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A NOTE ON EXACT BANACH FRAMES

S.K. Kaushik

Department of Mathematics
Kirorimal College
University of Delhi
Delhi, 110 007, INDIA
e-mail: shikk2003@yahoo.co.in

Abstract: Exact Banach frames of type w has been introduced and studied. It has been proved that an exact Banach frame is exact of type w. The converse need not be true. A characterization for Banach frames which are exact of type w has been given. Finally, it has been proved that a block perturbation of an exact Banach frame is an exact Banach frame.

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1. Introduction

Frames for Hilbert spaces were introduced by Duffin and Schaeffer [4], while addressing some deep problems in non-harmonic Fourier series. Gröchenig [6] generalized frames for Banach spaces and called them Banach frames. Banach frames were further studied in [1, 2, 3, 5, 7, 8].

In the present paper, exact Banach frames of type w have been introduced and studied. It has been proved that an exact Banach frame is always exact of type w. However, the converse need not be true. Examples has been given to establish various non-implications among Banach frames which are exact, weak exact and exact of type w. A characterization for Banach frames which are exact of type w has been given. Finally, it has been proved that a block perturbation of an exact Banach frame is an exact Banach frame.

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2. Preliminaries

Throughout this paper E will denote a Banach space over the scalar field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), E^* and E^{**}, respectively, the first and the second conjugate spaces of E, $[x_n]$ the closed linear span of $\{x_n\}$ in the norm topology of E, $\widetilde{[f_n]}$ the closed linear span of $\{f_n\}$ in the $\sigma(E^*, E)$-topology, E_d an associated Banach space of scalar valued sequences indexed by $\mathbb{N}$.

A sequence $\{f_n\}$ in E^* is total over E if $\{x \in E : f_n(x) = 0,\, n \in \mathbb{N}\} = \{0\}$.

Definition 2.1. (see [6]) Let E be a Banach space and E_d be an associated Banach space of scalar-valued sequences, indexed by $\mathbb{N}$. Let $\{f_n\} \subset E^*$ and $S : E_d \to E$ be given. The pair $(\{f_n\}, S)$ is called a *Banach frame for E with respect to E_d* if:

1. $\{f_n(x)\} \in E_d$, for each $x \in E$.
2. there exist positive constants A and B with $0 < A \leq B < \infty$ such that

$$A\|x\|_E \leq \|\{f_n(x)\}\|_{E_d} \leq B\|x\|_E\,, \qquad x \in E\,, \tag{2.1}$$

S is a bounded linear operator such that

$$S(\{f_n(x)\}) = x\,, \qquad x \in E\,.$$

The positive constants A and B, respectively, are called *lower* and *upper frame bounds* of the Banach frame $(\{f_n\}, S)$. The operator $S : E_d \to E$ is called the *reconstruction operator* (or, the *pre-frame operator*). The inequality (2.1) is called the *frame inequality*.

The Banach frame $(\{f_n\}, S)$ is called *tight* if $A = B$ and *normalized tight* if $A = B = 1$. If removal of one f_n renders the collection $\{f_n\} \subset E^*$ no longer a Banach frame for E, then $(\{f_n\}, S)$ is called an *exact Banach frame*.

We give below two results in the form of lemmas which will be used in this paper.

Lemma 2.2. (see [9]) *If E is a Banach space and $\{f_n\} \subset E^*$ is total over E, then E is linearly isometric to the BK-space $E_d = \{\{f_n(x)\} : x \in E\}$, where the norm is defined by $\|\{f_n(x)\}\|_{E_d} = \|x\|_E$, $x \in E$.*

Lemma 2.3. (see [7]) *Let $(\{f_n\}, S)$ $(\{f_n\} \subset E^*, S : E_d \to E)$ be a Banach frame for E with respect to E_d. Then $(\{f_n\}, S)$ is exact if and only if $f_n \notin \widetilde{[f_i]}_{i \neq n}$, for all n.*

3. Main Results

Definition 3.1. A Banach frame $(\{f_n\}, S)$ $(\{f_n\} \subset E^*, S : E_d \to E)$ for E with respect to E_d is said to be *exact of type w* if

$$\{\alpha_n\} \subset \mathbb{K},\ \sum_{n=1}^{\infty} \alpha_n f_n(x) = 0 \ \text{ for all } \ x \in E \Rightarrow \alpha_n = 0,\ \forall\, n \in \mathbb{N}\,.$$

Regarding the existence of Banach frames which are exact to type w, we give below the following examples.

Example 3.2. Let $E = c$ and let $\{e_n\}$ be the sequence of unit vectors in E. Define $\{g_n\} \subset E^*$ by

$$g_n(x) = \xi_n\,, \quad (x = \{\xi_n\} \subset E,\ n \in \mathbb{N})\,.$$

Then $\{g_n\}$ is total over E and $g_i(e_j) = \delta_{ij}$, for all $i, j \in \mathbb{N}$. Since $[e_n] = c_0 \subsetneqq c$, there exists a non-zero functional $g \in E^*$ and a $y \in E \setminus c_0$ such that $g(e_n) = 0$, for all $n \in \mathbb{N}$ and $g(y) \neq 0$. Define $\{f_n\} \subset E^*$ by

$$f_1 = g \quad \text{and} \quad f_n = g_{n-1}, \quad \text{for all} \quad n = 2, 3, \dots\,.$$

Then $\{f_n\}$ is total over E. Therefore, by Lemma 2.2, there exists an associated Banach space $E_d = \{\{f_n(x)\} : x \in E\}$ equipped with norm given by $\|\{f_n(x)\}\|_{E_d} = \|x\|_E$, $x \in E$. Define $S : E_d \to E$ by $S(\{f_n(x)\}) = x$, $x \in E$. Then S is a bounded linear operator such that $(\{f_n\}, S)$ is a normalized tight Banach frame for E with respect to E_d.

Let $\{\alpha_n\} \subset \mathbb{K}$ be such that $\sum\limits_{n=1}^{\infty} \alpha_n f_n(x) = 0$, for all $x \in E$. Then

$$\alpha_1 g(x) + \sum_{n=1}^{\infty} \alpha_{n+1} g_n(x) = 0, \quad \text{for all} \quad x \in E$$

Put $x = e_n$, $n \in \mathbb{N}$. Then $\alpha_n = 0$, $n = 2, 3, \dots$. Also for $x = y$, $\alpha_1 = 0$. Thus $(\{f_n\}, S)$ is exact of type w.

Example 3.3. Let $E = c_0$ and $\{e_n\}$ be the sequence of unit vectors in E^*. Define $\{f_n\} \subset E^*$ by

$$f_1 = e_1,\ f_n = \frac{1}{2} e_{n-1} - \frac{1}{2} e_n\,, \quad n = 2, 3, \dots\,.$$

Then, by Lemma 2.2, there exists an associated Banach space $E_d = \{\{f_n(x)\} : x \in E\}$ equipped with norm given by $\|\{f_n(x)\}\|_{E_d} = \|x\|_E$. Define $S : E_d \to E$

by $S(\{f_n(x)\}) = x$, $x \in E$. Then S is a bounded linear operator such that $(\{f_n\}, S)$ is a Banach frame for E with respect to E_d. Also since

$$\frac{1}{2} f_1(x) - \sum_{n=2}^{\infty} f_n(x) = 0, \quad x \in E\,,$$

the Banach frame $(\{f_n\}, S)$ is not exact of type w.

Theorem 3.4. *An exact Banach frame for a Banach space E is exact of type w.*

Proof. Let $(\{f_n\}, S)$ $(\{f_n\} \subset E, S : E_d \to E)$ be an exact Banach frame. Then, by Lemma 2.3, there exists a sequence $\{x_n\} \subset E$ such that $f_n(x_n) = 1$, $n \in \mathbb{N}$ and $f_i(x_n) = 0$ for each $i \in \mathbb{N}$ such that $i \neq n$. Let $\{\alpha_n\} \subset \mathbb{K}$ be such that $\sum\limits_{n=1}^{\infty} \alpha_n f_n(x) = 0$, for all $x \in E$. Then for each $n \in \mathbb{N}$, $\alpha_n = \sum\limits_{i=1}^{\infty} \alpha_i f_i(x_n) = 0$. □

Remark 1. The converse of Theorem 3.4 need not be true. Indeed, in Example 3.3, the Banach frame $(\{f_n\}, S)$ is exact of type w. But since $f_1 \in E^* = \widetilde{[g_n]} = \widetilde{[f_n]}_{i \neq 1}$, it follows from Lemma 2.3 that $(\{f_n\}, S)$ is not exact.

Definition 3.5. A Banach frame $(\{f_n\}, S)$ $(\{f_n\} \subset E^*, S : E_d \to E)$ for E with respect to E_d is said to be *weak exact* if there exists a sequence $\{\phi_n\} \subset E^{**}$ such that $\phi_i(f_j) = \delta_{ij}$, for all $i, j \in \mathbb{N}$.

In view of Lemma 2.3 one may observe that an exact Banach frame for E is weak exact. The converse need not be true (Example 3.6). However the converse is true for reflexive Banach spaces.

Since an exact Banach frame is exact of type w, it is interesting to investigate the relationship between weak exact Banach frames and Banach frames which are exact of type w. The following example shows that a weak exact Banach frame need not be exact of type w.

Example 3.6. Let $E = c_0$ and $(\{f_n\}, S)$ $(\{f_n\} \subset E^*, S : E_d \to E)$ be the Banach frame for E with respect to E_d as given in Example 3.3. Define $\{\phi_n\} \subseteq E^{**}$ by

$$\phi_1 = (1, 1, \ldots) \ \text{ and } \ \phi_n = (0, 0, \ldots, \underset{\substack{\downarrow \\ n\text{-th place}}}{-2}, -2, -2, \ldots), \quad n = 2, 3, \ldots .$$

Then $\phi_i(f_j) = \delta_{ij}$, for all $i, j \in \mathbb{N}$. Therefore $(\{f_n\}, S)$ is a weak exact Banach frame for E. Also since

$$\frac{1}{2} f_1(x) - \sum_{i=2}^{\infty} f_i(x) = 0, \quad \text{for all } \ x \in E\,,$$

$(\{f_n\}, S)$ is not exact of type w.

Also a Banach frame which is exact of type w need not be a weak exact Banach frame (Example 3.2).

Definition 3.7. Let $(\{f_n\}, S)$ $(\{f_n\} \subset E^*, S : E_d \to E)$ be a Banach frame for E which is exact of type w. The Banach frame $(\{f_n\}, S)$ is said to satisfy *property* (M) if there exists no Banach frame $(\{g_n\}, T)$ $(\{g_n\} \subset E^*, T : E_{d_0} \to E)$ with $\{f_n\} \subsetneqq \{g_n\}$ that is exact of type w.

The following example gives the existence of a Banach frame satisfying property (M).

Example 3.8. Let $E = c_0$ and let $\{e_n\}$ be the sequence of unit vector in E^*. Define $\{f_n\} \subset E^*$ by

$$f_1 = (1, 0 \ldots) \text{ and } f_n = ((-1)^{n+1}, 0, 0, \underset{\substack{\downarrow \\ n\text{-th place}}}{1}, 0, 0, \ldots), \quad n = 2, 3, \ldots .$$

Then, by Lemma 2.2, there exists an associated Banach space $E_d = \{\{f_n(x)\} : x \in E\}$ and a reconstruction operator $S : E_d \to E$ given by $S(\{f_n(x)\}) = x$, $(x \in E)$ such that $(\{f_n\}, S)$ is a Banach frame for E with respect to E_d. Let $\{\alpha_n\} \subset \mathbb{K}$ be such that $\sum_{n=1}^{\infty} \alpha_n f_n(x) = 0$, for all $x \in E$. Then

$$\alpha_1 f_1(x) + \sum_{n=2}^{\infty} \alpha_n f_n(x) = 0\,, \quad \text{for all } \ x \in E\,.$$

Taking $x = (1, 0, 0, \ldots)$, we get $\alpha_1 = 0$. Also taking $x = e_j$ $(j = 2, 3, \ldots)$, we obtain $\alpha_j = 0$ for each $j = 2, 3, \ldots$. Thus $(\{f_n\}, S)$ is exact of type w. Since $[f_n] = E^*$, $(\{f_n\}, S)$ satisfies property (M).

Regarding Banach frames for E which does not satisfy property (M), we give the following example.

Example 3.9. Let $E = \ell^2$ and let $\{e_n\}$ be the unit vector sequence in E. Define $\{g_n\} \subset E^*$ by $g_n = \frac{1}{2}(e_1 + e_{n+1})$, $n \in \mathbb{N}$. Then $[g_n] = E^*$. Therefore, by Lemma 2.2, there exists an associated Banach space $E_{d_1} = \{\{g_n(x)\} : x \in E\}$ and a reconstruction operator $S_1 : E_{d_1} \to E$ given by $S_1(\{g_n(x)\}) = x$, $x \in E$ such that $(\{g_n\}, S_1)$ is a Banach frame for E with respect to E_{d_1}. Also $(\{g_n\}, S_1)$ is exact of type w. Let $f \in E^*$ be such that it does not admit any expansion of the form $f(x) = \sum_{n=1}^{\infty} \alpha_n g_n(x)$, $x \in E$, where $\{\alpha_n\} \subset \mathbb{K}$. Define $\{f_n\} \subset E^*$ by

$f_1 = f$, $f_n = g_{n-1}$, $n = 2, 3, \ldots$. Then $\{f_n\}$ is total over E. So, by Lemma 2.2 again, there exists an associated Banach space $E_d = \{\{f_n(x)\} : x \in E\}$ and a reconstruction operator $S : E_d \to E$ given by $S(\{f_n(x)\}) = x$, $x \in E$ such that $(\{f_n\}, S)$ is a Banach frame for E with respect to E_d. Let $\{\beta_n\} \subset \mathbb{K}$ be such that $\sum_{n=1}^{\infty} \beta_n f_n(x) = 0$, for all $x \in E$. Then

$$\beta_1 f_1(x) + \sum_{n=1}^{\infty} \beta_{n+1} g_n(x) = 0\,, \quad \text{for all } x \in E\,.$$

If $\beta_1 = 0$, then $\sum_{n=1}^{\infty} \beta_{n+1} g_n(x) = 0$. Since $(\{g_n\}, S_1)$ is exact of type w, $\beta_{n+1} = 0$ for all $n \in \mathbb{N}$. Also if $\beta_1 \neq 0$, then $f(x) = \sum_{i=1}^{\infty} \gamma_i g_i(x)$, $x \in E$, where $\gamma_i = \beta_{i+1}/\beta_i$, $i \in \mathbb{N}$. This is a contradiction. Hence $(\{f_n\}, S)$ is exact of type w. Also $\{g_n\} \subsetneqq \{f_n\}$. Therefore $(\{g_n\}, S_1)$ is a Banach frame which does not satisfy property (M).

Theorem 3.10. *Let $(\{f_n\}, S)$ $(\{f_n\} \subset E^*, S : E_d \to E)$ be a Banach frame for E with respect to E_d which is exact of type w. Then $(\{f_n\}, S)$ satisfies property (M) if and only if for every $f \in E^*$ there exists a unique sequence $\{\alpha_n\} \subset \mathbb{K}$ such that*

$$f(x) = \sum_{i=1}^{\infty} \alpha_i f_i(x), \quad \text{for all } x \in E\,. \tag{3.1}$$

Proof. Suppose $f \in E^*$ does not admit any expansion of the form (3.1) for all $x \in E$. Define $\{g_n\} \subset E^*$ by $g_1 = f$ and $g_n = f_{n-1}$, $n = 2, 3, \ldots$. Then $\{f_n\} \subsetneqq \{g_n\}$. Let $\{a_n\} \subset \mathbb{K}$ be such that $\sum_{n=1}^{\infty} a_n g_n(x) = 0$, for all $x \in E$. Then

$$a_1 f(x) + \sum_{n=1}^{\infty} a_{n+1} f_n(x) = 0, \quad \text{for all } x \in E\,.$$

If $a_1 = 0$, then $a_{n+1} = 0$ for all $n \in \mathbb{N}$, since $(\{f_n\}, S)$ is exact of type w. If $a_1 \neq 0$, then f admits an expansion of the form (3.1) which is a contradiction. Hence $(\{f_n\}, S)$ does not satisfy property (M).

Conversely suppose $(\{f_n\}, S)$ does not satisfy property (M). Let $\{g_n\} \subset E^*$ be as defined above. Then f does not admit any expansion of the form (3.1) for all $x \in E$ because otherwise

$$f(x) + \sum_{n=1}^{\infty} -\alpha_n f_n(x) = 0\,, \quad \text{for all} \quad x \in E\,.$$

This is a contradiction. □

4. Block Perturbation of Exact Banach Frames

Let $\{f_n\}$ be sequence in E^* and let $\{m_n\}$, $\{p_n\}$ are increasing sequence of positive integers such that $m_0 = 0$ and $m_{n-1} + 1 \leq p_n \leq m_n$, for all $n \in \mathbb{N}$.

Define $\{g_n\} \subset E^*$ by

$$g_k = \begin{cases} f_k & \text{if } \ k \neq p_n \,, \\ f_{p_n} + h_n & \text{if } \ k = p_n, \ n \in \mathbb{N} \,, \end{cases} \tag{4.1}$$

where

$$h_n = \sum_{i=m_{n-1}+1}^{p_n - 1} \alpha_i f_i + \sum_{i=p_n+1}^{m_n} \alpha_i f_i \quad \text{and} \quad \|g_n\| \leq M < \infty \text{ for all } n \,.$$

$\{g_n\}$ is called *block-perturbation* of $\{f_n\}$.

Theorem 4.1. *Let $(\{f_n\}, S)$ $(\{f_n\} \subset E^*, S : E_d \to E)$ be a Banach frame for E with respect to E_d. Let $\{g_n\}$ be a block perturbation of $\{f_n\}$ given by* (4.1). *Then there exists an associated Banach space E_{d_1} and a reconstruction operator $S_1 : E_{d_1} \to E$ such that $(\{g_n\}, S_1)$ is a Banach frame for E with respect to E_{d_1}. Moreover, if $(\{f_n\}, S)$ is exact then $(\{g_n\}, S_1)$ is also exact.*

Proof. Suppose $g_k(x) = 0$ for all $k \in \mathbb{N}$. Then, by (4.1)

$$\begin{cases} f_k(x) = 0 \,, & \text{for all } k \neq p_n, n \in \mathbb{N} \,, \\ f_{p_n}(x) = -h_n(x) \,, & n \in \mathbb{N} \,. \end{cases}$$

This gives $f_k(x) = 0$ for all $k \in \mathbb{N}$. Therefore, by frame inequality for the Banach frame $(\{f_n\}, S)$, $x = 0$. Then, by Lemma 2.2, there exists an associated Banach space $E_{d_1} = \{\{g_n(x)\} : x \in E\}$ and a reconstruction operator $S_1 : E_{d_1} \to E$ given by $S_1(\{g_n(x)\}) = x$, $x \in E$ such that $(\{g_n\}, S_1)$ is a Banach frame for E with respect to E_{d_1}. Now since the Banach frame $(\{f_n\}, S)$ is exact, by Lemma 2.3, $f_n \notin \widetilde{[f_i]}_{i \neq n}$. Therefore there exists a sequence $\{x_n\} \subset E$ such that $f_i(x_j) = \delta_{ij}$, for all $i, j \in \mathbb{N}$. Define $\{y_n\} \subset E$ by $y_k = x_k - \alpha_k x_{p_k}$ for $k \neq p_n$, $m_{n-1} + 1 \leq k \leq m_n$, $n \in \mathbb{N}$ and $y_k = x_{p_n}$ for $k = p_n$, $n \in \mathbb{N}$. Then $g_i(y_j) = \delta_{ij}$, for all $i, j \in \mathbb{N}$. So $g_n \notin \widetilde{[g_i]}_{i \neq n}$. Therefore, again by Lemma 2.3, the Banach frame $(\{g_n\}, S_1)$ is exact. □

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