzy metric space. Then $(K_0(X),H_M,H_N,*,\Diamond)$ is precompact if and only if $(X,M,N,*,\Diamond)$ is precompact.*

Definition 9. (see [1) Let $(X,M,N,*,\Diamond)$ and $(Y,M',N',*',\Diamond')$ be two intuitionistic fuzzy metric spaces. A mapping $f:X\to Y$ is called an isometry if for each $x, y \in X$ and $t > 0$, $M(x,y,t) = M'(f(x), f(y), t)$ and $N(x,y,t) = N'(f(x), f(y), t)$. In this case, X and Y are called isometric.

Definition 10. (see [1] Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space. An intuitionistic fuzzy metric completion of $(X, M, N, *, \diamondsuit)$ is a complete intuitionistic fuzzy metric space $(Y, M', N', *', \diamondsuit')$ such that $(X, M, N, *, \diamondsuit)$ is isometric to a dense subspace of Y.

Definition 11. An intuitionistic fuzzy metric space $(X, M, N, *, \diamondsuit)$ is said to be completable if it admits an intuitionistic fuzzy metric completion.

Lemma 1. (see [1] *Suppose that* $(Y_1, M_1', N_1', *_1', \diamondsuit_1')$ *and* $(Y_2, M_2', N_2', *_2', \diamondsuit_2')$ *are two intuitionistic fuzzy metric completions of* $(X, M, N, *, \diamondsuit)$. *Then* Y_1 *and* Y_2 *are isometric. Thus, if an intuitionistic fuzzy metric space has an intuitionistic fuzzy metric completion, it is unique up to isometry.*

Lemma 2. *Let* $(Y, M', N', *', \diamondsuit')$ *be an intuitionistic fuzzy metric space and* X *be a dense subset of* Y. *Then* $F_0(X)$, *and hence* $K_0(X)$, *dense in* $(K_0(Y), H_{M'}, H_{N'}, *', \diamondsuit')$.

Proof. Let $A \in K_0(Y)$, and let $\varepsilon \in (0,1)$ and $t > 0$. We shall prove that there is $F \in F_0(X) \cap B_{(H_{M'}, H_{N'})}(A, \varepsilon, t)$. Choose $\delta \in (0, \varepsilon)$. Since $*'$ and $\diamondsuit'$ is continuous, we can find $\alpha \in (0,1)$ such that $1-\varepsilon < (1-\delta) *' (1-\alpha)$ and $\varepsilon > \delta \diamondsuit' \alpha$. Let $\{a_1, a_2, ..., a_n\} \subseteq A$ such that $A \subseteq \cup_{i=1}^{n} B_{(M',N')}(a_i, \alpha, \frac{t}{2})$. Since X is dense in Y, for each $i \in \{1, 2, ..., n\}$ there exists $x_i \in B_{(M',N')}(a_i, \delta, \frac{t}{2}) \cap X$. Put $F = \{x_1, x_2, ..., x_n\}$. Obviously $F \in F_0(X)$. We show that $F \in B_{(H_{M'}, H_{N'})}(A, \varepsilon, t)$, indeed, since for each $i \in \{1, 2, ..., n\}$,

$$\begin{aligned} M'(A, x_i, t) &\geq M'(a_i, x_i, t) \geq M'\left(a_i, x_i, \frac{t}{2}\right) > 1 - \delta \text{ and} \\ N'(A, x_i, t) &\leq N'(a_i, x_i, t) \leq N'\left(a_i, x_i, \frac{t}{2}\right) < \delta. \end{aligned}$$

We deduce that

$$\inf_{x \in F} M'(A, x, t) > 1 - \delta > 1 - \varepsilon \text{ and } \sup_{x \in F} N'(A, x, t) < \delta < \varepsilon.$$

On the other hand, given $a \in A$ there exists $i \in \{1, 2, ..., n\}$ such that

$$\begin{aligned} M'\left(a_i, a, \frac{t}{2}\right) &> 1 - \alpha \text{ and } N'\left(a_i, a, \frac{t}{2}\right) < \alpha. \text{ Then} \\ M'(a, F, t) &\geq M'(a, x_i, t) \geq M'\left(a, a_i, \frac{t}{2}\right) *' M'\left(a_i, x_i, \frac{t}{2}\right) \end{aligned}$$

$$
\begin{aligned}
& > (1-\alpha) *' (1-\delta), \\
N'(a,F,t) & \leq N'(a,x_i,t) \leq N'\left(a,a_i,\frac{t}{2}\right) \Diamond' N'\left(a_i,x_i,\frac{t}{2}\right) < \alpha \Diamond' \delta, \text{ so} \\
\inf_{a\in A} M'(a,F,t) & \geq (1-\alpha) *' (1-\delta) > 1-\varepsilon \quad \text{and} \\
\sup_{a\in A} N'(a,F,t) & \leq \alpha \Diamond' \delta < \varepsilon.
\end{aligned}
$$

Consequently

$$H_{M'}(A,F,t) > 1-\varepsilon \text{ and } H_{N'}(A,F,t) < \varepsilon.$$

Therefore the proof is completed. □

Theorem 7. *Let $(X,M,N,*,\Diamond)$ be an intuitionistic fuzzy metric space. Then $(K_0(X),H_M,H_N,*,\Diamond)$ is completable if and only if $(X,M,N,*,\Diamond)$ is completable.*

Proof. Suppose that $(X,M,N,*,\Diamond)$ is completable. Then X is dense in its (intuitionistic fuzzy metric) completion $\left(\widetilde{X},\widetilde{M},\widetilde{N},*',\Diamond'\right)$, so by Lemma 2, $K_0(X)$ is dense in $\left(K_0(\widetilde{X}),H_{\widetilde{M}},H_{\widetilde{N}},*',\Diamond'\right)$. Since by Theorem 5, $\left(K_0(\widetilde{X}),H_{\widetilde{M}},H_{\widetilde{N}},*',\Diamond'\right)$ is complete, we conclude that $(K_0(X),H_M,H_N,*,\Diamond)$ is completable and its completion is isometric to $(K_0(\widetilde{X}),H_{\widetilde{M}},H_{\widetilde{N}},*',\Diamond')$.

Conversely, suppose that $(K_0(X),H_M,H_N,*,\Diamond)$ is completable. Then, there is an isometry $i:(K_0(X),H_M,H_N,*,\Diamond)\to\left(K_0(\widetilde{X}),H_{\widetilde{M}},H_{\widetilde{N}},*',\Diamond'\right)$ such that $i(K_0(X))$ is dense in $K_0(\widetilde{X})$. Since for each $x\in X$, $\{x\}\in K_0(X)$, the restriction i_X of i to X is an isometry between $(X,M,N,*,\Diamond)$ and $(i_X(X),H_{\widetilde{M}},H_{\widetilde{N}},*',\Diamond')$. Then $\left(\overline{i_X(X)},H_{\widetilde{M}},H_{\widetilde{N}},*',\Diamond'\right)$ is a complete intuitionistic fuzzy metric space that has $i_X(X)$ as a dense subspace. We conclude that $(X,M,N,*,\Diamond)$ is completable. □

The following example shows that the Hausdorff intuitionistic fuzzy metric defined above does not provide an intuitionistic fuzzy metric when one consider the set $IF\text{-}CB_0(X)$ of nonempty closed and IF-bounded subsets of a given intuitionistic fuzzy metric space $(X,M,N,*,\Diamond)$.

We know that [19] a subset A of an intuitionistic fuzzy metric space $(X,M,N,*,\Diamond)$ is said to be IF-bounded if there is $t>0$ and $\varepsilon\in(0,1)$ such that $M(x,y,t)>1-\varepsilon$ and $N(x,y,t)<\varepsilon$ for all $x,y\in A$.

For each $A, B \in IF\text{-}CB_0(X)$ and $t > 0$, put

$$\begin{aligned} H_M(A,B,t) &= \min\left\{\inf_{a\in A} M(a,B,t), \inf_{b\in B} M(A,b,t)\right\}, \\ H_N(A,B,t) &= \max\left\{\sup_{a\in A} N(a,B,t), \sup_{b\in B} N(A,b,t)\right\}. \end{aligned}$$

Example 5. Denote by $*$ is a continuous t-norm, $\Diamond$ is a continuous t-conorm defined on $[0,1]\times[0,1]$ by $a*b = \max\{0, a+b-1\}$ and $a\Diamond b = \min\{1, a+b\}$ for all $a, b \in [0,1]$. Now let $(x_n)_{n=2}^{\infty}$ and $(y_n)_{n=2}^{\infty}$ be two sequences of distinc points such that $A \cap B = \emptyset$, where $A = \{x_n : n \geq 2\}$ and $B = \{y_n : n \geq 2\}$. Put $X = A \cup B$. Define two real valued function M and N on $X^2 \times (0, \infty)$ as follows:

$$\begin{aligned} M(x_n,x_m,t) &= M(y_n,y_m,t) = 1 - \left[\frac{1}{n\wedge m} - \frac{1}{n\vee m}\right], \\ N(x_n,x_m,t) &= N(y_n,y_m,t) = \frac{1}{n\wedge m} - \frac{1}{n\vee m} \quad \text{and} \\ M(x_n,y_m,t) &= M(y_m,x_n,t) = \frac{1}{n}\wedge\frac{1}{m}, \\ N(x_n,y_m,t) &= N(y_m,x_n,t) = 1 - \left[\frac{1}{n}\wedge\frac{1}{m}\right], \end{aligned}$$

for all $n, m \geq 2$. It was shown in [1], $(X, M, N, *, \Diamond)$ is an intuitionistic fuzzy metric space. Moreover, the topology $\tau_{(M,N)}$ is the discret topology on X. Clearly A and B are closed and IF-bounded subset of X (note that actually they satisfies the stronger property that for each $t > 0$, $M(x_n, x_m, t) > \frac{1}{2}$, $M(y_n, y_m, t) > \frac{1}{2}$, $N(x_n, x_m, t) < \frac{1}{2}$, $N(y_n, y_m, t) < \frac{1}{2}$ for all $n, m \geq 2$). Since for each $n \geq 2$ and each $t > 0$, $M(x_n, B, t) = M(A, y_n, t) = \frac{1}{n}$ and $N(x_n, B, t) = N(A, y_n, t) = 1 - \frac{1}{n}$, we deduce that

$$\begin{aligned} \min\left\{\inf_{a\in A} M(a,B,t), \inf_{b\in B} M(A,b,t)\right\} &= 0, \\ \max\left\{\sup_{a\in A} N(a,B,t), \sup_{b\in B} N(A,b,t)\right\} &= 1, \end{aligned}$$

so $(H_M, H_N, *, \Diamond)$ is not intuitionistic fuzzy metric on $IF\text{-}CB_0(X)$.

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