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REDUCED PROJECTIVE CURVES WITH NEGATIVE COTANGENT SHEAF

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Abstract: Here we study reduced projective curves (mainly nodal ones) whose cotangent sheaf (modulo its torsion) is negative.

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1. Negative Cotangent Sheaf

For any algebraic scheme X defined over an algebraically closed field $\mathbb{K}$ (resp. reduced complex analytic space X) let TX denote its tangent sheaf, i.e. the dual of the sheaf Ω^1_X of germs of regular (resp. holomorphic) 1-forms on X. It is well-known that Ω^1_X is locally free if and only X is smooth. We are interested in the case in which TX is locally free, but X is singular. Here, following [1], we study the case $\dim(X) = 2$, X projective or compact and TX is rather positive. We consider only the case $\dim(X) = 1$, but we show that taking a dual approach (i.e. looking at the negativity of the contangent sheaf) may give interesting results. Now assume that X is an integral projective curve. We always assume $\mathrm{Sing}(X) \neq \emptyset$. Let $\pi : Y \to X$ denote its normalization. If A is a rank one torsion free sheaf on X, then $\pi^*(A)$ has torsion if and only if A is not locally free, while $\pi^{[*]} := \pi^*(A)/\mathrm{Tors}(\pi^*(A))$ is locally free. The sheaf $\Omega^1_X/\mathrm{Tors}(\Omega^1_X)$ is a rank one torsion free sheaf on X and it would be natural to say that it is negative if $\deg(\Omega^1_X/\mathrm{Tors}(\Omega^1_X)) < 0$. We recall that the degree of a rank one tarsion-free sheaf F on X may be defined by the Riemann-Roch type formula: $\deg(F) = \chi(F) - \chi(\mathcal{O}_X)$. When X is Gorenstein

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we have $\deg(F^*) = -\deg(F)$ (and hence when X is Gorenstein $\deg(TX) > 0$ if and only if $\deg(\Omega^1_X/\mathrm{Tors}(\Omega^1_X)) < 0$). Notice that if X is Gorenstein, then $\Omega^1_X/\mathrm{Tors}(\Omega^1_X) = TX^*$. When F is locally free, then of course $\deg(F^*) = -\deg(F)$, even if X is not Gorenstein. When F is locally free, then $\pi^*(F)$ is locally free and $\deg(\pi^*(F)) = \deg(F)$ (and in paricular $\deg(F) > 0$ if and only if F is ample), but this is not true if F is not locally free (see Example 1). In this paper we will heavily use this well-known fact. If $\mathrm{char}(\mathbb{K}) = 0$, then TX is not locally free, because X is not normal (see [4]). If $\mathrm{char}(\mathbb{K}) > 0$ we gave examples of plane curves (and hence of Gorenstein curves) with TX locally free and very ample (see [1], §1).

Example 1. Assume $\mathrm{char}(\mathbb{K}) \neq 2$. Fix an integer $g > 0$. Let X be an integral projective curve such that $p_a(X) = g$ and X has exactly g singular points, say $P_1, \dots, P_g$, such that each of them is either an ordinary node or an ordinary cusp. Hence X is Gorenstein. If X has at least one ordinary cusp of X, then assume $\mathrm{char}(\mathbb{K}) \neq 3$. Since X is Gorenstein, the torsion-free sheaf $\Omega^1_X/\mathrm{Tors}(\Omega^1_X)$ is reflexive. The natural map $\Omega^1_X \to \omega_X$ shows that $\Omega^1_X/\mathrm{Tors}(\Omega^1_X)$ is the torsion-free sheaf $\mathcal{I}_{\{P_1,\dots,P_g\}} \otimes \omega_X$. Thus $\Omega^1_X/\mathrm{Tors}(\Omega^1_X)$ is not locally free and it has degree $g-2$. This degree is negative if and only if $g = 1$. Furthermore, its dual TX is not locally free, it has degree $2-g$ and, if $g = 1$, it contains $\mathcal{O}_X$ (use [2], Lemma 3.1.7 (a), and that X is Gorenstein). Hence if $g = 1$, then $h^0(X, TX) = 1$, $\deg(TX) = 1$ and TX is not spanned. Let $\pi : Y \to X$ be its normalization. Hence $Y \cong \mathbf{P}^1$. The line bundle $\pi^{[*]}(\Omega^1_X/\mathrm{Tors}(\Omega^1_X))$ has degree -2, while the line bundle $\pi^{[*]}(TX)$ has degree $2-2g$ (use [2], part 2 of Proposition 3.2.4). Notice that if $g = 1$, then $\pi^{[*]}(TX)$ is trivial, while the dual of $\pi^{[*]}(\Omega^1_X/\mathrm{Tors}(\Omega^1_X))$ is very ample. If $g = 1$ the torsion-free sheaf TX is an ample $\mathbb{Q}$, i.e. it is associated to a Weil divisor P_1 such that $\mathcal{O}_X(tP_1)$ (i.e. the sheaf $TX^{\otimes t}/\mathrm{Tors}(TX^{\otimes t})$) is an ample line bundle for all even positive integers. Notice that (under our assumption $g > 0$) there is no line bundle on X with negative degree containing $\Omega^1_X/\mathrm{Tors}(\Omega^1_X)$.

Example 2. Assume $\mathrm{char}(\mathbb{K}) \neq 2$. Fix integers $g > 0$ and $q > 0$. Let X be an integral projective curve such that $p_a(X) = g + q$ and X has exactly g singular points such that each of them is either an ordinary node or an ordinary cusp. If X has at least one ordinary cusp of X, then assume $\mathrm{char}(\mathbb{K}) \neq 3$. As in Example 1 we get that $\Omega^1_X/\mathrm{Tors}(\Omega^1_X)$ is not locally free, $\deg(\Omega^1_X/\mathrm{Tors}(\Omega^1_X)) = 2q+g-2$, TX is not locally free, $\deg(TX) = -2q-g+2$, that the line bundle $\pi^{[*]}(\Omega^1_X/\mathrm{Tors}(\Omega^1_X))$ has degree $2q-2$, while the line bundle $\pi^{[*]}(TX)$ has degree $-2q+2-2g$.

What happens when the projective curve X is reduced and connected, but not irreducible? The following result is very easy and well-known (see [5] for the two-dimensional case).

Proposition 1. *Let X be a reduced and connected Gorenstein projective curve. Assume ω_X^* ample and that X is not irreducible. Then X is isomorphic to a reducible plane conic.*

The negativity of the quotient of the cotangent sheaf by its torsion gives more interesting results and examples.

Proposition 2. *Let X be a reduced projective curve such that all its irreducible components are rational and $\pi : Y \to X$ its normalization. Then the dual of $L := \pi^*(\Omega_X^1)/\mathrm{Tors}(\pi^*(\Omega_X^1))$ is an ample line bundle, i.e. $\deg(L|C) < 0$ for all connected components C of X.*

Theorem 1. *Let X be a reduced and connected nodal projective curve.*

(a) If there are $L \in \mathrm{Pic}(X)$ such that $\deg(L|T) < 0$ for all irreducible components T of X and an injective map $j : \Omega_X^1/\mathrm{Tors}(\Omega_X^1) \to L$, then all irreducible components of X are smooth and rational.

(b) Assume $p_a(X) = 0$. Then there are $L \in \mathrm{Pic}(X)$ such that $\deg(L|T) < 0$ for all irreducible components T of X and an injective map

$$j : \Omega_X^1/\mathrm{Tors}(\Omega_X^1) \to L.$$

Proof of Proposition 1. We outline two different proofs. Let T be any of the irreducible component of X. Let Z be the closure of $X\backslash T$ in X. Since X is reducible and connected, $T \cap Z \neq \emptyset$. Since ω_X is locally free and ample. $\omega_X|T$ is a line bundle and $\deg(\omega_X|T) < 0$. Since ω_X is locally free, it is easy to check that ω_T is a subsheaf of $\omega_X|$ (this is a paricular case of the subadjunction formula given in [3], Chapter III, Example 7.2, or [5], Proposition 2.3 and Proposition 2.11). Hence the torsion-free sheaf ω_T has negative degree. Thus $T \cong \mathbf{P}^1$. Since T is smooth, it is easy to check that the support of $\omega_X|T/\omega_T$ is equal to the set $T \cap Z$. Since $\deg(\omega_T) = -2$ and $\deg(\omega_X|T) < 0$, we get $\sharp(T \cap Z) = 1$. Since this must be true for all irreducible components of X, we get that X has a unique singular point, P, and $m \geq 2$ irreducible components $X_1, \dots, X_m$, all of them smooth, rational and containing P. We also see that X_i is quasi-transversal to X_j for all $i \neq j$, i.e. $X_i \cup X_j$ is a plane conic for all $i \neq j$. Let F be a torsion-free sheaf on X such that its restriction to every irreducible component of X. We may define the degree $\deg(F)$ by the formula $\deg(F) = \chi(F) - \chi(\mathcal{O}_X)$. With this definition we may apply duality to get $\deg(\omega_X) = 2p_a(X) - 2$ as in the case of irreducible curves done in [2]. Since ω_X^* is ample, we have $\deg(\omega_X) \leq -m$. Hence we get $p_a(X) = 0$ (and in particular every irreducible component of X is smooth and rational) and $m = 2$. $\square$

Remark 1. Let X be a reduced connected curve, T an irreducible component of X, $\pi : Y \to X$ its normalization and C the connected component of Y such that $\pi(C) = T$. The functoriality properties of the sheaf of Kähler differentials gives maps $j_T : \Omega^1_X|T \to \Omega^1_T$ and $j_C : f^*(\Omega^1_X)|C \to \Omega^1_C$ which are an isomorphism over each point of $X_{reg} \cap T$. Thus j_T and j_C are injective modulo torsion. Thus $\deg((\Omega^1_X|T)/\mathrm{Tors}(\Omega^1_X|T)) \leq \Omega^1_T/\mathrm{Tors}(\Omega^1_T)$. Furthermore, $\deg((f^*(\Omega^1_X)|C)/\mathrm{Tors}(f^*(\Omega^1_X)|C) < 0$ if $C \cong \mathbf{P}^1$.

Proof of Proposition 2. Apply Remark 1. □

Proof of Theorem 1. Part (a) follows from Example 1 and the last sentence of Example 2. Now we will check part (b). Since X is nodal and connected and $p_a(X) = 0$, every irreducible component of X is smooth and rational and a line bundle on X is uniquely determined by the degrees of its restriction to the irreducible components of X. Fix an irreducible component T of X. Set $S := \{P \in \mathrm{Sing}(X) : P \in T_{reg}\}$. Thus $S = T \cap E$, where E is the closure of $X \backslash T$ in X. Notice that S is a Cartier divisor of T. We have $\omega_X|T = \omega_T(S)$. Since T is smooth at each point of S, a local calculation as in Example 1 shows that $(\Omega^1_X|T)/\mathrm{Tors}(\Omega^1_X|T) = \Omega^1_T/\mathrm{Tors}(\Omega^1_T)$. Let L be any line bundle on X such that $\deg(L|T) = -1$ for all irreducible components of X. Applying several times a Mayer-vietoris exact sequence we get $h^0(X, (\Omega^1_X/\mathrm{Tors}(\Omega^1_X) \otimes L^*) \neq 0$. Thus there is an injective map $\Omega^1_X/\mathrm{Tors}(\Omega^1_X) \to L$. □

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