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ON SOLVABILITY OF NONLINEAR QUASIVARIATIONAL INEQUALITIES

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Abstract: The aim of this paper is to introduce and study a class of nonlinear quasivariational inequalities in a real Hilbert space. A perturbed iterative algorithm for finding the approximate solutions of the nonlinear quasivariational inequality is suggested. The existence and uniqueness of solution for the nonlinear quasivariational inequality is proved and the convergence of iterative sequence generated by the perturbed iterative algorithm is established.

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1. Introduction

One of the most interesting and important problems in the variational inequality theory is the development of an efficient and implementable iterative algorithm [1-4, 6, 7]. Adly [1] investigated a class of general variational inclusions with maximal monotone mapping by using the resolvent operator technique. Re-

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cently, Huang [4], Wang, Liu, Feng and Kang [7] and others introduced and studied some classes of variational inequalities, and established the existence of solutions and the convergence of iterative algorithms by applying the resolvent operator technique, respectively.

In this paper, we introduce a new class of nonlinear quasivariational inequalities, and suggest a perturbed iterative algorithm by applying the resolvent operator technique. The solvability of the nonlinear quasivariational inequality and the convergence of iterative sequence generated by the perturbed iterative algorithm are established. The results presented in this paper extend, improve and unify the corresponding results due to Dong, Lee and Huang [2], Huang, Bai, Cho and Kang [3], Huang [4], Verma [6] and Wang, Liu, Feng and Kang [7].

2. Preliminaries

Let H be a real Hilbert space with a norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$, respectively. Assume that I and 2^H denote the identity mapping on H and the family of nonempty subsets of H, respectively. Let g, m, A, B, $D : H \to H$ and $N : H \times H \to H$ be mappings. Suppose that $W : H \times H \to 2^H$ is a mapping such that $W(\cdot, y) : H \to 2^H$ is a maximal monotone operator and $\mathrm{Range}(g-m) \cap \mathrm{dom}\ W(\cdot, y) \neq \emptyset$ for each $y \in H$. For a given $f \in H$, we consider the following problem: Find $x \in H$ such that $(g-m)(x) \in \mathrm{dom}\ (W(\cdot, x))$ and

$$f \in N(A(x), B(x)) + W((g-m)(x), D(x)), \tag{2.1}$$

which is called a *nonlinnear quasi-variational inequality*, where

$$(g-m)(x) = g(x) - m(x), \quad \forall x \in H.$$

We shall separate our future investigation into two special cases:

(A) If $f = 0$, $D = I$ and $N(A(x), B(x)) = A(x) - B(x)$ for all $x \in H$, then the problem (2.1) is equivalent to finding $x \in H$ such that $(g-m)(x) \in \mathrm{dom}\,(W(\cdot, x))$ and

$$0 \in A(x) - B(x) + W((g-m)(x), x),$$

which was introduced and studied by Huang, see [4].

(B) If $f = 0$, $N(A(x), B(x)) = A(x) - B(x)$ and $W((g-m)(x), D(x)) = W(g(x))$ for all $x \in H$, then the problem (2.1) is equivalent to finding $x \in H$ such that $(g-m)(x) \in \mathrm{dom}\,(W(\cdot, x))$ and

$$0 \in A(x) - B(x) + W(g(x)),$$

which was introduced and studied by Adly, see [1].

The following definitions and results play crucial roles in this paper.

Let $W : H \to 2^H$ be a maximal monotone mapping. For a given $\rho > 0$, the resolvent operator associated with W is defined by

$$J_\rho^W(u) = (I + \rho W)^{-1}(u), \quad \forall u \in H.$$

It is known that J_ρ^W is single-valued and nonexpansive.

Definition 2.1. Let $A : H \to H$ be a mapping. The mapping $N : H \times H \to H$ is said to be:

(1) *strongly monotone* with respect to A in the first argument if there exists a constant $\alpha > 0$ such that

$$\langle N(A(x), u) - N(A(y), u), x - y\rangle \geq \alpha\|x - y\|^2, \quad \forall x, y, u \in H;$$

(2) *Lipschitz continuous* with respect to the first argument if there exists a constant $a > 0$ such that

$$\|N(x, u) - N(y, u)\| \leq a\|x - y\|, \quad \forall x, y, u \in H.$$

Similarly, we can define the Lipschitz continuity of N with respect to the second argument.

Definition 2.2. A mapping $g : H \to H$ is said to be:

(1) *strongly monotone* if there exists a constant $\eta > 0$ satisfying

$$\langle g(x) - g(y), x - y\rangle \geq \eta\|x - y\|^2, \quad \forall x, y \in H;$$

(2) *Lipschitz continuous* if there exists a constant $b > 0$ satisfying

$$\|g(x) - g(y)\| \leq b\|x - y\|, \quad \forall x, y \in H.$$

Definition 2.3. Let g, $m : H \to H$ be mappings. The mapping m is said to be *relaxed monotone* with respect to g if there exists a constant $\lambda > 0$ satisfying

$$\langle m(x) - m(y), g(x) - g(y)\rangle \geq -\lambda\|x - y\|^2, \quad \forall x, y \in H.$$

Lemma 2.1. (see [9]) *Let $\{\alpha_n\}_{n\geq 0}$, $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_n\}_{n\geq 0}$ be three nonnegative real sequences satisfying the inequality*

$$\alpha_{n+1} \leq (1 - w_n)\alpha_n + \beta_n w_n + \gamma_n, \quad \forall n \geq 0,$$

where $\{w_n\}_{n\geq 0} \in [0, 1]$, $\sum_{n=0}^\infty w_n = \infty$, $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=0}^\infty \gamma_n < \infty$. Then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 2.2. *Let ρ be a positive constant. Then the completely generalized nonlinear quasivariational inclusion* (2.1) *has a solution $x \in H$ if and only* *if the mapping* $G : H \to H$ *defined by*

$$G(x) = x - (g - m)(x) + J_\rho^{W(\cdot,D(x))}[(g - m)(x) - \rho N(A(x), B(x)) + \rho f], \quad x \in H\,, \tag{2.2}$$

has a fixed point $x \in H$.

Proof. The nonlinear quasivariational inequality (2.1) has a solution $x \in H$ if and only if

$$\begin{aligned} & f \in N(A(x), B(x)) + W((g - m)(x), D(x)) \\ \Leftrightarrow\ & \rho f - \rho N(A(x), B(x)) + (g - m)(x) \in (g - m)(x) + \rho W((g - m)(x), D(x)) \\ \Leftrightarrow\ & x = x - (g - m)(x) + J_\rho^{W(\cdot,D(x))}[(g - m)(x) - \rho N(A(x), B(x)) + \rho f]. \end{aligned}$$

This completes the proof. □

Lemma 2.2 implies that the nonlinear quasivariational inequality (2.1) is equivalent to a fixed point problems. On the basic of this observation, we now suggest and analyze the following perturbed iterative algorithm for finding the approximate solutions of the nonlinear quasivariational inequality (2.1).

Algorithm 2.1. Let A, B, D, g, $m : H \to H$, $N : H \times H \to H$ be mappings. For each $n \geq 0$, $W_n : H \times H \to 2^H$ be a mapping. For given $f \in H$, $x_0 \in H$, compute sequence $\{x_n\}_{n\geq 0}$ by the following scheme

$$\begin{aligned} y_n = {} & (1 - \alpha'_n - \beta'_n)x_n + \alpha'_n\{x_n - (g - m)(x_n) \\ & + J_\rho^{W_n(\cdot,D(x_n))}[(g - m)(x_n) - \rho N(A(x_n), B(x_n)) + \rho f]\} + \beta'_n v_n, \\ x_{n+1} = {} & (1 - \alpha_n - \beta_n)x_n + \alpha_n\{y_n - (g - m)(y_n) \\ & + J_\rho^{W_n(\cdot,D(y_n))}[(g - m)(y_n) - \rho N(A(y_n), B(y_n)) + \rho f]\} + \beta_n u_n, \quad \forall n \geq 0, \end{aligned}$$

where ρ is a positive constant and $\{u_n\}_{n\geq 0}$, $\{v_n\}_{n\geq 0}$ are any bounded sequences in H and $\{\alpha_n\}_{n\geq 0}$, $\{\beta_n\}_{n\geq 0}$, $\{\alpha'_n\}_{n\geq 0}$ and $\{\beta'_n\}_{n\geq 0}$ are any sequences in $[0, 1]$.

3. Main Results

Now we establish the existence of solutions of the nonlinear quasivariational inequality (2.1) and the convergence of the sequence generated by Algorithm 2.1.

Theorem 3.1. *Let* A, B,D, g, $m : H \to H$ *be Lipschitz continuous with constants* a, b, d, p *and* q, *respectively,* $g - m$ *be strongly monotone with* *constant λ, and m be relaxed monotone with respect to g with constant δ. Assume that $N : H \times H \to H$ is Lipschitz continuous in the first and second arguments with constants t, k, respectively, and is strongly monotone with respect to A in the first argument with constant α, respectively. Suppose that $W : H \times H \to 2^H$ is a mapping such that $W(\cdot, y) : H \to 2^H$ is a maximal mapping and* $\text{Range}\,(g-m) \cap \text{dom}\,(W(\cdot,y)) \neq \emptyset$ *for each $y \in H$ and there exists a constant ξ satisfying*

$$\|J_\rho^{W(\cdot,x)}(z) - J_\rho^{W(\cdot,y)}(z)\| \leq \xi\|x-y\|, \quad \forall x, y, z \in H. \tag{3.1}$$

Let

$$L = 2\sqrt{1 - 2\lambda + q^2 + p^2 + 2\delta} + \xi d, \quad K = bk.$$

Suppose that there exists a constant $\rho > 0$ satisfying

$$L + \rho K < 1 \tag{3.2}$$

and one of the following conditions hold.

$$ta > K, \quad (K - KL - \alpha)^2 > (t^2a^2 - K^2)(2L - L^2),$$
$$\left|\rho - \frac{\alpha - K(1-L)}{t^2a^2 - K^2}\right| < \frac{\sqrt{(K-KL-\alpha)^2 - (t^2a^2-K^2)(2L-L^2)}}{t^2a^2 - K^2}; \tag{3.3}$$

$$ta < K,$$
$$\left|\rho - \frac{\alpha - K(1-L)}{t^2a^2 - K^2}\right| > \frac{\sqrt{(K-KL-\alpha)^2 - (t^2a^2-K^2)(2L-L^2)}}{K^2 - t^2a^2}. \tag{3.4}$$

Then for a given $f \in H$, the nonlinear quasivariational inequality (2.1) *has a unique solution $x^* \in H$.*

Proof. Let f, x, y be arbitrary points in H and G be defined by (2.2). In view of the Lipschitz continuity of g, m and strong monotonicity of $g - m$, we deduce that

$$\begin{aligned}\|x-y-((g-m)(x)-(g-m)(y))\|^2 &= \|x-y\|^2 - 2\langle x-y, (g-m)(x)-(g-m)(y)\rangle \\ &+ \|m(x)-m(y)\|^2 + \|g(x)-g(y)\|^2 - 2\langle m(x)-m(y), g(x)-g(y)\rangle \\ &\leq (1 - 2\lambda + q^2 + p^2 + 2\delta)\|x-y\|^2. \end{aligned} \tag{3.5}$$

Since $N : H \times H \to H$ is Lipschitz continuous with respect to the first and second arguments, respectively, and is strongly monotone with respect to A in the first argument, A and B are Lipschitz continuous, we conclude that

$$\begin{aligned}&\|x - y - \rho(N(A(x), B(x)) - N(A(y), B(x)))\|^2 \\ &\qquad \leq (1 - 2\alpha\rho + \rho^2t^2a^2)\|x-y\|^2. \end{aligned} \tag{3.6}$$

By virtue of (3.1), (3.5) and (3.6), we conclude that

$$\begin{aligned}\|G(x)-G(y)\| &\le \|x-y-((g-m)(x)-(g-m)(y))\| \\ &\quad + \|J_\rho^{W(\cdot,D(x))}[(g-m)(x)-\rho N(A(x),B(x))+\rho f] \\ &\quad - J_\rho^{W(\cdot,D(y))}[(g-m)(x)-\rho N(A(x),B(x))+\rho f]\| \\ &\quad + \|J_\rho^{W(\cdot,D(y))}[(g-m)(x)-\rho N(A(x),B(x))+\rho f] \\ &\quad - J_\rho^{W(\cdot,D(y))}[(g-m)(y)-\rho N(A(y),B(y))+\rho f]\| \\ &\le 2\|x-y-((g-m)(x)-(g-m)(y))\| + \xi\|D(x)-D(y)\| \\ &\quad + \|x-y-\rho(N(A(x),B(x))-N(A(y),B(x))\| \\ &\quad + \rho\|N(A(y),B(x))-N(A(y),B(y))\| \le \theta\|x-y\|,\end{aligned}$$

where $\theta = L+\rho K+\sqrt{1-2\alpha\rho+\rho^2 t^2 a^2}$.

It is easy to see that (3.2) and one of (3.3) and (3.4) ensure that $\theta < 1$. That is, the contraction mapping G has a unique fixed point $x^* \in H$. It follows from Lemma 2.2 that x^* is the unique solution of the nonlinear quasivariational inequality (2.1). This completes the proof. □

Theorem 3.2. *Let A, B, D, g, m, N, W, K and L be as in Theorem 3.1 and* (3.1) *hold. Suppose that for each $n \ge 0$, $W_n(\cdot,y) : H \to 2^H$ is maximal monotone mapping,* Range $(g-m)\cap \operatorname{dom} W_n(\cdot,y) \ne \emptyset$ *for each $y \in H$ and*

$$\|J_\rho^{W_n(\cdot,x)}(z) - J_\rho^{W_n(\cdot,y)}(z)\| \le \xi\|x-y\|, \quad \forall x,y,z \in H\,, \tag{3.7}$$

and

$$\lim_{n\to\infty} \|J_\rho^{W_n(\cdot,x)}(z) - J_\rho^{W(\cdot,x)}(z)\| = 0, \quad \forall x,z \in H, \tag{3.8}$$

where ξ is a positive constant. Assume that $\{\alpha_n\}_{n\ge0}$, $\{\beta_n\}_{n\ge0}$, $\{\alpha'_n\}_{n\ge0}$ and $\{\beta'_n\}_{n\ge0}$ satisfy

$$\alpha_n+\beta_n \le 1, \quad \alpha'_n+\beta'_n \le 1, \quad \forall n \ge 0; \tag{3.9}$$

$$\lim_{n\to\infty}\beta'_n = 0, \quad \sum_{n=0}^{\infty}\alpha_n = \infty, \quad \sum_{n=0}^{\infty}\beta_n < \infty. \tag{3.10}$$

If there exists a positive constant ρ satisfying (3.2) *and one of* (3.3) *and* (3.4), *then for a given $f \in H$, the nonlinear quasivariational inequality* (2.1) *has a unique solution $x^* \in H$ and the sequence $\{x_n\}_{n\ge0}$ defined by Algorithm 2.1 converges strongly to x^*.*

Proof. Let $f \in H$ be a fixed point. Theorem 3.1 means that the nonlinear quasivariational inequality (2.1) has a unique solution $x^* \in H$. In light of Lemma 2.2, we conclude that

$$
\begin{aligned}
x^* &= (1-\alpha_n'-\beta_n')x^* + \alpha_n'\{x^* - (g-m)(x^*) \\
&+ J_\rho^{W_n(\cdot,D(x^*))}[(g-m)(x^*) - \rho N(A(x^*),B(x^*)) + \rho M(E(x^*),F(x^*)) + \rho f]\} + \beta_n' x^* \\
&= (1-\alpha_n-\beta_n)x^* + \alpha_n\{x^* - (g-m)(x^*) + J_\rho^{W_n(\cdot,D(x^*))}[(g-m)(x^*) \\
&- \rho N(A(x^*),B(x^*)) + \rho M(E(x^*),F(x^*)) + \rho f]\} + \beta_n x^*, \quad \forall n \geq 0. \quad (3.11)
\end{aligned}
$$

According to Algorithm 2.1 and (3.11), we get that

$$
\begin{aligned}
&\|y_n - x^*\| \\
&\leq (1-\alpha_n'-\beta_n')\|x_n - x^*\| + \alpha_n'\|x_n - x^* - ((g-m)(x_n) - (g-m)(x^*))\| \\
&\quad + \alpha_n'\|J_\rho^{W_n(\cdot,D(x_n))}[(g-m)(x_n) - \rho N(A(x_n),B(x_n)) + \rho f] \\
&\quad - J_\rho^{W_n(\cdot,D(x^*))}[(g-m)(x_n) - \rho N(A(x_n),B(x_n)) + \rho f]\| \\
&\quad + \alpha_n'\|J_\rho^{W_n(\cdot,D(x^*))}[(g-m)(x_n) - \rho N(A(x_n),B(x_n)) + \rho f] \\
&\quad - J_\rho^{W_n(\cdot,D(x^*))}[(g-m)(x^*) - \rho N(A(x^*),B(x^*)) + \rho f]\| \\
&\quad + \alpha_n'\|J_\rho^{W_n(\cdot,D(x^*))}[(g-m)(x^*) - \rho N(A(x^*),B(x^*)) + \rho f] \\
&\quad - J_\rho^{W(\cdot,D(x^*))}[(g-m)(x^*) - \rho N(A(x^*),B(x^*)) + \rho f]\| \\
&\quad + \beta_n'\|v_n - x^*\|, \quad \forall n \geq 0.
\end{aligned} \tag{3.12}
$$

It follows from (3.7)-(3.9) and the proof of Theorem 3.1 that

$$
\|y_n - x^*\| \leq (1-\alpha_n'-\beta_n')\|x_n - x^*\| + \theta\alpha_n'\|x_n - x^*\| + M_n\alpha_n' + M\beta_n' \tag{3.13}
$$

for all $n \geq 0$, where

$$
\theta = L + \rho K + \sqrt{1 - 2\alpha\rho + \rho^2 t^2 a^2},
$$

$$
M = \sup\{\|v_n - x^*\|,\ \|u_n - x^*\| : \quad n \geq 0\} < \infty
$$

and

$$
\begin{aligned}
M_n = &\|J_\rho^{W_n(\cdot,D(x^*))}[(g-m)(x^*) - \rho N(A(x^*),B(x^*)) + \rho f] \\
&- J_\rho^{W(\cdot,D(x^*))}[(g-m)(x^*) - \rho N(A(x^*),B(x^*)) + \rho f]\|
\end{aligned}
$$

for each $n \geq 0$. In a similar way, we derive that

$$
\|x_{n+1} - x^*\| \leq (1-\alpha_n-\beta_n)\|x_n - x^*\| + \theta\alpha_n\|y_n - x^*\| + M_n\alpha_n + M\beta_n \tag{3.14}
$$

for any $n \geq 0$. Substituting (3.13) into (3.14), we infer that

$$
\|x_{n+1} - x^*\| \leq (1-(1-\theta)\alpha_n)\|x_n - x^*\| + (2M_n + M\beta_n')\alpha_n + M\beta_n \tag{3.15}
$$

for all $n \geq 0$. It follows from (3.9), (3.10), (3.15), Lemma 2.1 and $\theta < 1$ that $x_n \to x^*$ as $n \to \infty$. This completes the proof. □

Remark 3.1. Theorem 3.1 in [2], Theorem 4.1 in [3], Theorem 4.1 in [4], Theorem 2.2 in [6] and Theorem 3.1 in [7] are special cases of Theorem 3.1 and Theorem 3.2, respectively.

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