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PROPERTIES OF TOP MODULES

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Abstract: Let R be any ring with identity. A right R-module M is called a top module if $\mathrm{Spec}_r(M)$ is a space with Zariski topology. In this paper it will be showed that a right semi-simple module M over a right quasi-duo ring R is top if and only if it is distributive. If R is a right quasi-duo, right perfect ring and M is a right top R-module, then it is cyclic. In addition, several known results on the multiplication modules are extended to top modules.

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1. Introduction

All rings in this paper are assumed to be associative with identity and all modules are assumed to be unitary. Let R be a ring and M a right R-module. Denote the annihilator of M as $M^{\perp} = \{r \in R | Mr = 0\}$. Thus, for any submodule N of M, we have $(M/N)^{\perp} = \{r \in R | Mr \subseteq N\}$.

A proper right ideal (resp. ideal) P of R is called *right prime* (resp. *prime*) if $aRb \subseteq P$ implies $a \in P$ or $b \in P$. A proper ideal P of R is called *completely prime* if $ab \in P$ implies $a \in P$ or $b \in P$. We write $\mathrm{Spec}_r(R)$ (resp. $\mathrm{Spec}(R)$, $\mathrm{Max}_r(R)$, $\mathrm{Max}(R)$) for the set of all right prime ideals (resp. all prime ideals, all maximal right ideals, all maximal ideals) of R.

A proper right submodule K of M is called *prime* if $mRr \subseteq K$, for $r \in R$ and $m \in M$, implies $r \in (M/K)^{\perp}$ or $m \in K$. Clearly a right prime ideal of R is a prime submodule of R_R. A ring R is called *rpm* (i.e. right *pm*) (resp.

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pm) if every right prime (resp. prime) ideal is contained in a unique maximal right ideal (resp. maximal ideal). A right R-module is called *pm* if every prime submodule is contained in a unique maximal submodule of M. Some of the work on *pm*-rings, *rpm*-rings and *pm*-modules can be found in Demarco and Orsatti [3], Sun [5] and [11] and Zhang [15], [18], [19], [20].

Let M be a right R-module. We define the set (lattice) $\mathrm{Lat}(M) = \{N|N$ is a right submodule of $M\}$, $\mathrm{Spec}_r(M)$ as the space of all prime submodules of M and $\mathrm{Spec}_r^P(M) = \{K \in \mathrm{Spec}_r(M)|(M/K)^{\perp} = P\}$, where $P \in \mathrm{Spec}(R)$.

Let M be any right R-module and $U_r(N) = \{K \in \mathrm{Spec}_r(M) \,|\, K \not\supseteq N\}$, where $N \in \mathrm{Lat}(M)$. A submodule K of M is called P-prime submodule of M if $K \in \mathrm{Spec}_r^P(M)$, where $P \in \mathrm{Spec}(R)$.

Definition. M is called a *right top module* (i.e. a module with Zariski topology) if, for each pair of submodules L_1, L_2, there at least one $H \in \mathrm{Lat}(M)$ such that $U_r(L_1) \bigcap U_r(L_2) = U_r(H)$.

If M is a right top R-module and $\zeta(M)$ denotes the collection of all subsets $U = U_r(L)$ of $\mathrm{Spec}_r(M)$, where L is any submodule of the right R-module M, then $\zeta(M)$ contains the empty set and $\mathrm{Spec}_r(M)$, $\zeta(M)$ is closed under arbitrary unions since $\bigcup_{i\in\Gamma} U_r(L_i) = U_r(\sum_{i\in\Gamma} L_i)$, where L_i $(i \in \Gamma)$ are submodules of M, and $\zeta(M)$ is closed under finite intersections by the above definition. Hence $\mathrm{Spec}_r(M)$ is a space with Zariski topology. A ring R is called *right top* if R_R is a right top R-module.

Top rings and top modules were studied in McCasland et al [8], Zhang [14], [16], [15], [18], [19], [20]. A right R-module M is called a *multiplication module* if for each submodule N of M there exists an ideal I of R such that $N = MI$. Right multiplication modules are top modules, but top modules need not to be multiplication modules (see Example 2.3). There exist many papers containing results on multiplication modules (see Barnard (1981), El-Bast and Smith [1], Tuganbaev [12], Zhang et al [15] and Zhang [19]).

A right R-module M is called *uniserial* (see Tuganbaev [14]) if all submodules of it form a chain. Any uniserial module is a top module (see Zhang [20], Corollary 2.6). A right R-module M is called *distributive* (see Tuganbaev [14], [13] if $F \bigcap (G + H) = F \bigcap G + F \bigcap H$ for all submodules F, G and H of M. We shall show that any right semi-simple top R-module is distributive (Theorem 2.2). Especially, for a right quasi-duo ring R, a right semi-simple R-module M is top if and only if it is distributive (Theorem 2.2). For a right top, right perfect ring R, a right R-module M is top if and only if it is cyclic (Corollary 2.7).

2. Main Results

Given right R-modules M, M', M and M' are called *prime-compatible* if there does not exist any prime ideal P of R such that $\mathrm{Spec}_r^P(M)$ and $\mathrm{Spec}_r^P(M')$ are both non-empty. A collection of R-module M_i $(i \in I)$ is called *prime-compatible* if M_i and M_j are prime-compatible for each pair of distinct $i, j \in I$. Zhang [14], Proposition 2.12 gave the following result.

Lemma 2.1. *Let R be any ring and $M = \bigoplus_{i\in I} M_i$, where M_i $(i \in I)$ are right R-modules. Then M is a right top R-module if and only if M_i $(i \in I)$ are prime-compatible right top R-modules.*

R is called a right *quasi-duo ring* if every its maximal right ideal is 2-sided.

Theorem 2.2. *Let R be any ring and M a semi-simple right R-module with the decomposition $M = \bigoplus_{i\in I} M_i$, where M_i $(i \in I)$ are simple right R-modules, and let $P_i = M_i^{\perp}$ $(i \in I)$. Consider the following statements:*

(1) M is a right top R-module.

(2) M is a right distributive R-module.

(3) $P_i \neq P_j$ for each pair $i \neq j$ in I.

(4) $M_i \ncong M_j$ for each pair $i \neq j$ in I.

Then (1) $\Leftrightarrow$ (3), (2) $\Leftrightarrow$ (4) and (1) $\Rightarrow$ (2). Moreover, if R is a right quasi-duo ring, then the above conditions (1), (2), (3) and (4) are equivalent.

Proof. M_i $(i \in I)$ are top, distributive R-modules since M_i $(i \in I)$ are simple right R-modules.

(1) $\Leftrightarrow$ (3). Since M_i $(i \in I)$ are simple right R-modules, 0 is a right maximal submodule of M_i and hence 0 is a prime submodule of M_i. So $\mathrm{Spec}_r^{P_i}(M_i) = \{0\}$ for any $i \in I$. Thus $\mathrm{Spec}_r^P(M_i) = \emptyset$ for any $P \in \mathrm{Spec}(R)$ and $P \neq P_i$. By Lemma 2.1, M is a right top R-module if and only if M_i $(i \in I)$ are prime-compatible right top R-modules if and only if $P_i \neq P_j$ for each pair of distinct i, j in I.

(2) $\Leftrightarrow$ (4). For any sub-factor (i.e. any submodule of factor modules of M) $M_i' \bigoplus M_j'$ of M, where M_i', M_j' are a pair of simple right R-modules, by Facchini ([4], Theorem 2.14 and Proposition 2.15) there are M_i $(i \in J \subseteq I)$ such that

$$M_i' \bigoplus M_j' \cong \bigoplus_{j\in J\subseteq I} M_i.$$

Let $f : M_i' \bigoplus M_j' \mapsto \bigoplus_{j\in J} M_i$ be the isomorphism of right R-modules. Then, for any $0 \neq m_i \in M_i'$, $f(m_iR) = f(M_i') = f(m_i)R \subseteq \bigoplus_{j\in J} M_i$ and $f(m_i) \neq 0$. Hence there are M_i $(i \in J' \subseteq J)$ such that $M_i' \cong f(m_i)R \cong \bigoplus_{i\in J'} M_i$ by

Facchini ([4], Proposition 2.15). Since M_i', M_i $(i \in J')$ are simple right R-modules, $|J'| = 1$ and hence there is $i_0 \in J'$ such that $M_i' \cong M_{i_0}$. Similarly we can prove that there is $j_0 \in I$ such that $M_j' \cong M_{j_0}$. Tuganbaev [13], Lemma 5.7, showed that M is a distributive module if and only if M does not have sub-factors that are direct sums of two isomorphic simple modules. Hence M is a distributive module if and only if each pair M_i, M_j $(i \neq j)$ are not isomorphic.

(1) $\Rightarrow$ (2). Suppose that $M_i \cong M_j$ for $i \neq j$ in I. Then $P_i = P_j$. Thus (3) $\Rightarrow$ (4). We have proved that (1) $\Leftrightarrow$ (3) and (2) $\Leftrightarrow$ (4). Thus (1) $\Rightarrow$ (2).

If N is a right simple R-module, then N is a right cyclic R-module and hence there is a maximal right ideal P of R such that $N \cong R/P$. Hence P is a maximal ideal of R since R is a right quasi-duo ring and $P = N^{\perp}$. Thus, for each pair of simple R-modules M_i, M_j $(i, j \in I)$, $M_i \cong M_j$ if and only if $M_i^{\perp} = M_j^{\perp}$ (i.e. $P_i = P_j$). Thus (3) $\Leftrightarrow$ (4), and hence (1)-(4) are equivalent. □

Let I be a right ideal of R and let I^0 denote the largest 2-sided ideal contained in I. Then $(R/I)^{\perp} = I^0$. We have the following examples.

Example 2.3. Let R be any ring and $M = \bigoplus_{i\in\Gamma} M_i$ satisfying that $M_i = R/P_i$, P_i $(i \in \Gamma)$ are maximal right ideals of R, and $P_i^0 \neq P_j^0$ for each pair $P_i \neq P_j (i, j \in \Gamma)$. Then the following results hold:

(1) M is a semi-simple, top, distributive R-module.

(2) If R is a right quasi-duo ring and the set Γ is finite, then M is a multiplication R-module.

(3) If R is a right quasi-duo ring, the set Γ is infinite, and P_1 contains the ideal $\bigcap_{i\neq 1} P_i$ (for example, one can take $R = Z$ and $M_i = Z/Zp_i = Z_{p_i}$, where p_i $(i = 1, 2, \cdots,)$ are the infinite set of prime integers), then the semi-simple top R-module M is not a multiplication module.

Proof. (1) is from Theorem 2.2 since, for each pair of distinct $P_i, P_j (i, j \in \Gamma)$, $P_i^0 \neq P_j^0$ and $R/P_i \not\cong R/P_j$ (suppose that $R/P_i \cong R/P_j$, then $P_i^0 = (R/P_i)^{\perp} = (R/P_j)^{\perp} = P_j^0$, a contradiction).

(2) See Tuganbaev [12], Example 1.19 (2).

(3) By Theorem 2.2 and Tuganbaev [12], Example 1.19 (3). □

A ring R is called *right hereditary* if every its right ideal is a projective right R-module. A ring R is *von Neumann regular ring* provided that for every $x \in R$ there exists $y \in R$ such that $xyx = x$. Artinian semi-simple rings are (right and left) hereditary von Neumann regular rings.

Proposition 2.4. *Let R be a right hereditary von Neumann regular ring. Then the following statements are equivalent:*

(1) R is a right top ring.

(2) R is a right distributive ring.
(3) R is a right duo ring.

Proof. (1) $\Rightarrow$ (3). Since R is right top, R is right quasi-duo by Zhang et al [18], Corollary 2.8. Hence every (right) maximal ideal of R is completely prime by Sun ([11], Lemma 1.3) and $N'(R) \subseteq J(R) = 0$, where $N'(R)$ is the intersection of all completely prime ideals of R and $J(R)$ is a Jacobson radical of ring R . So R is a reduced ring and hence R is a Abelian regular ring by Rege ([10], Proposition 3.11). Thus R is a duo ring.

(2) $\Leftrightarrow$ (3). Tuganbaev [14], 4.7.1 showed that, for a hereditary ring R, R is right distributive if and only if it is right duo ring.

(3) $\Rightarrow$ (1). $\mathrm{Spec}_r(R) = \mathrm{Spec}(R)$ since R is a right duo ring. Hence R is a right top ring because $\mathrm{Spec}(R)$ is a space with the Zariski topology. □

Let M_R be semi-simple and Γ the set of all simple submodules of M. Then $\cong$ is an equivalence relation on Γ. Let the set of equivalence classes (i.e. isomorphism classes) be $\{\Omega_j | j \in J\}$. Then $B_j = \sum_{E \in \Omega_j} E$ is called a *homogeneous component* of M (see F. Kasch ([6], Definition 8.1.7)). We consider F. Kasch ([6], Theorem 8.2.4) and hence have the following results.

Theorem 2.5. *Let R be a nonzero Artinian semi-simple ring and M a right R-module, and let $R = R_1 \bigoplus R_2 \bigoplus \cdots \bigoplus R_n$ be the decomposition of R into homogeneous components, where R_i $(i = 1, 2, \cdots, n)$ are simple nonzero ideals (i.e. simple rings). Then M is a right top R-module if and only if MR_i $(i = 1, 2, \cdots, n)$ are right uniserial R-modules.*

Proof. If $n = 1$, then the results are clear from Zhang [20], Proposition 2.7. Let $n \geq 2$ in the following proof.

Since $R = R_1 \bigoplus R_2 \bigoplus \cdots \bigoplus R_n$, where R_i $(i = 1, 2, \cdots, n)$ are 2-sided ideals, by Kash ([6], Lemma 7.2.3), there are central idempotents $e_i \in R_i$ $(i = 1, 2, \cdots, n)$ such that $1 = \sum_{i=1}^n e_i$, $e_i^2 = e_i, e_i e_j = 0 (i \neq j)$, and $R_i = Re_i = Re_iR$. So $M = \sum_{i=1}^n MR_i = \sum_{i=1}^n Me_i$, where $MR_i = Me_i$. For any $m \in Me_i \bigcap \sum_{j \neq i} Me_j$, there are $a_i, a_j \in M$ such that $m = a_i e_i = \sum_{j \neq i} a_j e_j$. So $m = me_i = \sum_{j \neq i} a_j e_j e_i = 0$ and $M = MR_1 \bigoplus MR_2 \bigoplus \cdots \bigoplus MR_n$. Hence, by Lemma 2.1, MR_i $(i = 1, 2, \cdots, n)$ are right top R-modules since M is a right top R-module. If $MR_i = 0$, then the results are clear. If $MR_i \neq 0$, then

$$(MR_i)^{\perp} = R_1 \bigoplus \cdots \bigoplus R_{i-1} \bigoplus R_{i+1} \bigoplus \cdots \bigoplus R_n$$

is a maximal ideal of R. So MR_i is a right top R_i-module since

$$R_i \cong R/(R_1 \bigoplus \cdots \bigoplus R_{i-1} \bigoplus R_{i+1} \bigoplus \cdots \bigoplus R_n)$$

and hence MR_i is a right uniserial R_i-module by Zhang [20], Proposition 2.7. Thus MR_i is a right uniserial R-module.

Conversely, if MR_i $(i = 1, 2, \cdots, n)$ are right uniserial R-modules, then MR_i $(i = 1, 2, \cdots, n)$ are right top R-modules by Zhang [20], Corollary 2.6. Let $MR_i \neq 0$, N a proper right submodule of MR_i and

$$Q_i = R_1 \bigoplus \cdots \bigoplus R_{i-1} \bigoplus R_{i+1} \bigoplus \cdots \bigoplus R_n.$$

Then $[(MR_i)/N]^{\perp} = Q_i \neq R$. Since Q_i is a maximal ideal of R, N is a prime submodule of MR_i by Zhang [20], Lemma 2.2. So any proper submodule of MR_i is Q_i-prime and hence, for any $P \in \mathrm{Spec}(R)$ and $P \neq Q_i$, $\mathrm{Spec}_r^P(MR_i) = \emptyset$. Thus MR_i $(i = 1, 2, \cdots, n)$ are prime-compatible right top R-modules since $Q_i \neq Q_j$ $(i \neq j)$ and hence M is a top R-module by Lemma 2.1. □

Let J denote the Jacobson radical of R. A ring R is *semilocal* if $R/J(R)$ is a semisimple artinian ring. A ring R is called *right perfect* (resp. *semiperfect*) if every right R-module (resp. every finitely generated right R-module) has a projective cover. Semiperfect rings are semilocal (see Lam [7], p. 335).

Theorem 2.6. Let R be a right quasi-duo semilocal ring and M a right top R-module. Then M/MJ is a right cyclic R-module. Furthermore if $MJ \neq M$ and MJ is a superfluous submodule of M, then M is a right cyclic R-module.

Proof. Since R is a right quasi-duo semilocal ring, $\mathrm{Max}_r(R) = \mathrm{Max}(R)$ and hence there are distinct maximal ideals P_i $(i = 1, 2, \cdots, n)$ of R such that $P_1 \bigcap P_2 \bigcap \cdots \bigcap P_n = J$ (see Lam ([7], Proposition 20.2)) with $P_i \not\supseteq \bigcap_{j \neq i} P_j$ for any $i = 1, 2, \cdots, n$. M/MP_i is cyclic for any $i = 1, 2, \cdots, n$ by Theorem 2.10(B). Suppose that $n \geq 2$. Let $A = P_2 \bigcap P_3 \bigcap \cdots \bigcap P_n$ and $J = P_1 \bigcap A$. Note that $P_1 + A = R$. Hence $M = MP_1 + MA$. Moreover, $(MP_1) \bigcap (MA) = [(MP_1) \bigcap (MA)][P_1 + A] = [(MP_1) \bigcap (MA)]P_1 + [(MP_1) \bigcap (MA)]A \subseteq MAP_1 + MP_1A \subseteq MJ \subseteq (MP_1) \bigcap (MA)$ and hence $MJ = (MP_1) \bigcap (MA)$. Now, by Zhang [20], Theorem 2.10 (B), M/MP_1 is cyclic and hence there is a right ideal B of R such that $M/MP_1 \cong R/B$, and, by induction on n, M/MA is right cyclic so that $M/MA \cong R/C$ for some right ideal C with $A \subseteq C$. Thus $M/MJ \cong M/MP_1 \bigoplus M/MA \cong R/B \bigoplus R/C \cong R/(B \bigcap C)$ since $B + C = R$. It follows that M/MJ is cyclic. There exists $m \in M$ such that $M = MJ + mR$. If $M = 0$, then it is clear. Now let $M \neq 0$. If $MJ \neq M$ and MJ is a superfluous submodule of M, then $M = mR$. □

Corollary 2.7. *Let R be a right perfect ring. Then the following results hold:*

(1) If R is a right quasi-duo and M is a right top R-module, then M is a cyclic R-module.

(2) If R is a right top ring, then M is a right top R-module if and only if M is a right cyclic R-module.

Proof. (1). M/MJ is a cyclic right R-module by Theorem 2.6. Hence there exists $m \in M$ such that $M = MJ + mR$. If $M = 0$, then it is clear. Now let $M \neq 0$. Since J is right T-nilpotent, by Anderson and Fuller [2], Lemma 28.3, we know that, for every nonzero right R-module M, $MJ \neq M$ and MJ is a superfluous (or small) submodule of M. So $M = mR$.

(2). Since R is a right top ring, by Zhang et al [18], Corollary 2.8, R is a right quasi-duo ring. Thus, by (1), if M is a right top R-module, then M is cyclic. Conversely, if M is a right cyclic R-module, then M is the homomorphic image of R and hence, by Zhang [17], Proposition 2.3, it is a right top R-module since R is a right top ring. □

The intersection $J(M)$ of kernels of all homomorphisms from a right R-module M into simple modules is called the Jacobson radical of M. We note that either $J(M) = M$ (if $\mathrm{Max}_r(M) = \emptyset$) or $J(M)$ coincides with the intersection of all maximal right submodules of M. Facchini ([4], Lemma 1.3) showed that $MJ(R) \subseteq J(M)$.

Corollary 2.8. *If R is a right quasiduo, right semilocal ring and M is right finitely generated top R-module, then M is a cyclic right R-module.*

Proof. Since M is finitely generated, every proper submodule is contained in a maximal submodule and hence $MJ \neq M$, MJ is a superfluous submodule of M. Thus M is cyclic by Theorem 2.6. □

Corollary 2.9. *Let R be a nonzero right duo, right perfect ring. Then the following statements are equivalent:*

(1) M is a right top R-module.

(2) M is a right multiplication R-module.

(3) M is a cyclic right R-module.

Proof. (1) $\Leftrightarrow$ (3). is from Corollary 2.7 since any right duo ring is a right top ring.

(2) $\Rightarrow$ (1). is from Zhang [19], Proposition 2.1.

(3) $\Rightarrow$ (2). Since R is right duo and M is cyclic, there is an (right) ideal I such that $M \cong R/I$. For any R-submodule L/I of R/I (where L is a (right) ideal of R and $L \supseteq I$), $(R/I)L = L/I$. Thus R/I is a multiplication R-module. □

Proposition 2.10. *Let M be a right top Artinian R-module. Then the following results hold:*

(1) $M/J(M)$ is a cyclic semi-simple distributive module.

(2) If $J(M)$ is a superfluous submodule of M, then M is cyclic.

(3) If M is finitely generated, then M is cyclic.

Proof. (1) $M/J(M)$ is a top R-module by Zhang [17], Proposition 2.3 since M is a right top R-module. By Anderson and Fuller [2], Proposition 10.15 we know that $M/J(M)$ is a finitely generated semi-simple top module, and hence $M/J(M)$ is a finitely generated semi-simple distributive R-module by Theorem 2.2. Let

$$M/J(M) = M_1/J(M) \bigoplus M_2/J(M) \bigoplus \cdots \bigoplus M_n/J(M),$$

where $M_i/J(M)$ $(i = 1, 2, \cdots, n)$ are right simple R-submodules of $M/J(M)$. For every $i = 1, 2, \cdots, n$, there is an epimorphism $\alpha_i : R_R \twoheadrightarrow M_i/J(M)$. Let $N/J(M) = \sum_{i=1}^{n} \alpha_i(R_R) \subseteq M/J(M)$ and let $\pi_i(M/J(M)) \twoheadrightarrow M_i/J(M)$ be the natural projections. Then $\pi_i(N/J(M)) = \alpha_i(R_R) = M_i/J(M)$ and $N/J(M) = \bigoplus_{i=1}^{n}(N/J(M) \bigcap M_i/J(M))$ since $M/J(M)$ is a distributive module. Hence $M_i/J(M) = \pi_i(N/J(N)) = N/J(M) \bigcap M_i/J(M) \subseteq N/J(M)$ and $N/J(M) = M/J(M)$. Thus $\sum_{i=1}^{n} \alpha_i : R_R \twoheadrightarrow M/J(M)$ is an epimorphism and hence $M/J(M)$ is cyclic.

(2) By (1) there exists a cyclic submodule X of M such that $M = X + J(M)$. By assumption $J(M)$ is superfluous in M. Hence $M = X$.

(3) Since M is finitely generated, every proper submodule is contained in a maximal submodule of M and hence $J(M)$ is superfluous in M. By (2), M is cyclic. □

Corollary 2.11. *Let R be a right top ring and M a nonzero right Artinian R-module. Then the following statements are equivalent:*

(1) M is a right finitely generated top R-module.

(2) M is a right cyclic R-module.

(3) M is a right top R-module and $J(M)$ is a superfluous submodule of M.

Proof. (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (2) are clear from Proposition 2.10.

(2) $\Rightarrow$ (1) and (2) $\Rightarrow$ (3). Since M is a cyclic right R-module, there is a right ideal I of R such that $M \cong R/I \neq 0$. Since R is a right top ring, M is a right top R-module by Zhang [17], Proposition 2.3. Clearly $J(M)$ is superfluous in M since M is cyclic. □

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