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THE SPHERICAL BERNSTEIN WAVELET

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Abstract: In this work we introduce a new bandlimited spherical wavelet: The Bernstein wavelet. It possesses a couple of interesting properties. To be specific, we are able to construct bandlimited wavelets free of oscillations. The scaling function of this wavelet is investigated with regard to the spherical uncertainty principle, i.e., its localization in the space domain as well as in the momentum domain is calculated and compared to the well-known Shannon scaling function. Surprisingly, they possess the same localization in space although one is highly oscillating whereas the other one shows no oscillatory behavior. Moreover, the Bernstein scaling function turns out to be the first bandlimited scaling function known to the literature whose uncertainty product tends to the minimal value 1.

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1. Motivation

Over the last decade wavelets have found important applications in numerous areas of mathematics, physics, engineering and computer science. Wavelets form versatile tools for representing general functions or data sets. They es-

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pecially become more and more important in Earth sciences since most recent satellite missions deliver millions of data scattered around the globe. Meanwhile spherical wavelets introduced by [5, 7, 8, 9] play the fundamental role in the analysis of regional, high-frequent phenomena observed in geophysical, geodetic, meteorological and seismological applications (see, e.g., [2, 4, 5, 6] and the many references therein).

The spherical wavelets discussed here are based on expansions of Legendre polynomials. Hence, they form radial basis functions on the sphere whose argument depends only on the spherical distance between the localization of the wavelet and its evaluation point. Moreover, they are usually designed to fulfill the partial differential equations corresponding to the considered geophysical context (see [9]). In view of constructive approximation on the sphere the wavelet techniques have shown their strength in detecting local and regional (also time-dependent) effects (see [5, 6]) for more than one decade. For example, studies of local changes of mass in the gravitational field observed from satellite data or crustal field modeling from Earth's magnetic data represent recent examples of application. Also vectorial data are analyzed by an intrinsic extension of scalar wavelets to vectorial wavelets, e.g., for modeling atmospherical flows and oceanographic streams. Along this experience the wavelet techniques have been extended to (non-linear) Galerkin methods for solving spherical partial differential equations, such as the Navier-Stokes equation (see [2, 3]).

In the literature bandlimited and non-bandlimited wavelets are distinguished. Bandlimited wavelets are generated by a finite series of Legendre polynomials whereas for non-bandlimited wavelets this Legendre expansion is infinite. Examples for both types can be found in [5, 6, 9]. However, due to the construction as series of finitely many Legendre polynomials in frequency domain, it has been common belief that it is not possible to build spherical bandlimited wavelets that show no oscillations (except for one that is required to obtain a vanishing zero moment). Just at this point this work comes into play. We introduce a smooth, bandlimited, spherical wavelet: The spherical Bernstein wavelet. Its great advantage is that it possesses the appealing property of fast and stable numerical evaluation because it can be formulated in closed form by an elementary function. In fact, it has been constructed via its explicit formula, i.e., in the space domain whereas usually spherical wavelets are generated in the momentum domain via its Legendre coefficients. This different type of construction is also reflected by the proofs in this article – many properties of the Bernstein wavelet (especially its localization properties) are shown differently as for other well-known wavelets. As the Bernstein wavelet shows no oscillatory behavior in space domain it is also much easier to derive error estimates for spatial truncation as required in many numerical procedures, e.g., spherical panel clustering [10].

The paper is organized as follows: First, we briefly introduce the mathematical tools and notations that are needed later on. Second, we define the Bernstein wavelet in the space domain in closed form and prove that all essential properties of spherical wavelets are fulfilled. Surprisingly, when comparing the Bernstein scaling function with kernels known to the literature, we realize that its space uncertainty is exactly the same as in the Shannon case despite the huge oscillations of the latter. Finally, we show that the uncertainty product (known as Heisenberg's uncertainty product) converges to 1 in case of the bandlimited Bernstein wavelet.

2. Preliminaries

In the following we adopt the notation from the monograph [9]. The letters $\mathbb{N}, \mathbb{N}_0$ and $\mathbb{R}$ denote the set of positive integers, non-negative integers and real numbers. We write x, y to represent the elements of the three-dimensional space $\mathbb{R}^3$ endowed with the Euclidian canonical basis $\{\varepsilon^1, \varepsilon^2, \varepsilon^3\}$. Then the inner product is denoted by $x \cdot y = \sum_{i=1}^{3} x_i y_i$. The corresponding norm of x is given by $|x| = \sqrt{x \cdot x}$. The unit sphere is denoted by Ω, elements of it are usually written by Greek letters, e.g., ξ or η. As customary the space of all real, square-integrable functions F on Ω is called $L^2(\Omega)$. $L^2(\Omega)$ is a Hilbert space endowed with the inner product

$$\langle F, G \rangle_{L^2(\Omega)} = \int_\Omega F(\xi) G(\xi) \; d\omega(\xi),$$

and associated norm

$$\|F\|_{L^2(\Omega)} = \left(\int_\Omega F^2(\xi) \; d\omega(\xi) \right)^{1/2}.$$

It is well known that $L^2(\Omega)$ possesses an orthonormal basis consisting of the real-valued spherical harmonics $\{Y_{n,k}\}_{n=0,\ldots;k=-n,\ldots,n}$ of degree n and order k as defined, for example, in [14, 15]. Then the spherical addition theorem

$$\frac{2n+1}{4\pi} P_n(\xi \cdot \eta) = \sum_{k=-n}^{n} Y_{n,k}(\xi) Y_{n,k}(\eta)$$

connects the spherical harmonics $Y_{n,k}$ of degree n and the Legendre polynomial P_n (for more details see [1, 9, 13]).

2.1. Scaling Functions and Wavelets

First of all we recapitulate the definition of a so-called generator of a scaling function. As a matter of fact, the choice of this generator determines all properties of the spherical scaling function and its corresponding wavelet.

Definition 1. (Generator of a Scaling Function) A family $\{\{\Phi_J^\wedge(n)\}_{n\in\mathbb{N}_0}\}_{J\in\mathbb{N}_0}$ is called a generator of a scaling function, if it satisfies the following requirements:

i) For all $J \in \mathbb{N}_0$

$$\Phi_J^\wedge(0) = 1. \tag{1}$$

ii) For all $J, J' \in \mathbb{N}_0$ with $J \leq J'$ and all $n \in \mathbb{N}$

$$0 \leq \Phi_J^\wedge(n) \leq \Phi_{J'}^\wedge(n). \tag{2}$$

iii) For all $n \in \mathbb{N}$

$$\lim_{J\to\infty} \Phi_J^\wedge(n) = 1. \tag{3}$$

For fixed $J \in \mathbb{N}_0$ the sequence $\{\Phi_J^\wedge(n)\}_{n\in\mathbb{N}_0}$ is called the *symbol* of the corresponding scaling function Φ_J of scale J. According to [9] this scaling function of scale J is defined by

$$\Phi_J(\xi,\eta) = \sum_{n=0}^{\infty} \Phi_J^\wedge(n) \frac{2n+1}{4\pi} P_n(\xi\cdot\eta),$$

while the (linear) wavelet of scale J reads as follows

$$\Psi_J(\xi,\eta) = \sum_{n=0}^{\infty} \Psi_J^\wedge(n) \frac{2n+1}{4\pi} P_n(\xi\cdot\eta),$$

where $\Psi_J^\wedge(n)$ is given via the linear refinement equation $\Psi_J^\wedge(n) = \Phi_{J+1}^\wedge(n) - \Phi_J^\wedge(n)$, $n \in \mathbb{N}_0$. The whole set $\{\Phi_J\}_{J\in\mathbb{N}_0}$ is called a scaling function, the set $\{\Psi_J\}_{J\in\mathbb{N}_0}$ is called a wavelet. If for all scales J the symbol $\{\Phi_J^\wedge(n)\}_{n\in\mathbb{N}_0}$, respectively $\{\Psi_J^\wedge(n)\}_{n\in\mathbb{N}_0}$, is different from zero only for finitely many values of n, we call the corresponding scaling function $\{\Phi_J\}_{J\in\mathbb{N}_0}$, respectively the wavelet $\{\Psi_J\}_{J\in\mathbb{N}_0}$, *bandlimited*. The most simple representant of a scaling function is the well-known Shannon scaling function (see [5]).

Definition 2. (Shannon Scaling Function) Let $J \in \mathbb{N}_0$ and $\eta \in \Omega$ be fixed. Then the Shannon scaling function of scale J is given by

$$\Phi_J^S(\xi,\eta) = \sum_{n=0}^{2^J-1} \Phi_J^{S\wedge}(n) \frac{2n+1}{4\pi} P_n(\xi\cdot\eta),$$

where

$$\Phi_J^{S\wedge}(n) = \begin{cases} 1 & \text{if } n \in [0, 2^J), \\ 0 & \text{else.} \end{cases}$$

Obviously, its generator $\{\{\Phi_J^{S\wedge}(n)\}_{n\in\mathbb{N}_0}\}_{J\in\mathbb{N}_0}$ fulfills all three conditions of Definition 1 and the Shannon scaling function is bandlimited.

2.2. Space Localization

We start by formulating the space localization property mathematically: Suppose that F is real-valued and of class $L^2(\Omega)$. Moreover, assume without loss of generality, that $\|F\|_{L^2(\Omega)} = 1$. Then, we follow [5, 9, 16] and associate to F the normal (radial) field $\eta \mapsto \eta F(\eta), \eta \in \Omega$. This allows us to introduce the *expectation value* of F *in the space domain* by

$$e_F = \int_\Omega \eta F(\eta)^2 \, d\omega(\eta), \tag{4}$$

such that $e_F \in \mathbb{R}^3$. In that sense, $F(\eta)^2$ can be interpreted as probability density function with respect to the measure $d\omega(\eta)$. Immediately, we observe that $|e_F| \leq 1$. Additionally, the *variance in the space domain* σ_F^2 is given by

$$\sigma_F^2 = \int_\Omega (\eta - e_F)^2 F(\eta)^2 d\omega(\eta). \tag{5}$$

Hereby, σ_F can be regarded as the standard deviation of F. Let us clarify its geometrical meaning: Assume that $e_F \neq 0$, then F localizes in the spherical cap

$$C = \{\eta \in \Omega | 1 - \eta \cdot e_F/|e_F| \leq 1 - |e_F|\},$$

with radius σ_F. Clearly, we say F is "space localized" if the ratio

$$\Delta_F = \frac{\sigma_F}{|e_F|} \tag{6}$$

between the diameter σ_F of the spherical cap C and the absolute value of the expectation value $|e_F|$ is small (a table with examples can be found in [2]). For a concise description of the space localization as well as the uncertainty principle on the sphere we refer the reader to [5, 9, 16]. According to [9] we can simplify the expression for the expectation value e_F if F is a radial basis function. In this case, we assume without loss of generality that F is localized at $\varepsilon^3 = (0, 0, 1)^T$. Then (4) reads as

$$e_F = \left(2\pi \int_{-1}^{1} tF(t)^2 \, dt\right) \varepsilon^3. \tag{7}$$

and following [9] the variance in (5) can be rewritten as

$$\sigma_F^2 = 1 - |e_F|^2. \tag{8}$$

2.3. Localization in Momentum

In order to define the localization in momentum we follow the concepts of [5] and [16]. Thus, we consider the expectation value in the momentum domain which is in fact the expectation value of the surface curl operator L^* on the sphere Ω. Suppose again that F is real valued with $\|F\|_{L^2(\Omega)} = 1$, but this time F is an element of a space of Sobolev type $\mathcal{H}_2(\Omega)$, i.e., for $F \in \mathcal{H}_{2l}(\Omega)$ with $l \in \mathbb{N}$ there exists a function $G \in L^2(\Omega)$ whose Fourier coefficients $G^\wedge(n,k)$ fulfill $G^\wedge(n,k) = (-n(n+1))^l F^\wedge(n,k)$ for all $n = 0, 1, \ldots; k = -n, \ldots, n$. For a concise introduction to spherical Sobolev spaces see, e.g., [5, 6, 9].

The *expectation value in the momentum domain* of the function F is given by

$$e_F^{L^*} = \int_\Omega (L_\eta^* F(\eta)) F(\eta)\ d\omega(\eta), \tag{9}$$

which by definition leads to the value $e_F^{L^*} = 0 \in \mathbb{R}^3$. Its *variance in the momentum domain* $\left(\sigma_F^{L^*}\right)^2$ is defined as

$$\left(\sigma_F^{L^*}\right)^2 = \int_\Omega \left|(L_\eta^* - e_F^{L^*})F(\eta)\right|^2\ d\omega(\eta) = \int_\Omega (-\Delta_\eta^* F(\eta))F(\eta)\ d\omega(\eta), \tag{10}$$

where the last equation easily results from the surface theorem of Stokes. Again we can simplify this expression in the case of radial basis functions assuming without loss of generality that F is localized in ε^3

$$\left(\sigma_F^{L^*}\right)^2 = -2\pi \int_{-1}^{1} F(t) L_t F(t)\ dt, \tag{11}$$

where $L_t = \frac{d}{dt}(1-t^2)\frac{d}{dt}$ denotes the Legendre operator. The localization in momentum $\Delta_F^{L^*}$ is measured by the square root of the variance, i.e., $\Delta_F^{L^*} = \sigma_F^{L^*}$. In consequence, we are able to state the *uncertainty principle* for spherical functions (cf. [5, 9, 16]) which says that sharp localization in the space domain as well as in the momentum domain is not possible.

Theorem 3. (Uncertainty Principle) *Let $F \in \mathcal{H}_2(\Omega)$ with $\|F\|_{L^2(\Omega)} = 1$. Then $(\sigma_F)^2 \left(\sigma_F^{L^*}\right)^2 \geq |e_F|^2$ and (if $e_F \neq 0$)*

$$\Delta_F \Delta_F^{L^*} \geq 1\,. \tag{12}$$

The left hand side in (12) is called uncertainty product.

To avoid that the expectation value in momentum domain is vector valued and equal to 0 for all functions we start over considering the negative Beltrami operator $-\Delta^*$ instead of L^*. Therefore, we have to assume that $F \in \mathcal{H}_4(\Omega)$ with $\|F\|_{L^2(\Omega)} = 1$. This leads to the expectation value of $-\Delta^*$

$$e_F^{-\Delta^*} = \int_\Omega \left(-\Delta^*_\eta F(\eta)\right) F(\eta)\ d\omega(\eta) = \left(\sigma_F^{L^*}\right)^2 \tag{13}$$

which turns out to be the same as the variance of L^* (see (10)). The variance with respect to $-\Delta^*$ is defined as

$$\left(\sigma_F^{-\Delta^*}\right)^2 = \int_\Omega \left|\left((-\Delta^*_\eta) - e_F^{-\Delta^*}\right) F(\eta)\right|^2\ d\omega(\eta) = e_F^{(-\Delta^*)^2} - \left(e_F^{-\Delta^*}\right)^2 \tag{14}$$

and the localization in momentum in terms of these expressions reads as follows (if $e_F^{-\Delta^*} \neq 0$)

$$\Delta_F^{-\Delta^*} = \frac{\sigma_F^{-\Delta^*}}{\left(\frac{e_F^{(-\Delta^*)^2} - \left(e_F^{-\Delta^*}\right)^2}{e_F^{-\Delta^*}}\right)^{\frac{1}{2}}} = \left(e_F^{-\Delta^*}\right)^{\frac{1}{2}} = \Delta_F^{L^*}. \tag{15}$$

Note that this quantity does not change. Thus, the uncertainty principle (12) is left unmodified.

3. The Bernstein Wavelet

In this section we introduce the Bernstein scaling function and the corresponding Bernstein wavelet. The name *Bernstein* is motivated by the fact that it is proportional to the Bernstein polynomial $B^{2^J-1}_{2^J-1}$ on the interval $[-1, 1]$ (see, for instance, [17]). Later on, we also derive certain properties of this kernel.

Definition 4. (Bernstein Scaling Function) Let $\eta \in \Omega$ be fixed. The Bernstein scaling function of scale $J \in \mathbb{N}_0$ is given by

$$\Phi_J^B(\xi, \eta) = \frac{2^{J-2}}{\pi} \left(\frac{1 + \xi \cdot \eta}{2}\right)^{2^J-1}. \tag{16}$$

As usual in linear wavelet theory, we introduce the Bernstein wavelet of a certain scale by differences of scaling functions of two consecutive scales.

Definition 5. (Bernstein Wavelet) Let $\eta \in \Omega$ be fixed. The Bernstein wavelet of scale $J \in \mathbb{N}_0$ is given by

$$\Psi_J^B(\xi, \eta) = \Phi_{J+1}^B(\xi, \eta) - \Phi_J^B(\xi, \eta).$$

Clearly, we are able to understand Φ_J^B as bandlimited radial basis function with symbol $\{\Phi_J^{B\wedge}(n)\}_{n\in\mathbb{N}_0}$, such that

$$\Phi_J^B(\xi,\eta) = \Phi_J^B(t) = \sum_{n=0}^{2^J-1} \Phi_J^{B\wedge}(n)\frac{2n+1}{4\pi}P_n(t), \quad t = \xi\cdot\eta.$$

Interestingly, we find an explicit representation of the symbol by the following theorem.

Theorem 6. (Symbol of the Bernstein Scaling Function) *Suppose that $J \in \mathbb{N}_0$. Then the scaling function $\Phi_J^B(\cdot,\eta)$ of scale J possesses the symbol given by*

$$\Phi_J^{B\wedge}(n) = \begin{cases} \frac{(2^J)!(2^J-1)!}{(2^J-n-1)!(2^J+n)!} & \text{if } n \in [0,2^J), \\ 0 & \text{else.} \end{cases} \tag{17}$$

Before going into the details of the proof of Theorem 6, we borrow the following recursion formulae for the Legendre polynomials from [9, 13]: First, suppose that $n \geq 0$, then

$$(n+1)P_{n+1}(t) + nP_{n-1}(t) - (2n+1)tP_n(t) = 0, \tag{18}$$

for $t \in [-1,1]$ with $P_{-1}(t) = 0$. Second, let $n \geq 0$, then

$$P_n'(t) - P_{n-2}'(t) = (2n-1)P_{n-1}(t), \tag{19}$$

for $t \in [-1,1]$ with $P_{-1}(t) = P_{-2}(t) = 0$.

Based on these recursion formulae we are able to verify Theorem 6.

Proof. Formally, the symbol of Φ_J^B can be computed by solving the integral expression given by

$$\Phi_J^{B\wedge}(n) = 2\pi\int_{-1}^{1} \frac{2^{J-2}}{\pi}\left(\frac{1+t}{2}\right)^{2^J-1} P_n(t)\ dt.$$

For the following considerations we use the following abbreviation

$$I_{n,k} = \int_{-1}^{1} (1+t)^k P_n(t)\ dt. \tag{20}$$

Straightforward integration yields

$$I_{0,k} = \frac{2^{k+1}}{k+1}, \qquad I_{1,k} = I_{0,k}\frac{k}{k+2}.$$

From the recurrence formula (18) we obtain ($n \geq 1$)

$$(n+1)I_{n+1,k} + nI_{n-1,k} - (2n+1)I_{n,k+1} + (2n+1)I_{n,k} = 0.$$

Furthermore, we find by partial integration and (19) that

$$(2n+1)I_{n,k+1} = -(k+1)(I_{n+1,k} - I_{n-1,k}).$$

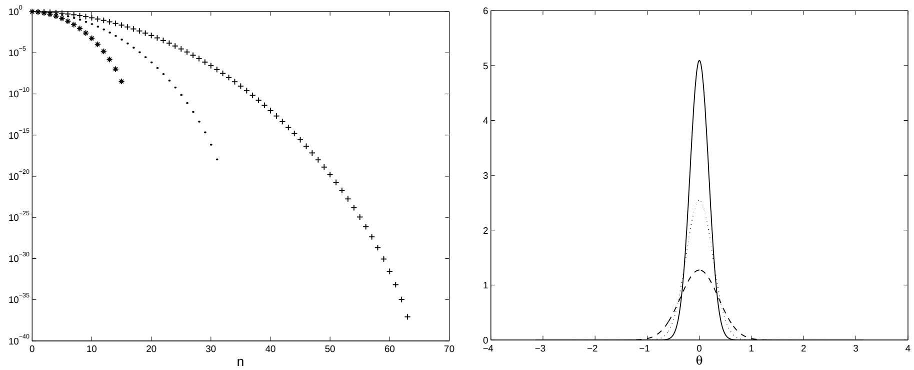

Figure 1: Left: Non-zero elements of the symbol of Φ_4^B (stars), Φ_5^B (dots), and Φ_6^B (pluses). Right: Bernstein scaling function $\Phi_4^B(\cos\theta)$ (dashed), $\Phi_5^B(\cos\theta)$ (dotted), and $\Phi_6^B(\cos\theta)$ (solid)

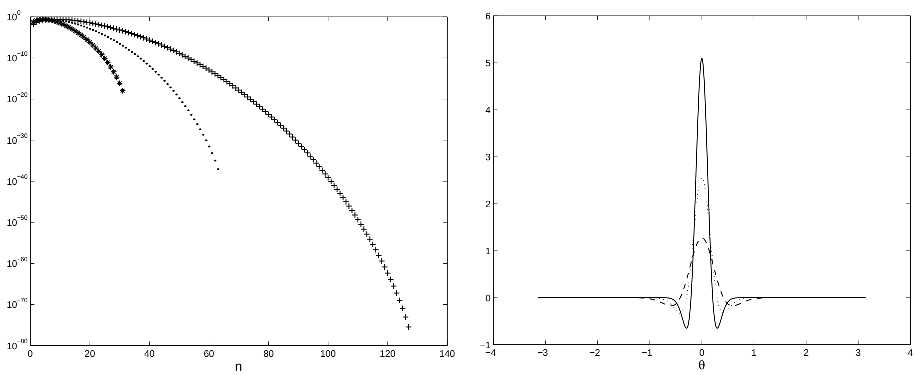

Figure 2: Left: Non-zero elements of the symbol of Ψ_4^B (stars), Ψ_5^B (dots), and Ψ_6^B (pluses). Right: Bernstein wavelet $\Psi_4^B(\cos\theta)$ (dashed), $\Psi_5^B(\cos\theta)$ (dotted), and $\Psi_6^B(\cos\theta)$ (solid)

Combining these results we arrive at the following recursion formula:

$$(n+k+2)I_{n+1,k} = -(2n+1)I_{n,k} + (k+1-n)I_{n-1,k}.$$

By letting $k = 2^J - 1$ this gives us the following result.

$$\Phi_J^{B\wedge}(0) = 1, \qquad \Phi_J^{B\wedge}(1) = \frac{2^J-1}{2^J+1},$$

$$(n+2^J+1)\Phi_J^{B\wedge}(n+1) = -(2n+1)\Phi_J^{B\wedge}(n) + (2^J-n)\Phi_J^{B\wedge}(n-1).$$

By induction over n we obtain the desired result. □

These considerations also help us to compute an explicit representation of the $L^2(\Omega)$-norm of the Bernstein scaling function.

Lemma 7. (Norm of the Bernstein Scaling Function) *Let $J \in \mathbb{N}_0$ and $\eta \in \Omega$ be fixed. Then the $L^2(\Omega)$-norm of the Bernstein scaling function Φ_J^B of scale J is given by*

$$\|\Phi_J^B(\cdot,\eta)\|_{L^2(\Omega)} = \frac{2^{J-1}}{\sqrt{\pi(2^{J+1}-1)}}. \tag{21}$$

Proof. The squared norm of Φ_J^B can be evaluated by aid of

$$\begin{aligned}\|\Phi_J^B(\cdot,\eta)\|^2_{L^2(\Omega)} &= 2\pi \int_{-1}^{1} \left(\frac{2^{J-2}}{\pi} \left(\frac{1+t}{2}\right)^{2^J-1} \right)^2 dt \\ &= 2\pi \int_{-1}^{1} \frac{2^{2J-4}}{\pi^2} \left(\frac{1+t}{2}\right)^{2^{J+1}-2} P_0(t)\, dt.\end{aligned}$$

Clearly, the latter integral is of the same type as in (20). Thus, we find by analogous considerations and $k = 2^{J+1}-2$ that

$$\|\Phi_J^B(\cdot,\eta)\|^2_{L^2(\Omega)} = \frac{2^{2J-2}}{\pi(2^{J+1}-1)}.$$ □

We denote the $L^2(\Omega)$-normalized Bernstein scaling function of scale J by

$$\tilde{\Phi}_J^B(\cdot,\eta) = \frac{\Phi_J^B(\cdot,\eta)}{\|\Phi_J^B(\cdot,\eta)\|_{L^2(\Omega)}}.$$

Based on the results obtained so far, we are now able to prove that the Bernstein scaling function forms an approximate identity.

3.1. Approximate Identity

Loosely spoken an approximate identity is a family of kernels depending on a parameter which converges to the Dirac distribution as this parameter tends to a limit value. In our case of scaling functions this parameter is the scale J which tends to infinity. The third property of the generator, i.e., (3) in Definition 1, ensures that any scaling function naturally forms an approximate identity.

In conclusion, we have to show in the following that this limit holds true for the Bernstein scaling function.

Theorem 8. (Approximate Identity) *The Bernstein scaling function*

$\{\Phi_J^B\}_{J\in\mathbb{N}_0}$ *forms an approximate identity.*

Proof. It is sufficient to prove that $\lim_{J\to\infty}\Phi_J^{B\wedge}(n)=1$ for fixed $n\in\mathbb{N}_0$. To accomplish this we take the logarithm of (17). Then

$$\log\Phi_J^{B\wedge}(n)=\sum_{k=1}^{2^J}\log k+\sum_{k=1}^{2^J-1}\log k-\sum_{k=1}^{2^J-n-1}\log k-\sum_{k=1}^{2^J+n}\log k$$
$$=\sum_{k=2^J-n}^{2^J}\log k-\sum_{k=2^J}^{2^J+n}\log k=\sum_{k=1}^{n}\log\left(\frac{2^J-k}{2^J+k}\right). \quad (22)$$

Since $n\in\mathbb{N}_0$ is fixed, the series in (22) is finite. Then we obtain

$$\lim_{J\to\infty}\log\Phi_J^{B\wedge}(n)=\lim_{J\to\infty}\sum_{k=1}^{n}\log\left(\frac{2^J-k}{2^J+k}\right)=0,$$

and, therefore, $\lim_{J\to\infty}\Phi_J^{B\wedge}(n)=1$. □

3.2. Monotonicity

Since the first property (1) constituting a scaling function is obviously fulfilled by $\Phi_J^{B\wedge}(0)$, we are left with the monotonicity condition (2).

Theorem 9. (Monotonicity) *The generator of the Bernstein scaling function $\{\Phi_J^B\}_{J\in\mathbb{N}_0}$ is monotonically increasing, i.e., for all $J,J'\in\mathbb{N}_0$ with $J\le J'$ and all $n\in\mathbb{N}$ we have*

$$0\le\Phi_J^{B\wedge}(n)\le\Phi_{J'}^{B\wedge}(n).$$

Proof. From Theorem 6 it is clear that $0\le\Phi_J^{B\wedge}(n)$ for all $J\in\mathbb{N}_0$ and all $n\in\mathbb{N}$. Therefore, it remains to prove the inequality between two different scales. It suffices to show that, for fixed $n\in\mathbb{N}_0$ and arbitrary $J,J'\in\mathbb{N}_0$ with $J\le J'$, the following ratio holds

$$\frac{\Phi_J^{B\wedge}(n)}{\Phi_{J'}^{B\wedge}(n)}\le 1.$$

Similar to Theorem 8 we consider the logarithms and obtain

$$\log\frac{\Phi_J^{B\wedge}(n)}{\Phi_{J'}^{B\wedge}(n)}=\sum_{k=1}^{n}\log\left(\frac{(2^J-k)(2^{J'}+k)}{(2^J+k)(2^{J'}-k)}\right)$$

which we want to be less or equal to 0. Thus, it is sufficient to show that the argument of each logarithm in this sum is smaller than 1. This can be easily

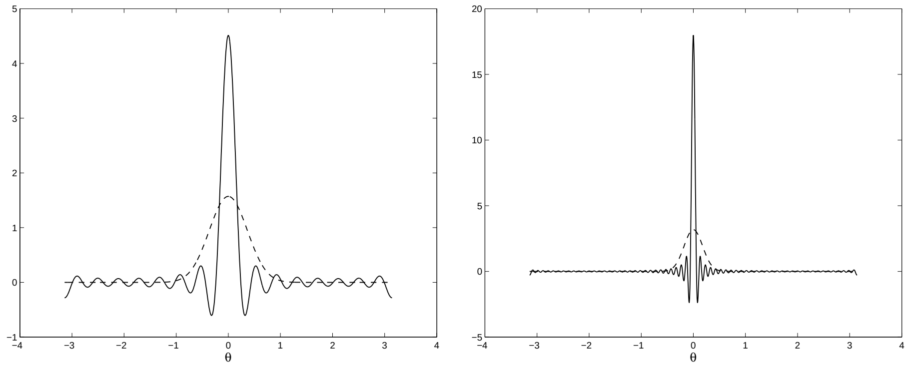

Figure 3: Left: The $L^2(\Omega)$-normalized Shannon scaling function $\tilde{\Phi}_4^S(t)$ (solid) and the $L^2(\Omega)$-normalized Bernstein scaling function $\tilde{\Phi}_4^B(t)$ (dashed). Right: The same for scale 6, $\tilde{\Phi}_6^S(t)$ (solid) and $\tilde{\Phi}_6^B(t)$ (dashed)

seen since from $J \leq J'$ follows $2^J \leq 2^{J'}$ and therefore,

$$\frac{(2^J - k)(2^{J'} + k)}{(2^J + k)(2^{J'} - k)} = \frac{2^{J+J'} - k^2 + k(2^J - 2^{J'})}{2^{J+J'} - k^2 + k(2^{J'} - 2^J)} \leq 1. \qquad \square$$

Now all conditions of Definition 1 are guaranteed and we can rightly speak of the Bernstein *scaling function*.

4. Oscillations

Throughout this section we are concerned with the oscillations of the Bernstein and Shannon scaling function, respectively. Before going into the mathematical details we like to motivate our considerations by comparing in Figure 3 both normalized kernels $\tilde{\Phi}_4^B$ and $\tilde{\Phi}_4^S$, and $\tilde{\Phi}_6^B$ and $\tilde{\Phi}_6^S$, respectively. Although both kernels are bandlimited to the same degree in each plot, we notice how smoothly the Bernstein kernel decays to zero whereas the Shannon kernel oscillates heavily.

Now, mathematically spoken, we understand *oscillations* as sign changes of the slope.

In the Shannon case this consideration can be simplified if we prove that this kernel possesses only simple zeros which imply slope changes. To be more specific, we borrow from [12] an alternative representation of $\Phi_J^S(t)$, i.e.,

$$\Phi_J^S(t) = \sum_{n=0}^{2^J-1} \frac{2n+1}{4\pi} P_n(t) \tag{23}$$

$$= \begin{cases} \frac{2^J}{4\pi} \frac{P_{2^J}(t) - P_{2^J-1}(t)}{t-1} & \text{for } t \in [-1,1) \\ \frac{2^{2J}}{4\pi} & \text{for } t = 1. \end{cases} \tag{24}$$

Theorem 10. (Zeros of the Shannon Kernel) *Let* $J \in \mathbb{N}_0$. *Then the Shannon scaling function* $\Phi_J^S(t)$ *possesses exactly* $2^J - 1$ distinct, simple *zeros on the interval* $[-1, 1]$.

Proof. From (23) we see that $\Phi_J^S(t)$ possesses at most $2^J - 1$ zeros. The case $J \leq 1$ is trivial. For $J \geq 2$ it is sufficient to show that the numerator $N_J(t) = P_{2^J}(t) - P_{2^J-1}(t)$ in (24) possesses $2^J - 1$ zeros in $(-1, 1)$. It is known that the Legendre polynomial P_n possesses n distinct (simple) zeros on $(-1, 1)$ (see [1]). Thus, we denote by $t_1, \ldots, t_{2^J}$ the zeros of P_{2^J} in increasing order, and with $s_1, \ldots, s_{2^J-1}$ the zeros of P_{2^J-1}, respectively. Furthermore, the zeros of P_{2^J-1} divide the zeros of P_{2^J} (see [1]). In detail, we have $s_i \in (t_i, t_{i+1})$. Clearly, the expression

$$N_J(s_i) = P_{2^J}(s_i) - P_{2^J-1}(s_i) = P_{2^J}(s_i)$$

has the same alternating sign as $P_{2^J}(s_i), i = 1, \ldots, 2^J - 1$. Therefore, we know from the Intermediate Value Theorem of classical analysis that $N_J(t)$ possesses at least $2^J - 2$ zeros in (t_1, t_{2^J}). Finally, we show that the remaining zero is placed in $(-1, t_1)$. This can be seen as follows: Keeping in mind that $P_n(-1) = (-1)^n$ we know that P_{2^J} has a negative sign in (t_1, t_2) since $P_{2^J}(-1) = 1$ and t_1 denotes the first zero. Thus, we obtain that $N_J(s_1)$ is negative and $N_J(-1) = 2$ is positive. Again, we can apply the Intermediate Value Theorem and arrive at the desired result. □

As an immediate consequence we find that the Shannon scaling function of scale $J \geq 2$ possesses $2^J - 2$ sign changes of its slope on $(-1, 1)$, i.e., oscillations. Obviously, the Bernstein scaling function as introduced in Definition 16 features no oscillations since -1 is a zero of multiplicity $2^J - 1$.

5. Space Uncertainty

In this section we derive an explicit representation of the space uncertainty of the Bernstein scaling function. In fact, we are led to the surprising result that the Shannon scaling function possesses the same space uncertainty.

Theorem 11. (Space Uncertainty of the Bernstein Scaling Function) *Suppose that $J \in \mathbb{N}$ and $\eta \in \Omega$ is fixed. Then the space uncertainty of the $L^2(\Omega)$-normalized Bernstein scaling function $\tilde{\Phi}_J^B(\cdot,\eta)$ of scale J is given by*

$$\Delta_{\tilde{\Phi}_J^B(\cdot,\eta)} = \frac{\sqrt{2^{J+1}-1}}{2^J-1},$$

where the expectation value and the variance in the space domain read as follows

$$e_{\tilde{\Phi}_J^B(\cdot,\eta)} = \frac{2^J-1}{2^J}\eta, \qquad \sigma^2_{\tilde{\Phi}_J^B(\cdot,\eta)} = \frac{2^{J+1}-1}{2^{2J}}.$$

Proof. Without loss of generality we set $\eta = \varepsilon^3$. From (7) we obtain

$$\begin{aligned} e_{\tilde{\Phi}_J^B(\cdot,\varepsilon^3)} &= \left(2\pi \int_{-1}^{1} t \left(\tilde{\Phi}_J^B(t)\right)^2 \, dt\right)\varepsilon^3 \\ &= \left(\frac{2\pi}{\|\Phi_J^B(\cdot,\varepsilon^3)\|^2_{L^2(\Omega)}} \int_{-1}^{1} t \left(\Phi_J^B(t)\right)^2 \, dt\right)\varepsilon^3. \end{aligned}$$

Substituting Φ_J^B by (16) and $\|\Phi_J^B(\cdot,\varepsilon^3)\|_{L^2(\Omega)}$ by (21) we deduce

$$\begin{aligned} e_{\tilde{\Phi}_J^B(\cdot,\varepsilon^3)} &= \left(\frac{(2^{J+2}-2)\pi^2}{2^{2J-2}} \int_{-1}^{1} t \left(\frac{2^{J-2}}{\pi}\left(\frac{1+t}{2}\right)^{2^J-1}\right)^2 \, dt\right)\varepsilon^3 \\ &= \left(\frac{2^{J+2}-2}{2^{2J-2}} \int_{-1}^{1} t \, 2^{2J-4}\left(\frac{1+t}{2}\right)^{2^{J+1}-2} \, dt\right)\varepsilon^3 \\ &= \left(\frac{2^{J+1}-1}{2} \int_{-1}^{1} \left(\frac{1+t}{2}\right)^{2^{J+1}-2} P_1(t) \, dt\right)\varepsilon^3. \end{aligned}$$

Again, we observe that the latter term is of the same type as studied in (20). By letting $n=1$ and $k = 2^{J+1}-2$ we find the desired result

$$e_{\tilde{\Phi}_J^B(\cdot,\varepsilon^3)} = \frac{2^J-1}{2^J}\varepsilon^3.$$

As an immediate consequence we deduce from (8) for the variance

$$\sigma^2_{\tilde{\Phi}_J^B(\cdot,\varepsilon^3)} = 1 - \frac{(2^J-1)^2}{2^{2J}} = \frac{2^{J+1}-1}{2^{2J}}, \tag{25}$$

such that the space uncertainty is given by

$$\Delta_{\tilde{\Phi}_J^B(\cdot,\varepsilon^3)} = \frac{\sqrt{2^{J+1}-1}}{2^J-1}. \qquad \square \tag{26}$$

Next we deal with the amazing result that the $L^2(\Omega)$-normalized Shannon scaling function possesses the same space uncertainty as the normalized Bernstein scaling function. First, we need a preliminary result on the norm of the Shannon scaling function of an arbitrary scale.

Lemma 12. (Norm of the Shannon Scaling Function of Scale J) *Suppose that $J \in \mathbb{N}_0$ and $\eta \in \Omega$ is fixed. Then the $L^2(\Omega)$-norm of Shannon scaling function $\Phi_J^B(\cdot, \eta)$ of scale J is given by*

$$\|\Phi_J^S(\cdot, \eta)\|_{L^2(\Omega)} = \frac{2^{J-1}}{\sqrt{\pi}}. \tag{27}$$

Proof. This result follows immediately from the definition of the Shannon generator (see Definition (2)) and the following norm identity

$$\|\Phi_J^S(\cdot, \eta)\|_{L^2(\Omega)}^2 = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \left(\Phi_J^{S\wedge}(n)\right)^2. \qquad \square$$

Based on the latter result we are now in the position to investigate the space uncertainty of the Shannon scaling function.

Theorem 13. (Space Uncertainty of the Shannon Scaling Function) *Assume that $J \in \mathbb{N}$ and $\eta \in \Omega$ is fixed. Then the space uncertainty of the $L^2(\Omega)$-normalized Shannon scaling function $\tilde{\Phi}_J^S(\cdot, \eta)$ of scale J is given by*

$$\Delta_{\tilde{\Phi}_J^S(\cdot,\eta)} = \frac{\sqrt{2^{J+1}-1}}{2^J - 1},$$

where the expectation value and the variance in the space domain read as follows

$$e_{\tilde{\Phi}_J^S(\cdot,\eta)} = \frac{2^J - 1}{2^J}\eta, \qquad \sigma^2_{\tilde{\Phi}_J^S(\cdot,\eta)} = \frac{2^{J+1}-1}{2^{2J}}.$$

Proof. We start with the explicit formula for the expectation value (see also [11])

$$\begin{aligned} e_{\tilde{\Phi}_J(\cdot,\varepsilon^3)} &= \left(\frac{1}{2\pi \|\Phi_J(\cdot, \varepsilon^3)\|_{L^2(\Omega)}^2} \sum_{n=0}^{\infty} (n+1)\, \Phi_J^\wedge(n) \Phi_J^\wedge(n+1) \right) \varepsilon^3 \\ &= \left(\frac{2 \sum\limits_{n=0}^{\infty} (n+1) \Phi_J^\wedge(n) \Phi_J^\wedge(n+1)}{\sum\limits_{n=0}^{\infty} (2n+1) \left(\Phi_J^\wedge(n)\right)^2} \right) \varepsilon^3. \end{aligned}$$

Without loss of generality we set $\eta = \varepsilon^3$. The variance $\sigma^2_{\tilde{\Phi}_J(\cdot,\eta)}$ can then be computed as in (8) and for the space uncertainty we have

$$\Delta_{\tilde{\Phi}_J(\cdot,\eta)} = \frac{\sigma_{\tilde{\Phi}_J(\cdot,\eta)}}{|e_{\tilde{\Phi}_J(\cdot,\eta)}|} = \left(\left(\frac{\sum\limits_{n=0}^{\infty} (2n+1) \left(\Phi_J^\wedge(n)\right)^2}{2 \sum\limits_{n=0}^{\infty} (n+1) \Phi_J^\wedge(n) \Phi_J^\wedge(n+1)} \right)^2 - 1 \right)^{\frac{1}{2}} .$$

The results of the theorem are obtained by combining Definition 2 with these formulae. □

Obviously, the last result indicates together with Theorem 10 that the quantity Δ_F for the space localization (see (6)) does not reflect the amount of oscillations. This is nicely illustrated in Figure 3.

6. Uncertainty in Momentum and the Uncertainty Product

We now derive the explicit formulae for the uncertainty in momentum and for the uncertainty product of the Bernstein scaling function. We again compare these quantities with their corresponding values in the Shannon case.

Theorem 14. (Uncertainty in Momentum of the Bernstein Scaling Function) *Let $J \in \mathbb{N}$ and $\eta \in \Omega$ be fixed. Then the uncertainty in momentum of the $L^2(\Omega)$-normalized Bernstein scaling function $\tilde{\Phi}_J^B(\cdot,\eta)$ of scale J is given by*

$$\Delta^{-\Delta^*}_{\tilde{\Phi}_J^B(\cdot,\eta)} = \Delta^{L^*}_{\tilde{\Phi}_J^B(\cdot,\eta)} = \frac{\sqrt{2^J - 1}}{\sqrt{2}} ,$$

where the expectation value (which coincides with the variance with respect to the L^* operator) and the variance in the momentum domain (with respect to the operator $-\Delta^*$) read as follows

$$e^{-\Delta^*}_{\tilde{\Phi}_J^B(\cdot,\eta)} = \frac{2^J - 1}{2} = \left(\sigma^{L^*}_{\tilde{\Phi}_J^B(\cdot,\eta)}\right)^2 ,$$

$$\left(\sigma^{-\Delta^*}_{\tilde{\Phi}_J^B(\cdot,\eta)}\right)^2 = \frac{(2^J - 1)^2}{4} \frac{3 \cdot 2^{J+1} - 1}{2^{J+1} - 3} .$$

Proof. Without loss of generality we set $\eta = \varepsilon^3$. For the calculation of $\left(\sigma^{L^*}_{\tilde{\Phi}_J^B(\cdot,\varepsilon^3)}\right)^2$ we use formula (11)

$$\left(\sigma^{L^*}_{\tilde{\Phi}_J^B(\cdot,\varepsilon^3)}\right)^2 = -2\pi \int_{-1}^{1} \tilde{\Phi}_J^B(t) L_t \tilde{\Phi}_J^B(t) \; dt$$

$$= \frac{-2\pi}{\|\Phi_J^B(\cdot,\varepsilon^3)\|^2_{L^2(\Omega)}} \int_{-1}^{1} \Phi_J^B(t) L_t \Phi_J^B(t)\ dt.$$

Using (16) and (21) we get

$$\begin{aligned}
&\left(\sigma^{L^*}_{\tilde\Phi_J^B(\cdot,\varepsilon^3)}\right)^2 \\
&\quad= \frac{-2\pi^2(2^{J+1}-1)}{2^{2J-2}} \int_{-1}^{1} \frac{2^{J-2}}{\pi}\left(\frac{1+t}{2}\right)^{2^J-1} L_t \frac{2^{J-2}}{\pi}\left(\frac{1+t}{2}\right)^{2^J-1}\ dt \\
&\quad= \frac{-2\pi^2(2^{J+1}-1)}{2^{2J-2}}\frac{2^{2J-4}}{\pi^2} \int_{-1}^{1} \left(\frac{1+t}{2}\right)^{2^J-1} L_t \left(\frac{1+t}{2}\right)^{2^J-1}\ dt \\
&\quad= \frac{-(2^{J+1}-1)}{2} \int_{-1}^{1} \left(\frac{1+t}{2}\right)^{2^J-1} L_t \left(\frac{1+t}{2}\right)^{2^J-1}\ dt. \qquad (28)
\end{aligned}$$

Simple differentiation yields the result of the application of the Legendre operator L_t

$$\begin{aligned}
L_t\left(\frac{1+t}{2}\right)^{2^J-1} &= \frac{d}{dt}(1-t^2)\frac{d}{dt}\left(\frac{1+t}{2}\right)^{2^J-1} \\
&= \frac{2^J-1}{2}\left(\frac{1+t}{2}\right)^{2^J-2}\left(2^J-2-2^Jt\right).
\end{aligned}$$

Substituting this in (28) we obtain by aid of the Legendre polynomials $P_0(t) = 1$ and $P_1(t) = t$

$$\begin{aligned}
&\left(\sigma^{L^*}_{\tilde\Phi_J^B(\cdot,\varepsilon^3)}\right)^2 \\
&= \frac{-(2^{J+1}-1)}{2} \int_{-1}^{1} \left(\frac{1+t}{2}\right)^{2^J-1} \frac{2^J-1}{2}\left(\frac{1+t}{2}\right)^{2^J-2}\left(2^J-2-2^Jt\right)\ dt \\
&= \frac{-(2^{J+1}-1)(2^J-1)}{2^2} \int_{-1}^{1} \left(\frac{1+t}{2}\right)^{2^{J+1}-3}\left(2^J-2-2^Jt\right)\ dt \\
&= \frac{-(2^{J+1}-1)(2^J-1)}{2^2}\left((2^J-2)\int_{-1}^{1} P_0(t)\left(\frac{1+t}{2}\right)^{2^{J+1}-3}\ dt\right. \\
&\qquad\left. - 2^J \int_{-1}^{1} P_1(t)\left(\frac{1+t}{2}\right)^{2^{J+1}-3}\ dt\right).
\end{aligned}$$

Once again we apply (20) to come up with

$$\left(\sigma^{L^*}_{\tilde\Phi_J^B(\cdot,\varepsilon^3)}\right)^2$$

$$= \frac{-(2^{J+1}-1)(2^J-1)}{2^2}\left((2^J-2)\frac{2}{2^{J+1}-2} - 2^J\frac{2(2^{J+1}-3)}{(2^{J+1}-2)(2^{J+1}-1)}\right)$$
$$= \frac{2^J-1}{2}.$$

From (13) we know that this also corresponds to the expectation value $e^{-\Delta^*}_{\tilde{\Phi}^B_J(\cdot,\varepsilon^3)}$. The uncertainty in momentum is given by

$$\Delta^{-\Delta^*}_{\tilde{\Phi}^B_J(\cdot,\varepsilon^3)} = \Delta^{L^*}_{\tilde{\Phi}^B_J(\cdot,\varepsilon^3)} = \sigma^{L^*}_{\tilde{\Phi}^B_J(\cdot,\varepsilon^3)} = \frac{\sqrt{2^J-1}}{\sqrt{2}}.$$

Finally, the variance $\left(\sigma^{-\Delta^*}_{\tilde{\Phi}^B_J(\cdot,\eta)}\right)^2$ is computed via (14)

$$\begin{aligned}\left(\sigma^{-\Delta^*}_{\tilde{\Phi}^B_J(\cdot,\eta)}\right)^2 &= e^{(-\Delta^*)^2}_{\tilde{\Phi}^B_J(\cdot,\eta)} - \left(e^{-\Delta^*}_{\tilde{\Phi}^B_J(\cdot,\eta)}\right)^2\\ &= \int_\Omega \left(\left(-\Delta^*_\xi\right)^2 \tilde{\Phi}^B_J(\xi,\eta)\right)\tilde{\Phi}^B_J(\xi,\eta)\; d\omega(\xi) - \left(\frac{2^J-1}{2}\right)^2\\ &= \int_\Omega \left(\Delta^*_\xi\tilde{\Phi}^B_J(\xi,\eta)\right)\left(\Delta^*_\xi\tilde{\Phi}^B_J(\xi,\eta)\right)\; d\omega(\xi) - \left(\frac{2^J-1}{2}\right)^2.\end{aligned}$$

Again, this integral can be simplified as we are considering the structure as radial basis function. Thus, without loss of generality we set $\eta = \varepsilon^3$.

$$\begin{aligned}\int_\Omega \left(\Delta^*_\xi\tilde{\Phi}^B_J(\xi,\varepsilon^3)\right)\left(\Delta^*_\xi\tilde{\Phi}^B_J(\xi,\varepsilon^3)\right)\; d\omega(\xi) &= 2\pi\int_{-1}^1 \left(L_t\tilde{\Phi}^B_J(t)\right)^2\; dt\\ &= \frac{2\pi}{\|\Phi^B_J(\cdot,\varepsilon^3)\|^2_{L^2(\Omega)}}\int_{-1}^1 \left(L_t\Phi^B_J(t)\right)^2\; dt.\end{aligned}$$

Inserting our known expressions for the norm (21) and for $L_t\Phi^B_J(t)$

$$\begin{aligned}&\int_\Omega \left(\Delta^*_\xi\tilde{\Phi}^B_J\;(\xi,\varepsilon^3)\right)^2\; d\omega(\xi)\\ &\quad= \frac{2\pi^2(2^{J+1}-1)}{2^{2J-2}}\left(\frac{2^{J-2}}{\pi}\right)^2\int_{-1}^1 \left(L_t\left(\frac{1+t}{2}\right)^{2^J-1}\right)^2\; dt\\ &\quad= \frac{2^{J+1}-1}{2}\int_{-1}^1 \left(\frac{2^J-1}{2}\left(\frac{1+t}{2}\right)^{2^J-2}(2^J-2-2^Jt)\right)^2\; dt\\ &\quad= \frac{(2^{J+1}-1)(2^J-1)^2}{2^3}\int_{-1}^1 \left(\frac{1+t}{2}\right)^{2^{J+1}-4}\left(2^J-2-2^Jt\right)^2\; dt.\end{aligned}$$

The last integral can be decomposed into 3 integrals of type (20) which compute to

$$e^{(-\Delta^*)^2}_{\tilde{\Phi}^B_J(\cdot,\varepsilon^3)} = \frac{(2^{J+1}-1)(2^J-1)^2}{2^3}\frac{2^3(2^J-1)}{(2^{J+1}-3)(2^{J+1}-1)} = \frac{(2^J-1)^3}{2^{J+1}-3}.$$

Therefore, we obtain

$$\left(\sigma^{-\Delta^*}_{\tilde{\Phi}^B_J(\cdot,\varepsilon^3)}\right)^2 = \frac{(2^J-1)^3}{2^{J+1}-3} - \left(\frac{2^J-1}{2}\right)^2 = \frac{(2^J-1)^2}{4}\frac{3\cdot 2^{J+1}-1}{2^{J+1}-3}. \qquad \square$$

For comparison we also calculate the uncertainty in momentum for the Shannon scaling function.

Theorem 15. (Uncertainty in Momentum of the Shannon Scaling Function) *Let $J \in \mathbb{N}$ and $\eta \in \Omega$ be fixed. Then the uncertainty in momentum of the $L^2(\Omega)$-normalized Shannon scaling function $\tilde{\Phi}^S_J(\cdot,\eta)$ of scale J is given by*

$$\Delta^{-\Delta^*}_{\tilde{\Phi}^S_J(\cdot,\eta)} = \Delta^{L^*}_{\tilde{\Phi}^S_J(\cdot,\eta)} = \frac{\sqrt{2^{2J}-1}}{\sqrt{2}},$$

where the expectation value (which coincides with the variance with respect to the L^* operator) and the variance in the momentum domain (with respect to the operator $-\Delta^*$) read as follows

$$e^{-\Delta^*}_{\tilde{\Phi}^S_J(\cdot,\eta)} = \frac{2^{2J}-1}{2} = \left(\sigma^{L^*}_{\tilde{\Phi}^S_J(\cdot,\eta)}\right)^2,$$

$$\left(\sigma^{-\Delta^*}_{\tilde{\Phi}^S_J(\cdot,\eta)}\right)^2 = \frac{(2^J-1)^2(2^J+1)^2}{12}.$$

Proof. We start with the following representation of $e^{-\Delta^*}_{\tilde{\Phi}_J(\cdot,\eta)}$ (see also [11])

$$\begin{aligned}\left(\sigma^{L^*}_{\tilde{\Phi}_J(\cdot,\eta)}\right)^2 &= \frac{1}{4\pi\|\Phi_J(\cdot,\eta)\|^2_{L^2(\Omega)}}\sum_{n=0}^{\infty}(2n+1)n(n+1)(\Phi_J^\wedge(n))^2 \\ &= \frac{\sum\limits_{n=0}^{\infty}(2n+1)n(n+1)(\Phi_J^\wedge(n))^2}{\sum\limits_{n=0}^{\infty}(2n+1)(\Phi_J^\wedge(n))^2}. \qquad (29)\end{aligned}$$

The uncertainty in momentum is then given by the square root of (29). Finally, the variance $\left(\sigma^{-\Delta^*}_{\tilde{\Phi}_J(\cdot,\eta)}\right)^2$ is deduced from

$$\left(\sigma^{-\Delta^*}_{\tilde{\Phi}_J(\cdot,\eta)}\right)^2 = e^{(-\Delta^*)^2}_{\tilde{\Phi}_J(\cdot,\eta)} - \left(e^{-\Delta^*}_{\tilde{\Phi}_J(\cdot,\eta)}\right)^2$$

$$= \frac{\sum\limits_{n=0}^{\infty}(2n+1)n^2(n+1)^2(\Phi_J^\wedge(n))^2}{\sum\limits_{n=0}^{\infty}(2n+1)(\Phi_J^\wedge(n))^2} - \left(\frac{\sum\limits_{n=0}^{\infty}(2n+1)n(n+1)(\Phi_J^\wedge(n))^2}{\sum\limits_{n=0}^{\infty}(2n+1)(\Phi_J^\wedge(n))^2}\right)^2 .$$

The results of the theorem are obtained by combining Definition 2 with these formulae. □

Now we have all the tools to discuss the uncertainty products for the two scaling functions.

It is of particular interest whether these products converge to the minimal value 1 as the scale J tends to infinity, since a value of 1 indicates that the functions are well localized in space and frequency.

Theorem 16. (Uncertainty Products of the Bernstein and of the Shannon Scaling Function) *The uncertainty product of the $L^2(\Omega)$-normalized Bernstein scaling function $\tilde{\Phi}_J^B(\cdot,\eta)$ of scale J is given by*

$$\Delta_{\tilde{\Phi}_J^B(\cdot,\eta)}\,\Delta^{L^*}_{\tilde{\Phi}_J^B(\cdot,\eta)} = \left(\frac{2^{J+1}-1}{2^{J+1}-2}\right)^{\frac{1}{2}},$$

and for $J \longrightarrow \infty$ this product tends to 1.

The uncertainty product of the $L^2(\Omega)$-normalized Shannon scaling function $\tilde{\Phi}_J^S(\cdot,\eta)$ of scale J reads as follows

$$\Delta_{\tilde{\Phi}_J^S(\cdot,\eta)}\,\Delta^{L^*}_{\tilde{\Phi}_J^S(\cdot,\eta)} = \left(\frac{(2^{J+1}-1)(2^J+1)}{2^{J+1}-2}\right)^{\frac{1}{2}}$$

and it diverges for $J \longrightarrow \infty$.

Proof. The product for the Bernstein scaling function of scale J can easily be computed from Theorem 11 and Theorem 14. In the Shannon case the result follows from Theorem 13 and Theorem 15. In both cases the behavior for $J \longrightarrow \infty$ is obvious. □

7. Concluding Remarks

In this article we have demonstrated that there exist bandlimited scaling functions without oscillations. Furthermore, the Bernstein scaling function represents an example of a family of kernels that minimizes its uncertainty product in the limit, but unlike the Gauß-Weierstraß kernels which also possess this property (cf. [5, 9]) this minimizer is bandlimited. In detail, the uncertainty product of the Bernstein scaling function tends to 1. To our knowledge it is the first bandlimited kernel presented in literature for which this property has been proven. An uncertainty product of 1 indicates that the space as well as frequency localization is well balanced. Note that the dyadic arrangement of the scales is not associated with the properties presented in this work. All of them hold true also for other types of scales. Finally, it should be emphasized that the Bernstein scaling function possesses a closed representation by elementary expressions. Especially the latter allows a fast and stable evaluation. The numerical properties of this new spherical scaling function and wavelet will be subject to further studies.

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