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SOME GENERALIZED INEQUALITIES AND THEIR APPLICATIONS

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Abstract: Shannon inequalities are well known in information theory. In this paper, we have proposed some generalized inequalities in terms of independent variable s, and applied these inequalities in Shannon's entropy, Renyi's entropy and mutual information. Also Dragomir [2, 3] inequalities becomes the particular cases of our proposed inequalities.

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1. Introduction

Dragomir and Ionescu [4] in 1994 proved the following converse of Jensen's discrete inequality for convex mappings for real variable.

Theorem 1. *Let $f : I \subseteq \Re \to \Re$ be a differentiable convex function on the interval $I, x_i \in \dot{I}$ ($\dot{I}$ is the interior of I), $p_i \geq 0\,(i = 1, 2, ..., n)$ and $\sum_{i=1}^{n} p_i = 1$. Then, we have the inequality*

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$$0 \le \sum_{i=1}^{n} p_i f(x_i) - f\left\{\sum_{i=1}^{n} p_i x_i\right\}$$
$$\le \sum_{i=1}^{n} p_i x_i f'(x_i) - \sum_{i=1}^{n} p_i x_i \sum_{i=1}^{n} p_i f'(x_i) . \quad (1.1)$$

They also pointed out some natural applications of (1.1) in connection to the arithematic mean, geometric mean inequality, the generalized triangular inequality, etc.

Let X be a random variable with the range $R = \{x_1, x_2, ..., x_n\}$ and the probability distribution $p_1, p_2, ..., p_n \, (p_i > 0, i = 1, 2, ..., n)$. The Shannon's [6] entropy is $H(X) = -\sum\limits_{i=1}^{n} p_i \ln p_i$.

Let X be a random variable taking values in $R = \{x_1, x_2, ..., x_n\}$ and having the probability distribution $p_1, p_2, ..., p_n$. Consider the Renyi's [5] entropy of order $\alpha \, (\alpha \in (0,1) \cup (1,\infty))$ as $H_\alpha(X) = \frac{1}{1-\alpha} \ln\left(\sum\limits_{i=1}^{n} p_i^\alpha\right)$, $\alpha > 0$, $\alpha \ne 1$.

Mutual information is a measure of the amount of information that one random variable contains about another random variable. It is the reduction in the uncertianity of one random variable due to the knowledge of the other. It is given as [1, p. 18] that $I(X;Y) = \sum\limits_{x \in X} \sum\limits_{y \in Y} r(x,y) \ln\left[\frac{r(x,y)}{p(x)q(y)}\right]$, where X and Y are two random variables with a joint probability mass function $r(x,y)$ and marginal probability mass function $p(x)$ and $q(y)$.

2. Main Results

In this section, we proposed the theorems which are the generalizations of the results due to Dragomir, see [2], [3].

Theorem 2. *Let $f : [a,b] \to \Re$ be twice differentiable on (a,b), continuous in $[a,b]$ and $m \le x^{2-s} f''(x) \le M \forall x \in (a,b)$ and $s \in \Re$. If $x_i \in [a,b], i = 1, 2, ..., n$ and $p = p_i, i = 1, 2, ..., n$ is a probability distribution, then*

$$m\left[\sum_{i=1}^{n} p_i \phi_s(x_i) - \phi_s\left(\sum_{i=1}^{n} p_i x_i\right)\right] \le \sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right)$$
$$\le M\left[\sum_{i=1}^{n} p_i \phi_s(x_i) - \phi_s\left(\sum_{i=1}^{n} p_i x_i\right)\right] . \quad (2.1)$$

Proof. Let us consider the function

$$\phi_s(x) = \left\{ \begin{array}{c} [s(s-1)]^{-1}[x^s - 1 - s(x-1)]; \quad s \neq 0,1 \\ x - 1 - \ln x; \quad s = 0 \\ 1 - x + \ln x; \quad s = 1 \end{array} \right\}. \tag{2.2}$$

Then,

$$\phi_s'(x) = \left\{ \begin{array}{c} (s-1)^{-1}\left[x^{s-1} - 1\right]; \quad s \neq 0,1 \\ 1 - x^{-1}; \quad s = 0 \\ \ln x; \quad s = 1 \end{array} \right\}, \tag{2.3}$$

and

$$\phi_s''(x) = \left\{ \begin{array}{c} x^{s-2}; s \neq 0,1 \\ x^{-2}; s = 0 \\ x^{-1}; s = 1 \end{array} \right\}. \tag{2.4}$$

Here $\phi''(x) > 0$, $\forall x > 0$, hence $\phi''(x)$ is strictly convex for all $x > 0$.

Let us consider the function $g : [a,b] \to \Re$

$$g(x) = f(x) - m\phi_s(x), \quad x \in (a,b), \ s \in \Re,$$

$$g''(x) = f''(x) - m\phi_s''(x) = x^{s-2}\left(x^{2-s}f''(x) - m\right) \geq 0,$$

which shows that the mapping $g(\cdot)$is convex on $[a,b]$.

Applying Jensen's discrete inequality for the convex mapping $g(\cdot)$, i.e: $g\left(\sum\limits_{i=1}^{n} p_i x_i\right) \leq \sum\limits_{i=1}^{n} p_i g(x_i)$. Therefore

$$f\left(\sum_{i=1}^{n} p_i x_i\right) - m\phi_s\left(\sum_{i=1}^{n} p_i x_i\right) \leq \sum_{i=1}^{n} p_i\left[f(x_i) - m\phi_s(x_i)\right],$$

$$\sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) \geq m\left[\sum_{i=1}^{n} p_i \phi_s(x_i) - \phi_s\left(\sum_{i=1}^{n} p_i x_i\right)\right].$$

The first inequality in (2.1) is proved.

The proof of the second inequality goes likewise for the mapping $h : [a,b] \to \Re, h(x) = M\phi_s(x) - f(x)$ which is convex on $[a,b]$. □

Corollary 1. *Let $x_i, w_i > 0, (i = 1, 2, ...n)$ and put $W_n = \sum\limits_{i=1}^{n} w_i$. Also, consider arithmetic mean $A_n(w,a)$, i.e. we recall $A_n(w,a) = \frac{1}{W_n}\sum\limits_{i=1}^{n} w_i a_i$.*

If $x_i \in [m,M] \subset (0,\infty), i = 1, 2, .., n$ also $s \in \Re$, then we have the inequalities

$$\exp\left[\frac{1}{MW_n}\left\{\sum_{i=1}^{n} w_i\phi_s\left(x_i\right)-\phi_s\left(\sum_{i=1}^{n} w_i x_i\right)\right\}\right] \le \frac{\left[\prod_{i=1}^{n} x_i^{w_i x_i}\right]^{\frac{1}{W_n}}}{\left[A_n\left(w,a\right)\right]^{A_n\left(w,a\right)}}$$

$$\le \exp\left[\frac{1}{mW_n}\left\{\sum_{i=1}^{n} w_i\phi_s\left(x_i\right)-\phi_s\left(\sum_{i=1}^{n} w_i x_i\right)\right\}\right]. \quad (2.5)$$

Proof. Consider the mapping $f(x) = x\ln x, x > 0$. Then:

$$f'(x) = \ln x + 1, \quad x \in (0,\infty), \quad f''(x) = \frac{1}{x}, \quad x \in (0,\infty),$$

which shows that f is strictly convex on the interval $(0,\infty)$.

$$\inf_{x\in[m,M]} f''(x) = \frac{1}{M}, \qquad \sup_{x\in[m,M]} f''(x) = \frac{1}{m}.$$

Applying theorem 2, for this mapping and $p_i = \frac{w_i}{W_n}$ $(i = 1,2,...n)$, we deduce (2.5).

The case of equality follows by the strict convexity of the mappings $g(x) = x\ln x - \frac{1}{M}\phi_s(x), h(x) = \frac{1}{m}\phi_s(x) - x\ln x$ on (m, M). We shall omit the detail. □

Theorem 3. *Let $f : [a,b] \to \Re_+$ be twice differentiable on (a,b), contineous in $[a,b]$ and $m \le x^{2-s}f''(x) \le M$ for all $x \in [a,b]$ and $s \in \Re$. If $x_i \in [a,b], (i = 1,2,...,n)$ and $p = p_i\,(i = 1,2,...n)$ is a probability distribution, then we have the inequalities*

$$\frac{1}{2}\sum_{ij} p_i p_j\left(x_i-x_j\right)\left(f'\left(x_i\right)-f'\left(x_j\right)\right)+M\left[\sum_{i=1}^{n} p_i\phi_s\left(x_i\right)-\phi_s\left(\sum_{i=1}^{n} p_i x_i\right)\right.$$

$$\left.-\frac{1}{2}\sum_{ij} p_i p_j\left(x_i-x_j\right)\left(\phi_s'\left(x_i\right)-\phi_s'\left(x_j\right)\right)\right] \le \sum_{i=1}^{n} p_i f\left(x_i\right)-f\left(\sum_{i=1}^{n} p_i x_i\right)$$

$$\le \frac{1}{2}\sum_{ij} p_i p_j\left(x_i-x_j\right)\left(f'\left(x_i\right)-f'\left(x_j\right)\right)+m\left[\sum_{i=1}^{n} p_i\phi_s\left(x_i\right)-\phi_s\left(\sum_{i=1}^{n} p_i x_i\right)\right.$$

$$\left.-\frac{1}{2}\sum_{ij} p_i p_j\left(x_i-x_j\right)\left(\phi_s'\left(x_i\right)-\phi_s'\left(x_j\right)\right)\right]. \quad (2.6)$$

Proof. Consider the mapping

$$g:[a,b]\to\Re,\quad g(x)=f(x)-m\phi_s(x),\quad s\in\Re,x\in(a,b)\ ,$$

where $\phi_s(x)$ is given in (2.2).

Then g is twice differetiable on (a,b) and

$$g''(x)=f''(x)-m\phi_s''(x)=f''(x)-mx^{s-2}\,,$$
$$g''(x)=x^{s-2}\left[x^{2-s}f''(x)-m\right]\geq 0,\quad \forall x\in(a,b)\,,\ s\in\Re\,,$$

which shows that the mapping is convex on $x\in[a,b],s\in\Re$. Also $\phi_s''(x)$ is given by (2.4). We apply inequality (1.1) for the convex mapping g, i.e.

$$0\leq\sum_{i=1}^{n}p_ig(x_i)-g\left(\sum_{i=1}^{n}p_ix_i\right)\leq\frac{1}{2}\sum_{ij}p_ip_j(x_i-x_j)\left(g'(x_i)-g'(x_j)\right)\ .$$

To obtain

$$\begin{aligned}0&\leq\sum_{i=1}^{n}p_i\left[f(x_i)-m\phi_s(x_i)\right]-\left[f\left(\sum_{i=1}^{n}p_ix_i\right)-m\phi_s\left(\sum_{i=1}^{n}p_ix_i\right)\right]\\&\leq\frac{1}{2}\sum_{ij}p_ip_j(x_i-x_j)\left[f'(x_i)-f'(x_j)-m\phi_s'(x_i)+m\phi_s'(x_j)\right]\\&=\frac{1}{2}\sum_{ij}p_ip_j(x_i-x_j)\left(f'(x_i)-f'(x_j)\right)\\&\quad-\frac{m}{2}\sum_{ij}p_ip_j(x_i-x_j)\left(\phi_s'(x_i)-\phi_s'(x_j)\right)\ .\end{aligned}$$

Thus

$$\begin{aligned}&\sum_{i=1}^{n}p_if(x_i)-f\left(\sum_{i=1}^{n}p_ix_i\right)\\&\leq\frac{1}{2}\sum_{ij}p_ip_j(x_i-x_j)\left(f'(x_i)-f'(x_j)\right)+m\left[\sum_{i=1}^{n}p_i\phi_s(x_i)-\phi_s\left(\sum_{i=1}^{n}p_ix_i\right)\right.\\&\qquad\left.-\frac{1}{2}\sum_{ij}p_ip_j(x_i-x_j)\left(\phi_s'(x_i)-\phi_s'(x_j)\right)\right]\end{aligned}$$

and the second inequality is proved.

The proof of the first inequality goes likewise for the mapping $h:[a,b]\to\Re$, $h(x)=M\phi_s(x)-f(x)$. $\square$

The following corollary holds.

Corollary 2. *Let* $x_i\in[m,M]\subset(0,\infty)$ *and* $p_i>0\,(i=1,2,...,n)$ *with* $\sum\limits_{i=1}^{n}p_i=1$. *Then we have the inequality*

$$\frac{1}{2}\sum_{ij}p_ip_j\frac{(x_i-x_j)^2}{x_{ix_j}}+\frac{1}{m^2}\left[\sum_{i=1}^{n}p_i\phi_s(x_i)-\phi_s\left(\sum_{i=1}^{n}p_ix_i\right)\right.$$
$$\left.-\frac{1}{2}\sum_{ij}p_ip_j(x_i-x_j)\left(\phi_s'(x_i)-\phi_s'(x_j)\right)\right]\le\ln\left(\sum_{i=1}^{n}p_ix_i\right)-\sum_{i=1}^{n}p_i\ln x_i$$
$$\le\frac{1}{2}\sum_{ij}p_ip_j\frac{(x_i-x_j)^2}{x_{ix_j}}+\frac{1}{M^2}\left[\sum_{i=1}^{n}p_i\phi_s(x_i)-\phi_s\left(\sum_{i=1}^{n}p_ix_i\right)\right.$$
$$\left.-\frac{1}{2}\sum_{ij}p_ip_j(x_i-x_j)\left(\phi_s'(x_i)-\phi_s'(x_j)\right)\right]. \quad (2.7)$$

Proof. Consider the mapping $f:[m,M]\subset(0,\infty)\to\Re$ given by $f(x)=-\ln x$. Then

$$f'(x)=-\frac{1}{x},\quad f''(x)=\frac{1}{x^2}.$$

Also,

$$\inf_{x\in[m,M]}f''(x)=\frac{1}{M^2},\qquad\sup_{x\in[m,M]}f''(x)=\frac{1}{m^2}.$$

Apply (2.6) for this mapping, we can write

$$\frac{1}{2}\sum_{ij}p_ip\frac{(z_i-x_j)^2}{x_ix_j}+\frac{1}{m^2}\left[\sum_{i=1}^{n}p_i\phi_s(x_i)-\phi_s\left(\sum_{i=1}^{n}p_ix_i\right)\right.$$
$$\left.-\frac{1}{2}\sum_{ij}p_ip_j(x_i-x_j)\left(\phi_s'(x_i)-\phi_s'(x_j)\right)\right]\le\ln\left(\sum_{i=1}^{n}p_ix_i\right)-\sum_{i=1}^{n}p_i\ln x_i$$
$$\le\frac{1}{2}\sum_{ij}p_ip_j\frac{(x_i-x_j)^2}{x_{ix_j}}+\frac{1}{M^2}\left[\sum_{i=1}^{n}p_i\phi_s(x_i)-\phi_s\left(\sum_{i=1}^{n}p_ix_i\right)\right.$$
$$\left.-\frac{1}{2}\sum_{ij}p_ip_j(x_i-x_j)\left(\phi_s'(x_i)-\phi_s'(x_j)\right)\right],$$

which is equivalent to (2.7). The case of equality follows by the strict convexity of the mapping, $g(x) = -\ln x - \frac{1}{M^2}\phi_s(x), h(x) = \frac{1}{m^2}\phi_s(x) + \ln x$ on the interval $[m, M]$. We shall omit the details. □

Corollary 3. *Let $x_i \in [m, M] \subset (0, \infty)$, also $s \in \Re$, $p_i > 0$ $(i = 1, 2, .., n)$ with $\sum_{i=1}^{n} p_i = 1$. Then we have the inequaity*

$$\exp\left\{\frac{1}{2}\sum_{ij} p_i p \frac{(z_i - x_j)^2}{x_i x_j} + \frac{1}{m^2}\left[\sum_{i=1}^{n} p_i \phi_s(x_i) - \phi_s\left(\sum_{i=1}^{n} p_i x_i\right)\right.\right.$$

$$\left.\left. -\frac{1}{2}\sum_{ij} p_i p_j (x_i - x_j)\left(\phi_s'(x_i) - \phi_s'(x_j)\right)\right]\right\} \le \frac{A_n(p, x)}{G_n(p, x)}$$

$$\le \exp\frac{1}{2}\left\{\sum_{ij} p_i p \frac{(z_i - x_j)^2}{x_i x_j} + \frac{1}{M^2}\left[\sum_{i=1}^{n} p_i \phi_s(x_i) - \phi_s\left(\sum_{i=1}^{n} p_i x_i\right)\right.\right.$$

$$\left.\left. -\frac{1}{2}\sum_{ij} p_i p_j (x_i - x_j)\left(\phi_s'(x_i) - \phi_s'(x_j)\right)\right]\right\}. \quad (2.8)$$

The equality holds in (2.8) iff $x_1 = x_2 = ... = x_n$.

The proof is obvious by (2.7). Also, $A_n(p, x) = \sum_{i=1}^{n} p_i x_i$ (arithmetic mean), $G_n(p, x) = \prod_{i=1}^{n} x_i^{p_i}$ (geometric mean).

If in (2.8), we put intead of x, $\frac{1}{x}$, we obtain the following corollary.

Corollary 4. *Let x_i, p_i $(i = 1, 2, ...n)$ be as in Corollary 3. Then we have the inequality*

$$\exp\left\{\frac{1}{2}\sum_{ij} p_i p \frac{(z_i - x_j)^2}{x_i x_j} + \frac{1}{m^2}\left[\sum_{i=1}^{n} p_i \phi_s(x_i) - \phi_s\left(\sum_{i=1}^{n} p_i x_i\right)\right.\right.$$

$$\left.\left. -\frac{1}{2}\sum_{ij} p_i p_j (x_i - x_j)\left(\phi_s'(x_i) - \phi_s'(x_j)\right)\right]\right\} \le \frac{G_n(p, x)}{H_n(p, x)}$$

$$\le \exp\left\{\frac{1}{2}\sum_{ij} p_i p \frac{(z_i - x_j)^2}{x_i x_j} + \frac{1}{M^2}\left[\sum_{i=1}^{n} p_i \phi_s(x_i) - \phi_s\left(\sum_{i=1}^{n} p_i x_i\right)\right.\right.$$

$$-\left.\left.\frac{1}{2}\sum_{ij} p_i p_j (x_i - x_j)\left(\phi_s'(x_i) - \phi_s'(x_j)\right)\right]\right\}. \quad (2.9)$$

The equality holds in (2.9), iff $x_1 = x_2 = ... = x_n$.
Also, $H_n(p,x) = \frac{1}{\sum\limits_{i=1}^{n} \frac{p_i}{x_i}}$ *(harmonic mean).*

3. Application in Shannon's Entropy

The following analytic inequality for the logarithemic mapping holds.

Lemma 1. *Let* $\xi_i \in [m, M] \subset (0, \infty)$, $p_i > 0$, $(i = 1, 2, ...n)$ *with* $\sum\limits_{i=1}^{n} p_i = 1$ *and* $s \in \Re$. *Then we have inequality*

$$\frac{1}{M}\left[\sum_{i=1}^{n} p_i \phi_s(\xi_i) - \phi_s\left(\sum_{i=1}^{n} p_i \xi_i\right)\right] \le \sum_{i=1}^{n} p_i \xi_i \ln \xi_i - \sum_{i=1}^{n} p_i \xi_i \ln\left(\sum_{i=1}^{n} p_i \xi_i\right) \le \frac{1}{m}\left[\sum_{i=1}^{n} p_i \phi_s(\xi_i) - \phi_s\left(\sum_{i=1}^{n} p_i \xi_i\right)\right]. \quad (3.1)$$

The case of equality holds iff $\xi_1 = \xi_2 = ... = \xi_n$.

The proof is obvious by Theorem 2 for the convex mapping

$$f : [0, \infty) \to \Re, f(x) = x \ln x.$$

Corollary 5. *Under the assumptions for* $\xi_i\,(i = 1, 2, ..., n)$, *we have*

$$0 \le \frac{1}{M}\left[\sum_{i=1}^{n} \phi_s(\xi_i) - \phi_s\left(\sum_{i=1}^{n} \xi_i\right)\right] \le \sum_{i=1}^{n} \xi_i \ln \xi_i - \sum_{i=1}^{n} \xi_i \ln\left(\sum_{i=1}^{n} \frac{1}{n} \xi_i\right) \le \frac{1}{m}\left[\sum_{i=1}^{n} \phi_s(\xi_i) - \phi_s\left(\sum_{i=1}^{n} \xi_i\right)\right], \quad (3.2)$$

equality holds iff $\xi_1 = \xi_2 = ... = \xi_n$.

The proof is obvious by Lemma 1, choosing $p_i = \frac{1}{n}\,(i = 1, 2, ..., n)$.

Theorem 4. *Let* X *be a random variable with probability distribution* $p_i\,(i = 1, 2, ..., n)$.

Assume that $p = \min\{p_i/i = 1, 2, ..., n\} > 0$ *and* $P = \max\{p_i/i = 1, 2, ..., n\} < 1$, *then:*

$$\frac{1}{2}\sum_{ij} p_i p_j (p_j - p_i)^2 + P^2 \left[\sum_{i=1}^{n} p_i \phi_s \left(\frac{1}{p_i} \right) - \phi_s (n) \right.$$
$$\left. -\frac{1}{2}\sum_{ij} (p_j - p_i) \left\{ \phi_s' \left(\frac{1}{p_i} \right) - \phi_s' \left(\frac{1}{p_j} \right) \right\} \right] \le \ln (n) - H (X)$$
$$\le \frac{1}{2}\sum_{ij} p_i p_j (p_j - p_i)^2 + p^2 \left[\sum_{i=1}^{n} p_i \phi_s \left(\frac{1}{p_i} \right) - \phi_s (n) \right.$$
$$\left. -\frac{1}{2}\sum_{ij} (p_j - p_i) \left\{ \phi_s' \left(\frac{1}{p_i} \right) - \phi_s' \left(\frac{1}{p_j} \right) \right\} \right] . \quad (3.3)$$

Proof. If we choose $x_i = \frac{1}{p_i} \in \left[\frac{1}{P}, \frac{1}{p} \right]$ in (2.7), we can deduce with $\left(m = \frac{1}{P}, M = \frac{1}{p} \right)$ (3.3). □

4. Application in Renyi's Entropy

(i) If $x_i = p_i^{\alpha-1} (i = 1, 2, ..., n) ; \alpha \in (0, 1)$, then $P^{\alpha-1} \le x_i \le p_i^{\alpha-1}$, then by (2.7), we deduce that:

$$\frac{1}{2}\sum_{ij} p_i p_j \frac{\left(p_i^{\alpha-1} - p_j^{\alpha-1} \right)}{p_i^{\alpha-1} p_j^{\alpha-1}} + \frac{1}{P^{2(\alpha-1)}} \left[\sum_{i=1}^{n} p_i \phi_s \left(p_i^{\alpha-1} \right) - \phi_s \left(\sum_{i=1}^{n} p_i^{\alpha} \right) \right.$$
$$\left. -\frac{1}{2}\sum_{ij} p_i p_j \left(p_i^{\alpha-1} - p_j^{\alpha-1} \right) \left(\phi_s' \left(p_i^{\alpha-1} \right) - \phi_s' \left(p_j^{\alpha-1} \right) \right) \right]$$
$$\le \ln \left(\sum_{i=1}^{n} p_i^{\alpha} \right) - \sum_{i=1}^{n} p_i \ln p_i^{\alpha-1}$$
$$\le \frac{1}{2}\sum_{ij} p_i p_j \frac{\left(p_i^{\alpha-1} - p_j^{\alpha-1} \right)}{p_i^{\alpha-1} p_j^{\alpha-1}} + \frac{1}{p^{2(\alpha-1)}} \left[\sum_{i=1}^{n} p_i \phi_s \left(p_i^{\alpha-1} \right) - \phi_s \left(\sum_{i=1}^{n} p_i^{\alpha} \right) \right.$$
$$\left. - \frac{1}{2}\sum_{ij} p_i p_j \left(p_i^{\alpha-1} - p_j^{\alpha-1} \right) \left(\phi_s' \left(p_i^{\alpha-1} \right) - \phi_s' \left(p_j^{\alpha-1} \right) \right) \right]$$

$$\frac{1}{2}\sum_{ij} p_i p_j \frac{\left(p_i^{\alpha-1}-p_j^{\alpha-1}\right)}{p_i^{\alpha-1}p_j^{\alpha-1}} + \frac{1}{P^{2(\alpha-1)}}\left[\sum_{i=1}^{n} p_i \phi_s\left(p_i^{\alpha-1}\right) - \phi_s\left(\sum_{i=1}^{n} p_i^{\alpha}\right)\right.$$

$$\left. -\frac{1}{2}\sum_{ij} p_i p_j \left(p_i^{\alpha-1}-p_j^{\alpha-1}\right)\left(\phi_s'\left(p_i^{\alpha-1}\right)-\phi_s'\left(p_j^{\alpha-1}\right)\right)\right]$$

$$\leq (1-\alpha)\left[H_\alpha(X)-H(X)\right]$$

$$\leq \frac{1}{2}\sum_{ij} p_i p_j \frac{\left(p_i^{\alpha-1}-p_j^{\alpha-1}\right)}{p_i^{\alpha-1}p_j^{\alpha-1}} + \frac{1}{p^{2(\alpha-1)}}\left[\sum_{i=1}^{n} p_i \phi_s\left(p_i^{\alpha-1}\right) - \phi_s\left(\sum_{i=1}^{n} p_i^{\alpha}\right)\right.$$

$$\left. -\frac{1}{2}\sum_{ij} p_i p_j \left(p_i^{\alpha-1}-p_j^{\alpha-1}\right)\left(\phi_s'\left(p_i^{\alpha-1}\right)-\phi_s'\left(p_j^{\alpha-1}\right)\right)\right].$$

(ii) If $x_i = p_i^{\alpha-1}\,(i=1,2,...,n)\,,\alpha\in(1,\infty)$, then $p^{\alpha-1}\leq x_i\leq P^{\alpha-1}$, then by (2.7), we deduce that:

$$\frac{1}{2}\sum_{ij} p_i p_j \frac{\left(p_i^{\alpha-1}-p_j^{\alpha-1}\right)}{p_i^{\alpha-1}p_j^{\alpha-1}} + \frac{1}{P^{2(\alpha-1)}}\left[\sum_{i=1}^{n} p_i \phi_s\left(p_i^{\alpha-1}\right) - \phi_s\left(\sum_{i=1}^{n} p_i^{\alpha}\right)\right.$$

$$\left. -\frac{1}{2}\sum_{ij} p_i p_j \left(p_i^{\alpha-1}-p_j^{\alpha-1}\right)\left(\phi_s'\left(p_i^{\alpha-1}\right)-\phi_s'\left(p_j^{\alpha-1}\right)\right)\right]$$

$$\leq (\alpha-1)\left[H(X)-H_\alpha(X)\right]$$

$$\leq \frac{1}{2}\sum_{ij} p_i p_j \frac{\left(p_i^{\alpha-1}-p_j^{\alpha-1}\right)}{p_i^{\alpha-1}p_j^{\alpha-1}} + \frac{1}{p^{2(\alpha-1)}}\left[\sum_{i=1}^{n} p_i \phi_s\left(p_i^{\alpha-1}\right) - \phi_s\left(\sum_{i=1}^{n} p_i^{\alpha}\right)\right.$$

$$\left. -\frac{1}{2}\sum_{ij} p_i p_j \left(p_i^{\alpha-1}-p_j^{\alpha-1}\right)\left(\phi_s'\left(p_i^{\alpha-1}\right)-\phi_s'\left(p_j^{\alpha-1}\right)\right)\right].$$

5. Application in Mutual Information

Theorem 5. *Let X and Y be two random variables with a joint probability mass function $r(x,y)$ and marginal probability mass function $p(x)$ and $q(y)$ respectively, also $0<m\leq \frac{r(x,y)}{p(x)q(y)}\leq M<\infty$ for all $(x,y)\in X\times Y$. Then we have the inequalities*

$$\frac{1}{M}\left[\sum_{(x,y)\in X\times Y} p(x)\,q(y)\,\phi_s\left(\frac{r(x,y)}{p(x)\,q(y)}\right)\right] \le I(X;Y) \le \frac{1}{m}\left[\sum_{(x,y)\in X\times Y} p(x)\,q(y)\,\phi_s\left(\frac{r(x,y)}{p(x)\,q(y)}\right)\right]. \quad (5.1)$$

Proof. Choosing $p_i = p(x)\,q(y)$, $\xi_i = \frac{r(x,y)}{p(x)q(y)}$ $(x, y \in X \times Y)$ in Lemma 1, and taking $\sum_{(x,y)\in X\times Y} p(x)\,q(y)\,\frac{r(x,y)}{p(x)q(y)} \ln \frac{r(x,y)}{p(x)q(y)} = I(X;Y)$.

Also, $\phi_s\left(\sum_{(x,y)\in X\times Y} p(x)\,q(y)\,\frac{r(x,y)}{p(x)q(y)}\right) = \phi_s(1) = 0$.

Then we have the desired inequality. □

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