International Journal of Pure and Applied Mathematics
Volume 31 No. 2 2006, 185-202

ON SOME RESULTS IN WEIGHTED SPACES UNDER CHICCO TYPE CONDITIONS

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Abstract: In the present paper we consider weighted spaces where the weight is related to the *distance* function from a fixed subset S of $\partial\Omega$.

In unbounded domains we study Dirichlet problem for linear elliptic equations in nondivergence form with discontinuous coefficients when the class of discontinuity is of Chicco type.

In particular we state some local and non local a priori bounds and study the dependence of the constants in the estimates. The coefficients of lower terms in the differential operator belong to weighted spaces and the principal coefficients are 'near' to functions satisfying a condition of Chicco type. The conditions we impose on the coefficients allow us to apply embedding results to get local estimates.

AMS Subject Classification: 35J25, 46E35
Key Words: elliptic equations, embedding theorems, a priori bounds, weighted spaces, Chicco condition

1. Introduction

In literature we can find several papers on Dirichlet problem for linear second order elliptic equations in nondivergence form in bounded open sets. We refer to the well known paper of C. Miranda [26], where the derivatives of the coefficients a_{ij} belong to the L^n spaces. Subsequent results were stated, for example, in [22], [24], [33].

Received: July 10, 2006

Generalizations of Miranda's result can be found in [2], [16], [17], [18] in wider classes of spaces while different classes of discontinuous operators were studied in [19], [20], [21], [28].

When Ω is an unbounded open set, the problem was studied in more general spaces than L^n spaces in [29], in spaces of Morrey type in [11], [12], [13] and in weighted spaces in [4], [5], [6], [9], [15].

Basic tools for proving existence and, sometimes, uniqueness of solutions of elliptic boundary value problems in Sobolev spaces are a priori bounds.

In this paper we state some a priori bounds for solutions of the problem

$$\begin{cases} Lu = f\,, \qquad f \in L^2_s(\Omega)\,, \\ u \in W^2_s(\Omega) \cap \overset{o}{W}{}^1_{s-1}(\Omega)\,, \end{cases} \tag{1.1}$$

where L is the operator

$$Lu = -\sum_{i,j=1}^{n} a_{ij}\, u_{x_i x_j} + \sum_{i=1}^{n} a_i\, u_{x_i} + a\, u \tag{1.2}$$

and $W^2_s(\Omega)$, $\overset{o}{W}{}^1_{s-1}(\Omega)$ and $L^2_s(\Omega)$ are some weighted spaces defined in Section 2.

When Ω is an unbounded domain it is necessary to give conditions at infinity and these conditions can be described in a very convenient form in terms of weight functions. The weight functions ρ are related to the *distance* from a fixed subset $S \subset \partial\Omega$, where Ω is an unbounded open set with singular boundary. Geometric properties of the domain (for example a domain Ω whose boundary $\partial\Omega$ has singularities as corners or edges) can be often very suitably characterized by a weight type power of the distance from the *singularity* set on $\partial\Omega$. The coefficients of the lower terms in the operator L belong to a class of spaces introduced in [7] which are more general than L^n spaces and are reduced to L^n spaces in the case Ω is a bounded open set. If the weight ρ is a positive constant and Ω is an unbounded open set, these spaces correspond to the spaces introduced in [29].

In this paper we consider a wide class of discontinuity: the functions which satisfy Chicco type conditions [19], [21]. We remark that continuous functions, the class of discontinuities considered at first in [26] and of Cordes type belong to this class.

In previous papers [4], [5] we studied the same problem under hypotheses on coefficients considerably weakened with respect to the assumptions we can find in the papers until now. The idea was to approximate a_{ij} by some functions

e_{ij} 'near' to a_{ij} in bounded open sets and by more regular functions at infinity and 'near' to a subset S of $\partial\Omega$. The functions e_{ij} satisfy conditions of Chicco type and the assumptions we impose on their derivatives are very 'weak'. We require only that $(e_{ij})_{x_h} \in L^q_{loc}(\overline{\Omega}\setminus S)$ and we can apply locally some embedding results without further assumptions on $(e_{ij})_{x_h}$. We remark that an hypothesis of Chicco type as above is not sufficient to get local estimates near to a singular subset S of $\partial\Omega$ and for $|x|$ large enough without further assumptions. We need in [4] to introduce functions regular 'enough' suitable connected to a_{ij} to obtain the results. We observe that local a priori estimates allow us to prove a priori bounds for solutions of problem (1.1).

In subsequent paper [6] we state local a priori bound under Cordes conditions on coefficients of the operator L without to introduce more regular functions close to a_{ij}. The reason is that Cordes conditions allow us to approximate a_{ij} by means of functions which do not introduce derivatives and, so, further hypotheses on derivatives to use embedding results.

In this paper we obtain local and non local bounds under Chicco conditions with different assumptions with respect to [4].

In the first place the idea is to introduce functions with derivatives equal to zero as coefficients of suitable operators. So we can apply a result stated in [4] (see Lemma 5.1 in Section 5) and use embedding results (see Section 3). This force us to assume Chicco conditions with a suitable choice of functions e_{ij}.

On the other hand we can obtain the same result with an additional assumption on derivatives of e_{ij} to apply embedding results.

In both cases we can able to get local estimates near to S and for $|x|$ large enough without assuming existence of functions more regular which approximate a_{ij}.

This is the principal novelty in this paper with respect to the other papers until now. We remark that the results stated in this paper and the ones contained in [4] and [5] are based on different assumptions. A priori bounds (see Theorem 6.1 in Section 6) are obtained using embedding theorems and the local a priori bounds stated in Section 5.

2. Notations and Function Spaces

Let E be a Lebesgue measurable subset of R^n and $\Sigma(E)$ the σ-algebra of Lebesgue measurable subsets of E.

We denote by $\mathcal{D}(A)$ the class of restrictions to A, $A \in \Sigma(E)$, of functions $\zeta \in C^\infty_o(R^n)$ such that supp $\zeta \cap \overline{A} \subset A$ and by $L^p_{loc}(A)$ the class of functions

$f : A \to C$ such that $\zeta f \in L^p(A)$ for any $\zeta \in \mathcal{D}(A)$. We set

$$|f|_{p,A} = \|f\|_{L^p(A)}\,, \qquad 1 \le p \le +\infty\,.$$

Let $B(x,r)$, $x \in R^n$, $r \in R_+$, be the open ball with center in x and radius r.

Let Ω be an open subset of R^n. We set

$$\Omega(x,r) = \Omega \cap B(x,r)\,, \qquad \forall\, x \in \Omega\,, \quad \forall\, r \in R_+\,.$$

We denote by $\mathcal{A}(\Omega)$ the class of functions $\rho : \Omega \to R_+$ satisfying

$$\sup_{x,y\in\Omega,\, |x-y|<\rho(y)} \left|\log \frac{\rho(x)}{\rho(y)}\right| < +\infty\,.$$

It is easy to see that $\rho \in \mathcal{A}(\Omega)$ if and only if $\rho : \Omega \to R_+$ and there exists a constant $c \in R_+$ such that

$$c^{-1}\rho(y) \le \rho(x) \le c\,\rho(y)\,, \qquad \forall\, x \in \Omega\,, \qquad \forall\, y \in \Omega(x,\rho(x))\,.$$

We observe that $\mathcal{A}(\Omega)$ contains the class of positive Lipschitz functions with Lipschitz constant less than 1.

Some examples of functions $\rho \in \mathcal{A}(\Omega)$ are given in [7], [31]. For any $\rho \in \mathcal{A}(\Omega)$ we set

$$S_\rho = \{y \in \partial\Omega \,:\, \lim_{x\to y} \rho(x) = 0\}\,.$$

S_ρ is a closed subset in $\partial\Omega$ (see [14]). Moreover if $S_\rho \ne \emptyset$ it results (see [7], [31])

$$\rho(x) \le \text{dist}\,(x, S_\rho)\,, \qquad \forall\, x \in \Omega\,.$$

It is well-known (see, *e.g.*, Theorem 2 in [27], and Lemma 3.6.1 in [33]) that there exists $\alpha \in C^\infty(\Omega) \cap C^{0,1}(\overline{\Omega})$, $c_1, c_2 \in R_+$ such that

$$c_1\,\text{dist}\,(x, S_\rho) \le \alpha(x) \le c_2\,\text{dist}\,(x, S_\rho)\,, \qquad \forall\, x \in \Omega\,.$$

We put

$$\Omega_k = \{x \in \Omega :\ |x| < k,\ \alpha(x) > 1/k\}\,, \qquad \forall\, k \in N\,.$$

If $f \in \mathcal{D}(\overline{R}_+)$ is a fixed function such that

$$0 \le f \le 1\,, \qquad f(t) = 1 \quad \text{if} \quad t \le 1/2\,, \qquad f(t) = 0 \quad \text{if} \quad t \ge 1\,,$$

we define the functions

$$\psi_k :\ x \in \overline{\Omega} \longrightarrow \Big(1 - f(k\alpha(x))\Big)\, f\big(|x|/2k\big)\,, \qquad \forall\, k \in N\,.$$

We remark that, for any $k \in N$, ψ_k belongs to $\mathcal{D}(\overline{\Omega} \setminus S_\rho)$ and the following conditions hold

$$0 \le \psi_k \le 1\,, \qquad \psi_{k|_{\overline{\Omega}_k}} = 1\,, \qquad \operatorname{supp} \psi_k \subset \overline{\Omega}_{2k}\,.$$

Let $\mathcal{A}_o(\Omega)$ be the class of measurable functions $\rho \in \mathcal{A}(\Omega)$. If $\rho \in \mathcal{A}_o(\Omega)$, then (see [7], [14]) we get

$$\rho \in L^\infty_{loc}(\overline{\Omega})\,, \qquad \rho^{-1} \in L^\infty_{loc}(\overline{\Omega} \setminus S_\rho)\,. \tag{2.1}$$

Further examples and properties of functions of $\mathcal{A}(\Omega)$ can be found in [7], [31].

If $r \in N$, $1 \le p \le +\infty$, $s \in R$ and $\rho \in \mathcal{A}_o(\Omega)$, we denote by $W^{r,p}_s(\Omega)$ the space of distributions u on Ω such that $\rho^{s+|\alpha|-r}\,\partial^\alpha u \in L^p(\Omega)$ for $|\alpha| \le r$ endowed with the norm

$$\|u\|_{W^{r,p}_s(\Omega)} = \sum_{|\alpha|\le r} |\rho^{s+|\alpha|-r}\,\partial^\alpha u|_{p,\Omega}\,.$$

Moreover we denote by $\overset{o}{W}{}^{r,p}_s(\Omega)$ the closure of $C^\infty_o(\Omega)$ in $W^{r,p}_s(\Omega)$. We put

$$W^{0,p}_s(\Omega) = L^p_s(\Omega)\,, \qquad W^{r,2}_s(\Omega) = W^r_s(\Omega)\,, \qquad \overset{o}{W}{}^{r,2}_s(\Omega) = \overset{o}{W}{}^r_s(\Omega)\,.$$

For some properties of weighted Sobolev spaces, where the weight functions are a power of a function $\rho \in \mathcal{A}(\Omega)$, see, e.g. [3], [8], [23], [25], [30], [32].

If $1 \le p < +\infty$, $s \in R$ and $\rho \in \mathcal{A}_o(\Omega)$, we set

$$\Omega(x) = \Omega(x, \rho(x)) \qquad \forall\, x \in \Omega\,, \tag{2.2}$$

and consider the spaces $K^p_s(\Omega)$, $\tilde{K}^p_s(\Omega)$, $\overset{o}{K}{}^p_s(\Omega)$ defined in [7] in correspondence of the family of open sets defined by (2.2). Let us recall that: $K^p_s(\Omega)$ is the space of functions $g \in L^p_{loc}(\overline{\Omega} \setminus S_\rho)$ such that

$$\|g\|_{K^p_s(\Omega)} = \sup_{x\in\Omega} \Big(\rho^{s-n/p}(x)\; |g|_{p,\Omega(x)}\Big) < +\infty\,, \tag{2.3}$$

endowed with the norm defined by (2.3), $\tilde{K}^p_s(\Omega)$ is the closure of $L^\infty_s(\Omega)$ in $K^p_s(\Omega)$, $\overset{o}{K}{}^p_s(\Omega)$ is the closure of $C^\infty_o(\Omega)$ in $K^p_s(\Omega)$.

The following inclusions hold

$$L^q_{s-\frac{n}{q}}(\Omega) = W^{0,q}_{s-\frac{n}{q}}(\Omega) \subset \overset{o}{K}{}^q_s(\Omega) \subset \tilde{K}^q_s(\Omega) \subset K^q_s(\Omega). \tag{2.4}$$

For some other properties of the spaces $K^p_s(\Omega)$, $\tilde{K}^p_s(\Omega)$ and $\overset{o}{K}{}^p_s(\Omega)$ we refer to [7].

Remark 2.1. Let us fix $\rho \in \mathcal{A}_o(\Omega)$, $1 \le p < +\infty$, $s \in R$. We observe that if $g \in L^p_{loc}(\overline{\Omega} \setminus S_\rho)$, then, for any $\zeta \in \mathcal{D}(\overline{\Omega} \setminus S_\rho)$, we have $\zeta g \in \overset{o}{K}{}^p_s(\Omega)$ (see Remark 1.1 in [9]).

3. Embedding results

Let us fix $\rho \in \mathcal{A}_o(\Omega)$ such that $S = S_\rho \ne \emptyset$ and $\lim\limits_{|x| \to +\infty} \rho(x) = 0$ and consider the following conditions:

i_1) There exists an open subset Ω^* of R^n with uniform C^1- regularity property satisfying $\Omega \subset \Omega^*$, $\partial\Omega \setminus S \subset \partial\Omega^*$.

i_2) $s, q \in R$ and q is such that $q > 2$ if $n = 2$, $q = n$ if $n \ge 3$.

Remark 3.1. By hypothesis i_1) (see [7] and [14, Remark 3.1]) there exists $\theta \in]0, \frac{\pi}{2}[$ such that $\forall\, x \in \Omega$, $\exists\, C_\theta(x)$ $\overline{C_\theta(x, \rho(x))} \subset \Omega$, where $C_\theta(x)$ is an open cone with the vertex in x, opening θ and $C_\theta(x, r)$, $r \in R_+$, is the intersection of $C_\theta(x)$ and $B(x, r)$.

In [7] the authors proved the following result (see also [10] in which the authors emphisize the dependence of the constant in the final bound).

Lemma 3.1. *If the hypotheses i_1) and i_2) are verified, then for any $g \in K^q_{-s+1}(\Omega)$ and any $u \in W^1_s(\Omega)$ we get $gu \in L^2(\Omega)$ and*

$$|g\, u|_{2,\Omega} \le H\, \|g\|_{K^q_{-s+1}(\Omega)}\, \|u\|_{W^1_s(\Omega)}\,, \tag{3.1}$$

where $H = H(n, \theta, \rho, s, q)$ is a positive constant.

Let us define the modulus of continuity of a function $g \in \tilde{K}^p_s(\Omega)$ introduced in [4].

Let $\Sigma(\Omega)$ be the σ-algebra of Lebesgue measurable subsets of Ω. If $p \in [1, +\infty[$, $s \in R$ and $g \in K^p_s(\Omega)$, we set

$$\tau^p_s[g](t) = \sup_{\sup_x \frac{|E \cap B(x,\rho(x))|}{\rho^n(x)} \le t} \|g\, \chi_E\|_{K^p_s(\Omega)}\,, \qquad E \in \Sigma(\Omega)\,, \quad t \in R_+\,,$$

where χ_E is the characteristic function of E.

It is known (see [7]) that $g \in \tilde{K}^p_s(\Omega)$ if and only if $g \in K^p_s(\Omega)$ and

$$\lim_{t \to 0} \tau^p_s[g](t) = 0\,.$$

We define the modulus of continuity of $g \in \tilde{K}^p_s(\Omega)$ as a function $\tau[g] : R_+ \to R_+$ satisfying

$$\tau^p_s[g](t) \le \tau[g](t)\,, \qquad \forall t \in R_+\,, \qquad \lim_{t\to 0} \tau\,[g](t) = 0\,.$$

In the case $g : \Omega \to R$, we put

$$A_r(g) = \{x \in \Omega : \rho^{-s+1}\,|g(x)| \ge r\}\,, \qquad r \in R_+\,.$$

If $g \in K^p_{-s+1}(\Omega), p \in [1, +\infty[$, we get $\lim_{r\to+\infty} \frac{|A_r(g)\cap B(x,\rho(x))|}{\rho^n(x)} = 0$. Let us denote, for all $k \in N$, by $r_k = r_k(g)$ a real number such that

$$\frac{|A_r(g) \cap B(x, \rho(x))|}{\rho^n(x)} \le \frac{1}{k} \tag{3.2}$$

and by $r[g]$ the function

$$r[g] : k \in N \to r[g](k) = r_k \in R_+\,.$$

Now we recall the following lemma which we will use later (see Lemma 3.2 in [4]).

Lemma 3.2. *If the hypotheses $i_1)$ and $i_2)$ are verified and $g \in \tilde{K}^q_{-s+1}(\Omega)$, then for any $k \in N$ we have*

$$|g\,u|_{2,\Omega} \le H\,\tau[g]\left(\frac{1}{k}\right)\,\|u\|_{W^1_s(\Omega)} + r[g](k)\,\|u\|_{L^2_{s-1}(\Omega)}\,, \qquad \forall\, u \in W^1_s(\Omega)\,,$$

where H is the constant in (3.1), $\tau[g]$ is a modulus of continuity of g in $\tilde{K}^q_{-s+1}(\Omega)$ and $r[g]$ is the function defined by (3.3).

4. Hypotheses

Let us set

$$B_+ = \{x \in B_1 : x_n > 0\}\,, \qquad B_o = \{x \in B_1 : x_n = 0\}\,,$$

and suppose that there exists an open subset Ω^* of R^n such that:

$h_1)$ there are a $d \in R_+$, an open cover $\{U_i\}_{i\in I}$ of $\partial\Omega^*$ and, for any $i \in I$, a C^2-diffeomorphism $\psi_i : \overline{U}_i \to \overline{B}_1$ such that:

— $\psi_i(U_i \cap \Omega^*) = B_+\,, \qquad \psi_i(U_i \cap \partial\Omega^*) = B_o\,;$

— the components of ψ_i and ψ_i^{-1} and of their first and second derivatives are bounded by a constant independent of i;

— for any $x \in \Omega^*_d$ there exists an $i \in I$ such that $B(x,d) \subset U_i$ and, for any $x \in \Omega^* \setminus \Omega^*_d$, we get $B(x,d) \subset \Omega^*$, where $\Omega^*_d = \{x \in \Omega^* : \mathrm{dist}(x, \partial\Omega^*) < d\}$;

— $\Omega \subset \Omega^*$, $\qquad \partial\Omega \setminus S \subset \partial\Omega^*$.

Remark 4.1. It is easy to prove that h_1) hold when Ω^* has the uniform C^2 - regularity property defined in [1].

Remark 4.2. By Theorem 3.2 in [30] and hypothesis h_1) it follows that there exists $\sigma \in \mathcal{A}(\Omega) \cap C^\infty(\Omega) \cap C^{0,1}(\overline{\Omega})$ such that

$$c_1\, \rho(x) \le \sigma(x) \le c_2\, \rho(x)\,, \qquad \forall\, x \in \Omega\,, \quad \sigma_x\,,\, \sigma\, \sigma_{xx} \in L^\infty(\Omega)\,, \tag{4.1}$$

where the constants c_1, $c_2 \in R_+$ are independent of x.

Let us consider in Ω the second order linear differential operator

$$Lu = -\sum_{i,j=1}^{n} a_{ij}\, u_{x_i x_j} + \sum_{i=1}^{n} a_i\, u_{x_i} + a\, u \tag{4.2}$$

with the following conditions on the coefficients:

h_2) $a_{ij} = a_{ji} \in L^\infty(\Omega)$, $i, j = 1, \ldots, n$.

h_3) $a_i \in \tilde{K}^q_1(\Omega)$, $i = 1, \ldots, n$, $a \in \tilde{K}^t_2(\Omega)$, where $q > 2$ if $n = 2$, $q = n$ if $n \ge 3$, and $t = 2$ if $2 \le n < 4$, $t > 2$ if $n = 4$, $t = \frac{n}{2}$ if $n > 4$.

Let us denote by $E(\nu, \Omega)$ the class of $n \times n$ real matrix-valued functions (e_{ij}) such that

h_4) $e_{ij} = e_{ji} \in L^\infty(\Omega)\,, \quad i, j = 1, \ldots, n,$

$$(e_{ij})_{x_h} \in L^q_{loc}(\overline{\Omega} \setminus S)\,, \qquad i, j, h = 1, \ldots, n\,,$$

$$\textstyle\sum_{i,j=1}^{n} e_{ij}\, \xi_i\, \xi_j \ge \nu\, |\xi|^2 \qquad \forall\, \xi \in R^n\,, \quad \text{a.e. in } \Omega\,,$$

where ν is a positive constant independent of x and ξ.

Moreover we set

$$\mathcal{G}(\Omega) = \{g \in L^\infty \ : \ \operatorname*{ess\,inf}_{\Omega} g > 0\}\,.$$

and suppose that (a_{ij}) satisfies the following conditions:

h_5) (Chicco Type Condition) There exists $\nu \in R_+$, $(e_{ij}) \in E(\nu, \Omega)$ and $g \in \mathcal{G}(\Omega)$ such that

$$\operatorname*{ess\,sup}_{\Omega} \sum_{i,j=1}^{n} \Big(e_{ij} - g a_{ij}\Big)^2 < \nu^2\,.$$

For example one can choose

$$g = \frac{\sum_{i,j=1}^{n} e_{ij}\, a_{ij}}{\sum_{i,j=1}^{n} a_{ij}^2} \in \mathcal{G}(\Omega)\,. \tag{4.3}$$

h_5') (Chicco Type Condition) With e_{ij}, for $i, j = 1 \dots n$, constant functions satisfying h_4).

Let us set

$$u_x = \left(\sum_{i=1}^{n} u_{x_i}^2 \right)^{1/2}, \quad u_{xx} = \left(\sum_{i,j=1}^{n} u_{x_i x_j}^2 \right)^{1/2}.$$

We consider a function $\beta : \Omega \to R_+$ such that the following hypothesis holds:

h_6) $\beta \in \tilde{K}_2^t(\Omega)$ and $\exists\, \delta \in \tilde{K}_1^q(\Omega)$ such that $\beta_x \le \beta\, \delta$.

For example, some functions which satisfy the hypothesis h_6) are given by $\beta = \frac{1}{\sigma^2}$, where σ is the function introduced in Remark 4.2, or

$$\beta(x) = \frac{1}{(1 + |x|^2)^\tau}, \qquad x \in \Omega, \quad \tau > 0$$

(for details see [9]).

The following condition can substitute h_5') in Lemma 5.2 and Theorem 6.1 (see Section 5 and Section 6 respectively).

h_7) $(e_{ij})_{x_h} \in \tilde{K}_1^q(\Omega)$, $i, j, h = 1, \dots, n$.

Remark 4.3. Let us note that h_4) and h_5) imply operator L defined in (4.2) is uniformly elliptic in Ω.

Remark 4.4. If in h_4) the functions $e_{ij} = \frac{1}{k}\delta_{ij}$, $k \in R_+$, and $g = \frac{\sum_{i=1}^{n} e_{ij} a_{ij}}{\sum_{i,j=1}^{n} a_{ij}^2}$, as in (4.3), conditions of Chicco type reduce to Cordes type conditions (we refer to [6] for some recent results in weighted spaces under Cordes condition). If in h_4) $e_{ij} = \frac{1}{k}\delta_{ij}$, $k \in R_+$ but $g \in \mathcal{G}(\Omega)$ is different from (4.3), we have a particular case of Chicco condition (see h_5')).

Remark 4.5. One can show that under hypotheses $h_1) - h_3)$ and h_6), it follows that for any $s, \lambda \in R$ the operator

$$u \in W_s^2(\Omega) \;\to\; Lu + \lambda\, \beta\, u \in L_s^2(\Omega)$$

is bounded.

5. Local A Priori Bounds

Let us set:

$$L_o u = - \sum_{i,j=1}^{n} a_{ij}\, u_{x_i x_j},$$

$$\text{and} \quad \tilde{f} = 1 + \sum_{i,j=1}^{n} |e_{ij}|\,\delta + \sum_{i,j=1}^{n} (e_{ij})_x\,, \qquad \text{if} \quad h_5) \quad \text{holds} \;,$$

$$\text{or} \quad \tilde{f} = \sum_{i,j=1}^{n} |e_{ij}|\,\delta\,, \qquad \text{if} \quad h_5') \quad \text{holds}\,,$$

where δ is the function defined in h_6) and e_{ij} are the functions which belong to the class $E(\nu,\Omega)$ (see h_4)).

Let us fix a bounded open subset V of R^n such that

$$V \subset \Omega\,, \qquad \text{or} \qquad V \cap \partial\Omega \neq \emptyset\,, \quad \text{and} \quad V \subset U_i \setminus S \quad \text{for some } i \in I\,.$$

Lemma 5.1. *If the hypotheses $h_1), h_2), h_4), h_6)$ hold, then for any $\lambda \geq 0$, for any function v satisfying*

$$v \in W^2(\Omega) \cap \overset{o}{W}{}^1(\Omega)\,, \qquad \operatorname{supp} v \subset V\,,$$

and for any $\epsilon \in R_+$ we have the bound

$$(\nu^2 - \epsilon^2)|v_{xx}|^2_{2,\Omega} \leq \left| - \sum_{i,j=1}^{n} e_{ij} v_{x_i x_j} + \lambda\,\beta\,v \right|^2_{2,\Omega} + c(\epsilon)|\tilde{f}\,v_x|^2_{2,\Omega}\,. \tag{5.1}$$

Moreover if also h_5) is verified we get

$$|v_{xx}|_{2,\Omega} \leq c\left(|L_o v + \lambda\,g^{-1}\beta\,v|_{2,\Omega} + |\tilde{f}\,v_x|_{2,\Omega}\right)\,, \tag{5.2}$$

where $c = c\,(\Omega,\nu,\|a_{ij}\|_\infty,\|e_{ij}\|_\infty)$.

The proof of Lemma 5.1 can be found in [4, Lemma 5.1].

Remark 5.1. If e_{ij} are constant functions, one can proves Lemma 5.1 as in [6, Lemma 5.1]. In particular inequality (5.1) takes the form

$$\nu^2|v_{xx}|^2_{2,\Omega} \leq \left| - \sum_{i,j=1}^{n} e_{ij} v_{x_i x_j} + \lambda\,\beta\,v \right|^2_{2,\Omega} + |\tilde{f}\,v_x|^2_{2,\Omega}\,,$$

where $\tilde{f}$ is defined at beginning of the section. As a consequence, by Chicco type condition h_5') we deduce (5.2).

Now we are able to prove the following lemma using as tools Lemma 5.1 and Lemma 3.2. A similar lemma was proved in [4] and in [6] under different hypotheses on coefficients of the operator L.

Lemma 5.2. *If the conditions $h_1) - h_7)$ or $h_1) - h_4)$, $h_5') - h_6)$ hold and λ_1 is a real number, then there exists a constant $c \in R_+$ such that for any $\lambda \in [\lambda_1, +\infty[$ and for any function v satisfying*

$$v \in W^2(\Omega) \cap \overset{o}{W}{}^1(\Omega)\,, \qquad \operatorname{supp} v \subset V\,,$$

we get

$$|v_{xx}|_{2,\Omega} \leq c\left(|Lv + \lambda\, g^{-1}\,\beta\, v|_{2,\Omega} + |\rho^{-1} v_x|_{2,\Omega} + |\rho^{-2} v|_{2,\Omega}\right), \tag{5.3}$$

where c is a positive constant depending on Ω, ν, n, ρ, θ, q, $\|a_{ij}\|_\infty$, $\|e_{ij}\|_\infty$, $\|g\|_\infty$, $\tau[\delta]$, $\tau[\psi_k(e_{ij})_x]$, $\tau[\beta]$, $\tau[a_i]$, $\tau[a]$, $r[\delta]$, $r[\psi_k(e_{ij})_x]$, $r[\beta]$, $r[a_i]$, $r[a]$.

Proof. Step 1. (Estimates for $|x|$ Large Enough and Near to S) Let us suppose $\lambda \geq 0$ and consider the functions ψ_k, $k \in N$, introduced in Section 2. Applying (5.1) in Lemma 5.1 to the function $(1 - \psi_k)\, v$, we get

$$(\nu - \epsilon)\big|\big((1-\psi_k)\,v\big)_{xx}\big|_{2,\Omega} \leq \Bigg(\bigg|- \sum_{i,j=1}^{n} e_{ij}\big((1-\psi_k)\,v\big)_{x_i x_j} + \lambda\,\beta\,(1-\psi_k)\,v\bigg|_{2,\Omega} + c_1|\tilde{f}((1-\psi_k)\,v)_x|_{2,\Omega}\Bigg) \tag{5.4}$$

with $\epsilon = 0$ when derivatives of functions e_{ij} are equal to zero (see Remark 5.1).

The first term on the right hand in (5.4) is bounded as follows

$$\begin{aligned}
&\bigg|- \sum_{i,j=1}^{n} e_{ij}\,\big((1-\psi_k)\,v\big)_{x_i x_j} + \lambda\,\beta\,(1-\psi_k)\,v\bigg|_{2,\Omega} \\
&\leq \Big|g\,L_o\big((1-\psi_k)\,v\big) + \lambda\,\beta\,(1-\psi_k)\,v\Big|_{2,\Omega} + \bigg|- \sum_{i,j=1}^{n} (e_{ij} - g a_{ij})\,\big((1-\psi_k)\,v\big)_{x_i x_j}\bigg|_{2,\Omega} \\
&\leq c_2\Bigg(\Big|(1-\psi_k)\,\big(L_o v + \lambda\,g^{-1}\,\beta\,v\big)\Big|_{2,\Omega} \\
&\quad + \Big|(1-\psi_k)_x\,v_x\Big|_{2,\Omega} + \Big|(1-\psi_k)_{xx}\,v\Big|_{2,\Omega}\Bigg) + h\Big|\big((1-\psi_k)\,v\big)_{xx}\Big|_{2,\Omega},
\end{aligned} \tag{5.5}$$

where g is the function in hypothesis $h_5)$ or in $h_5')$ and $h = \operatorname{ess\,sup}_\Omega \left(\sum_{i,j=1}^{n} |e_{ij} - g a_{ij}|^2\right)^{1/2}$ (recall that $\nu - h > 0$ by Chicco type condition). Now since $\tilde{f} \in \tilde{K}_1^q(\Omega)$ by h_6) and h_7) (respectively by h_6) when derivatives of functions e_{ij} are equal to zero), we can use Lemma 3.2 to estimate the last term in (5.4). So we obtain by (5.4) and (5.5), since $\rho \in L^\infty_{loc}(\overline{\Omega})$,

$$(\nu - \epsilon)\Big|\big((1-\psi_k)\,v\big)_{xx}\Big|_{2,\Omega} \le c_3\Bigg(\big|L_o\, v + \lambda\, g^{-1}\,\beta\, v\big|_{2,\Omega} + |\rho^{-1}\, v_x|_{2,\Omega} + |\rho^{-2}\, v|_{2,\Omega})\Bigg) + \left(c_1\, H\, \tau[\tilde{f}]\, \left(\frac{1}{k}\right) + h\right) \Big|\big((1-\psi_k)\,v\big)_{xx}\Big|_{2,\Omega}, \quad (5.6)$$

with $\epsilon = 0$ if h_5') holds.

Step 2. (Estimates on Bounded Sets and Far from S) We can apply Lemma 5.1 to the function $\psi_k v$ to get

$$|(\psi_k v)_{xx}|_{2,\Omega} \le c_4\Big(\big|L_o(\psi_k v) + \lambda\,\beta\, g^{-1}\,\psi_k v\big|_{2,\Omega} + |\tilde{f}(\psi_k v)_x|_{2,\Omega}\Big)\,. \qquad (5.7)$$

Proceeding as in the Step 1 we use Lemma 3.2 to obtain

$$|(\psi_k v)_{xx}|_{2,\Omega} \le c_5\Bigg(|L_o\, v + \lambda\, g^{-1}\,\beta\, v|_{2,\Omega} + |\rho^{-1} v_x|_{2,\Omega} + |\rho^{-2} v|_{2,\Omega} + H\,\tau[\psi_k\,\tilde{f}]\left(\frac{1}{k}\right)|(\psi_k v)_{xx}|_{2,\Omega}\Bigg)\,. \qquad (5.8)$$

If $\epsilon < \nu - h$, for a suitable choice $k = k_0 \in N$, by Chicco type condition and by definition of modulus of continuity given in Section 3, we get from (5.6) and (5.8)

$$\begin{aligned}\|v_{xx}\|_{2,\Omega} &\le \big\|(1-\psi_{k_0})v)\big\|_{2,\Omega} + \big\|\big(\psi_{k_0})v\big)_{xx}\big\|_{2,\Omega} \\ &\le c_6\left(|L_o\, v + \lambda\, g^{-1}\beta\, v|_{2,\Omega} + |\rho^{-1} v_x|_{2,\Omega} + |\rho^{-2} v|_{2,\Omega}\right). \qquad (5.9)\end{aligned}$$

We complete the proof reasoning as in [4, Lemma 5.2].

If $\lambda_1 < 0$, we fix $\lambda \in [\lambda_1, 0[$. Using h_6) and applying to β Lemma 3.2 we get the bound

$$|\lambda\, g^{-1}\,\beta\, v|_{2,\Omega} \le c_7 |\lambda_1|\,(\operatorname{ess\,inf} g)^{-1}\,\Big(|\rho^{-1} v_x|_{2,\Omega} + |\rho^{-2} v|_{2,\Omega}\Big). \qquad (5.10)$$

Now if we consider the inequality (5.9) with $\lambda = 0$, from (5.10) we easly deduce (5.3) with L_o instead of L.

Finally, applying Lemma 3.2 to the functions a_i and a verifying hypothesis h_3) we obtain the result. □

Remark 5.2. In the proof of Lemma 5.2, Step 2, we can apply embedding results without using hypothesis h_7). In fact, for any $k \in N$, let $r \geq 2k$ so that $\psi_{r_{|supp\,\psi_k}} = 1$. The function $\psi_r\,\tilde{f}$ belongs to the space $\tilde{K}^q_1(\Omega)$ (see Remark 2.1 and inclusions (2.4)) and then we can use Lemma 3.2 to estimate the last term in (5.7).

Instead in Step 1 we need a further assumption to apply embedding results if derivatives of e_{ij} are not equal to zero. In [4] we supposed the existence of more regular functions near to a_{ij}.

6. A Priori Bounds

We assume that the following further hypotheses hold:

h_8) There exists a function $\gamma : N \to R_+$ such that

$$\operatorname*{ess\,sup}_{\Omega\setminus\Omega_k} \sum_{i,j=1}^{n} |\gamma_{ij} - g a_{ij}| \leq \gamma(k)\,, \quad \forall k \in N\,, \qquad \lim_{k\to+\infty} \gamma(k) = 0\,,$$

where γ_{ij}, for $i, j = 1, ..., n$, are constant functions satisfying h_4).

h_9) The function σ which satisfies (4.1) is such that $\sigma_x \in \overset{o}{K}{}^q_0(\Omega)\,$, where q is the number defined in the hypothesis h_3);

h_{10}) $a_i \in \overset{o}{K}{}^q_1(\Omega)\,,\ i = 1, ..., n\,,\ a \geq \mu\rho^{-2}$ a.e. in $\Omega\,$, where $\mu \in R_+$ is independent of x.

An example of function $\rho \in \mathcal{A}_o(\Omega)$ satisfying the condition h_8) can be found in [15].

Local a priori bounds stated in Lemma 5.2 allow us to prove the following result under different hypotheses with respect to the same result obtained in [4, Teorem 6.1]. The tecnique used to prove the estimates on bounded open sets and far from S is the same which we can find in [4], but the local bounds used are obtained with different assumptions. About the estimates at infinity and near to S, we give the proof and correct some little printer's errors in the proof of Theorem 6.1 in [4].

Theorem 6.1. *If $h_1) - h_{10})$, or $h_1) - h_4)$, $h'_5) - h_6)$ and $h_8) - h_{10})$ hold, then there exist a constant $c \in R_+$ and a bounded open set $\Omega_o \subset\subset \overline{\Omega} \setminus S$ such that*

$$\|u\|_{W^2_s(\Omega)} \leq c\bigg(\|Lu + \lambda\, g^{-1}\,\beta\, u\|_{L^2_s(\Omega)} + |u|_{2,\Omega_o}\bigg)\,,$$

$$\forall\, u \in W^2_s(\Omega) \cap \overset{o}{W}{}^1_{s-1}(\Omega)\,, \qquad \forall\, \lambda \geq 0\,, \quad (6.1)$$

where c *is a positive constant depending on* Ω, ν, n, ρ, θ, q, s, t, a_i, γ_{ij}, $\|a_{ij}\|_\infty$, $\|e_{ij}\|_\infty$, $\|g\|_\infty$, $\tau[\delta]$, $\tau[\psi_k(e_{ij})_x]$, $\tau[\beta]$, $\tau[a]$, $r[\delta]$, $r[\psi_k(e_{ij})_x]$, $r[\beta]$, $r[a]$.

Proof. Step 1. (Estimates at Infinity and Near to S) If the principal coefficients of L are suitable constants, we can use Theorem 4.1 in [9] to get the bound (6.2).

Indeed if

$$\tilde{L}_o = -\sum_{i,j=1}^{n} \gamma_{ij}\, \frac{\partial^2}{\partial x_i \partial x_j}\,,$$

and g is the function in h_5), we have that there exists a bounded open subset Ω_0' of Ω such that

$$\|(1-\psi_k)\,u\|_{W^2_s(\Omega)} \leq c_1\Big(\big\|\tilde{L}_o((1-\psi_k)\,u) + (ga+\lambda\beta)(1-\psi_k)\,u\big\|_{L^2_s(\Omega)} + |(1-\psi_k)\,u|_{2,\Omega_0'}\Big)\,, \quad (6.2)$$

from which

$$\begin{aligned}\|(1-\psi_k)\,u\|_{W^2_s(\Omega)} &\leq c_1\bigg(\bigg\|-\sum_{i,j=1}^{n}\big(\gamma_{ij}-ga_{ij}\big)\big((1-\psi_k)\,u\big)_{x_ix_j} \\ &\quad - g\sum_{i,j=1}^{n} a_{ij}\big((1-\psi_k)\,u\big)_{x_ix_j} + (ga+\lambda\,\beta)(1-\psi_k)\,u\bigg\|_{L^2_s(\Omega)} + |(1-\psi_k)\,u|_{2,\Omega_0'}\bigg) \\ &\leq c_1\Big(\|g\|_\infty\|L_o\big((1-\psi_k)\,u\big) + (a+\lambda\,g^{-1}\,\beta)(1-\psi_k)\,u\|_{L^2_s(\Omega)} + \\ &\quad + |(1-\psi_k)\,u|_{2,\Omega_0'} + \gamma(k)\|((1-\psi_k)\,u)_{xx}\|_{L^2_s(\Omega)}\Big)\,. \quad (6.3)\end{aligned}$$

By a suitable choice $k = k_0 \in N$ we get from (6.3)

$$\|(1-\psi_{k_0})\,u\|_{W^2_s(\Omega)} \leq \Big(\|L_o\big((1-\psi_{k_0})\,u\big) + (a+\lambda\,g^{-1}\,\beta)(1-\psi_{k_0})\,u\|_{L^2_s(\Omega)} + |(1-\psi_{k_0})\,u|_{2,\Omega_0'}\Big)\,. \quad (6.4)$$

Step 2. (Estimates on Bounded Sets and Far from S) Locally we can apply Lemma 5.2 with $L = L_0 + a$. Then, reasoning as in [4] we get the estimate

$$\|\psi_{k_0} u\|_{W^2_s(\Omega)} \le c_3 \big(\big\| (L_o(\psi_{k_0} u) + (a + \lambda g^{-1} \beta) \psi_{k_0} u \big\|_{L^2_s(\Omega)} + \|\psi_{k_0} u\|_{L^2_{s-2}(\Omega)} \big) . \quad (6.5)$$

Remarking that, using (2.1)

$$\big\| (\psi_{k_0})_x u_x \big\|_{L^2_s(\Omega)} \le c_4 \, |u_x|_{2, supp \psi_{k_0}} ,$$

from well known inequality (see [1])

$$|u_x|_{2, supp\psi_{k_0}} \le K \left(\epsilon |u_{xx}|_{2, supp\psi_{k_0}} + \epsilon^{-1} |u|_{2, supp\psi_{k_0}} \right) ,$$

where $K = K(n, \Omega)$ and $0 < \epsilon < \epsilon_0$, $\epsilon_0 > 0$, inequalities (6.4) and (6.5) imply

$$\|u\|_{W^2_s(\Omega)} \le c_5 \big(\big\| L_o \, u + (a + \lambda \, g^{-1} \, \beta) \, u \big|_{L^2_s(\Omega} + |u|_{2, \Omega''_o} \big) , \qquad (6.6)$$

with $\Omega''_0 = \Omega'_0 \cap$ supp ψ_{k_0}.

Moreover from Corollary 2 in [7] we have that for any $\epsilon \in R_+$ there exist $c(\epsilon) \in R_+$ and an open set $\Omega_\epsilon \subset\subset \Omega$ such that

$$\sum_{i=1}^{n} \|a_i u_{x_i}\|_{L^2_s(\Omega)} \le \epsilon \, \|u\|_{W^2_s(\Omega)} + c(\epsilon) \, |u|_{2, \Omega_\epsilon} . \qquad (6.7)$$

From (6.6) and (6.7) we deduce the assertion with $\Omega_o = \Omega''_o \cup \Omega_\epsilon$. □

Remark 6.1. A different assumption in Theorem 6.1 could be the convergence of a_{ij} to more regular functions α_{ij} at infinity and near to S. For example functions such that $(\alpha_{ij})_{x_h}$ belong to the space $\overset{o}{K}{}^q_1(\Omega)$. Then we can modify the proof in Step 1 setting $\tilde{L}_o = - \sum_{i,j=}^{n} \alpha_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$, and using h_8) with α_{ij} in place of γ_{ij}.

Remark 6.2. We observe that in Theorem 6.1 we can suppose in place of the condition $a \ge \mu \rho^{-2}$, $\mu > 0$, in h_{10})

$$a = a' + a'' , \quad a' \in \overset{o}{K}{}^t_2(\Omega) , \quad a'' \ge \mu_o \rho^{-2} , \; \mu_o \in R_+ , \quad \text{a.e. in } \; \Omega .$$

From Theorem 6.1 in a standard way (see, *e.g.*, proof in [9, Corollary 4.1] we have the following result.

Corollary 6.2. *In the same hypotheses of Theorem 6.1 and if*

$$\beta^{-1} \in L^{\infty}_{loc}(\overline{\Omega} \setminus S)\,,$$

then for any $s \in R$ *there exist* c, $\lambda_o \in R_+$ *such that*

$$\|u\|_{W^2_s(\Omega)} \le c\|Lu + \lambda\, g^{-1}\,\beta\, u\|_{L^2_s(\Omega)}\,, \quad \forall\, u \in W^2_s(\Omega) \cap \overset{o}{W}{}^1_{s-1}(\Omega)\,,\ \forall\, \lambda \ge \lambda_o\,,$$

where c *has the same dependence of the andconstant in Theorem 6.1.*

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