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NORM INEQUALITIES WITH NON-ISOTROPIC KERNELS

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Abstract: In this study, we establish norm inequalities with non-isotropic kernels depending on $\lambda-$distance for maximal operators and fractional integrals generated by the $\lambda-$distance.

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1. Introduction

The $\lambda-$distance between points $x=(x_1,...,x_n)$ and $y=(y_1,...,y_n)$ is defined by the following formula given in [1], [5]-[7], [9]:

$$|x-y|_\lambda := (|x_1-y_1|^{\frac{1}{\lambda_1}}+|x_2-y_2|^{\frac{1}{\lambda_2}}+...+|x_n-y_n|^{\frac{1}{\lambda_n}})^{\frac{|\lambda|}{n}},$$

where $\lambda=(\lambda_1,\lambda_2,...,\lambda_n)$, $\lambda_k\geq\frac{1}{2}$, $k=1,2,...,n$, $|\lambda|=\lambda_1+\lambda_2+...+\lambda_n$. Note that this distance has the following properties of homogeneity for any positive t,

$$\left(\left|t^{\lambda_1}x_1\right|^{\frac{1}{\lambda_1}}+...+\left|t^{\lambda_n}x_n\right|^{\frac{1}{\lambda_n}}\right)^{\frac{|\lambda|}{n}}=t^{\frac{|\lambda|}{n}}\left|x\right|_\lambda,\ \ t>0.$$

From this, relation it follows that the $\lambda-$distance is the $a-$homogeneous function [1], [5]-[9], where $a=\frac{|\lambda|}{n}$. So the non-isotropic λ-distance has the following properties:

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1 $|x|_\lambda = 0 \Leftrightarrow x = \theta, \;\; \theta = (0, 0, ..., 0)$.
2. $\left|t^\lambda x\right|_\lambda = |t|^{\frac{|\lambda|}{n}} |x|_\lambda$.
3. $|x + y|_\lambda \leq k(|x|_\lambda + |y|_\lambda)$.
Here $k = 2^{\left(1+\frac{1}{\lambda_{\min}}\right)\frac{|\lambda|}{n}}$, $\lambda_{\min} = \min\{\lambda_1, \lambda_2, ..., \lambda_n\}$.
Here we consider $\lambda-$spherical coordinates by the following formulas:

$$x_1 = (\rho \cos \varphi_1)^{2\lambda_1}, ..., x_n = (\rho \sin \varphi_1 \sin\varphi_2 ... \sin \varphi_{n-1})^{2\lambda_n}.$$

We obtain that $|x|_\lambda = \rho^{\frac{2|\lambda|}{n}}$. It can be seen that the Jacobian $J_\lambda(\rho, \varphi)$ of this transformation is $J_\lambda(\rho, \varphi) = \rho^{2|\lambda|-1}\Omega_\lambda(\varphi)$, where $\Omega_\lambda(\varphi)$ is the bounded function, which only depends on angles $\varphi_1, \varphi_2, ..., \varphi_{n-1}$. It is clear that if $\lambda_i = \frac{1}{2}, \; i = 1, ..., n$, then the λ-distance is Euclidean distance.

Let μ be a non-negative $n-$dimensional Borel mesure on $\mathbb{R}^d$, that is a mesure satisfying

$$\mu(Q_\lambda(x, r)) \leq \ell(Q_\lambda(x, r))^{2|\lambda|}$$

for $\lambda-$cube $Q_\lambda(x, r) = \left\{y : \; |x_1 - y_1| \leq r^{2\lambda_1}, ..., |x_n - y_n| \leq r^{2\lambda_n}\right\} \subset \mathbb{R}^d$ with sides paralled to the coordinates axes centerd at x with side length $r^{2\lambda_i}$, $i = 1, 2, ..., n$, where $\lambda_1 \geq \frac{1}{2}, ..., \lambda_n \geq \frac{1}{2}, \; |\lambda| = \lambda_1 + ... + \lambda_n$, $\ell(Q_\lambda)$ stands for the side length of Q_λ and n is a fixed reel number such that $0 < n \leq d$.

Let $0 < \alpha < n$. For $f \in L_\infty(\mathbb{R}^d)$ a boundedly supported function we defined the fractional integral generated by $\lambda-$distance of order α as

$$I_{\alpha,\lambda} f(x) = \int\limits_{\mathbb{R}^d} |x - y|_\lambda^{(\alpha-n)\frac{2|\lambda|}{n}} f(y) d\mu(y)\,.$$

Furthermore, we define maximal operators depending on $\lambda-$dictance for $0 \leq \alpha < n$, as follows

$$M_\alpha f(x) = \sup_{x \in Q_\lambda} \frac{1}{\ell(Q_\lambda)^{(n-\alpha)\frac{2|\lambda|}{n}}} \int\limits_{Q_\lambda} |f(y)|\, d\mu(y)\,.$$

If $\alpha = 0$, we consider the Hardy-Littlewood radial maximal operators as

$$M_0 f(x) = Mf(x) = \sup_{x \in Q_\lambda} \frac{1}{\ell(Q_\lambda)^{2|\lambda|}} \int\limits_{Q_\lambda} |f(y)|\, d\mu(y).$$

The important properties of the fractional integrals, maximal operators and theirs generalizations were studied by many authors. We refer to papers

[2]-[7], [9]. The aim of this paper is to study norm inequalities for fractional integrals and maximal operators generated by the $\lambda-$distance.

Lemma 1. *Let $0 < \alpha < n$ and $f \in L_\infty(\mathbb{R}^d)$ a boundedly supported function. Then, the following integral is absolutely convergent*

$$I_{\alpha,\lambda} f(x) = \int\limits_{\mathbb{R}^d} |x-y|_\lambda^{(\alpha-n)\frac{2|\lambda|}{n}} f(y) d\mu(y).$$

Proof. Since the support of f is bounded, there is no problem of integrabilitiy at infinity. Although the kernel of the operator is singular at the diagonal $x = y$, we have the following inequality

$$|I_{\alpha,\lambda} f(x)| \leq \int\limits_{|x-y|_\lambda \leq 1} |x-y|_\lambda^{(\alpha-n)\frac{2|\lambda|}{n}} |f(y)| \, d\mu(y)$$
$$\leq \|f\|_{L_\infty(\mu)} \int\limits_{|x-y|_\lambda \leq 1} |x-y|_\lambda^{(\alpha-n)\frac{2|\lambda|}{n}} d\mu(y)$$
$$= \|f\|_{L_\infty(\mu)} \sum_{k=0}^{\infty} \int\limits_{2^{-k-1} \leq |x-y|_\lambda < 2^{-k}} |x-y|_\lambda^{(\alpha-n)\frac{2|\lambda|}{n}} d\mu(y)$$
$$= \|f\|_{L_\infty(\mu)} \sum_{k=0}^{\infty} \frac{\mu\left(Q_\lambda(x, 2^{-k+1})\right)}{2^{(k+1)(\alpha-n)\frac{2|\lambda|}{n}}} \leq 2^{(n-\alpha)\frac{2|\lambda|}{n} + 2|\lambda|} \|f\|_{L_\infty(\mu)} \sum_{k=0}^{\infty} 2^{-k\alpha\frac{2|\lambda|}{n}} < \infty$$

and hence the integral which defines $I_{\alpha,\lambda}$ is absolutely convergent. □

Teorem 1. *Let $0 < \alpha < n$ and f be a bounded function with compact support. Then, for $1 \leq p < \frac{n}{\alpha}$ the following inequality holds*

$$|I_{\alpha,\lambda} f(x)| \leq C \|f\|_{L_{p(\mu)}}^{\frac{p\alpha}{n}} Mf(x)^{1-\frac{p\alpha}{n}}. \tag{1.1}$$

Proof. Let $\rho > 0$. Then

$$|I_{\alpha,\lambda} f(x)| \leq \int\limits_{|x-y|_\lambda < \rho} |x-y|_\lambda^{(\alpha-n)\frac{2|\lambda|}{n}} |f(y)| \, d\mu(y)$$
$$+ \int\limits_{|x-y|_\lambda \geq \rho} |x-y|_\lambda^{(\alpha-n)\frac{2|\lambda|}{n}} |f(y)| \, d\mu(y) = I_1 + I_2.$$

For I_1 we have

$$I_1 = \sum_{k=0}^{\infty} \int\limits_{2^{(-k-1)}\rho \le |x-y|_\lambda < 2^{-k}\rho} |x-y|_\lambda^{(\alpha-n)\frac{2|\lambda|}{n}} |f(y)|\, d\mu(y)$$

$$\le 2^{4|\lambda| - \alpha\frac{2|\lambda|}{n}} \rho^{\alpha\frac{2|\lambda|}{n}} \sum_{k=0}^{\infty} 2^{-k\alpha\frac{2|\lambda|}{n}} \frac{1}{(2^{-k+1}\rho)^{2|\lambda|}} \int\limits_{\mathbf{Q}} |f(y)|\, d\mu(y) = C\rho^{\alpha\frac{2|\lambda|}{n}} Mf(x).$$

On the other hand if $p = 1$, then we have the following inequality for I_2

$$I_2 = \int\limits_{|x-y|_\lambda \ge \rho} |x-y|_\lambda^{(\alpha-n)\frac{2|\lambda|}{n}} |f(y)|\, d\mu(y)$$

$$\le \frac{1}{\rho^{(n-\alpha)\frac{2|\lambda|}{n}}} \int\limits_{|x-y|_\lambda \ge \rho} |f(y)|\, d\mu(y) = C\rho^{(\alpha-n)\frac{2|\lambda|}{n}} \|f\|_{L_1}.$$

For $1 \le p < \frac{n}{\alpha}$, put $\beta = \frac{2|\lambda|}{n}(n-\alpha)p' - n$. Then $\beta > 0$ and we have

$$I_2 = \int\limits_{|x-y|_\lambda \ge \rho} |x-y|_\lambda^{(\alpha-n)\frac{2|\lambda|}{n}} |f(y)|\, d\mu(y)$$

$$\le \|f\|_{L_{p(\mu)}} \left(\int\limits_{|x-y|_\lambda \ge \rho} |x-y|_\lambda^{(\alpha-n)\frac{2|\lambda|}{n}p'} d\mu(y) \right)^{\frac{1}{p'}} \le C \|f\|_{L_{p(\mu)}} \rho^{-(\frac{2|\lambda|}{p} - \frac{2|\lambda|}{n}\alpha)}.$$

Thus, there is the following inequality for any $\rho > 0$

$$|I_{\alpha,\lambda} f(x)| \le I_1 + I_2 \le C(\rho^{\frac{2|\lambda|}{n}\alpha} Mf(x) + \rho^{-2|\lambda|(\frac{1}{p} - \frac{\alpha}{n})} \|f\|_{L_{p(\mu)}}).$$

Minimizing the right-hand side with respect to ρ, we see that its minimum is reached at

$$\rho_{\min} = C_1 Mf(x)^{\frac{-p}{2|\lambda|}} \|f\|_{L_{p(\mu)}}^{\frac{p}{2|\lambda|}},$$

and easy evaluations give

$$|I_{\alpha,\lambda} f(x)| \le C \|f\|_{L_{p(\mu)}}^{\frac{p\alpha}{n}} Mf(x)^{1-\frac{p\alpha}{n}}. \qquad \square$$

Remark 1. Let $0 < \alpha < n$, $x \in \mathbb{R}^d$ and $x \in Q_\lambda$. Then, for every $y \in Q_\lambda$ we have $|x-y|_\lambda \le \sqrt{d}\ell(Q_\lambda)$ and so

$$\frac{1}{\ell(Q_\lambda)^{(n-\alpha)\frac{2|\lambda|}{n}}}\int_{\mathbf{Q}}|f(y)|\,d\mu(y)\leq d^{(n-\alpha)\frac{|\lambda|}{n}}\int_{Q_\lambda}|x-y|_\lambda^{(\alpha-n)\frac{2|\lambda|}{n}}|f(y)|\,d\mu(y)$$
$$\leq d^{(n-\alpha)\frac{|\lambda|}{n}}I_{\alpha,\lambda}\left(|f|\right)(x).$$

By taking the supremum over all $\lambda-$cube Q_λ containing x, we obtain that

$$M_\alpha f(x)\leq d^{(n-\alpha)\frac{|\lambda|}{n}}I_{\alpha,\lambda}\left(|f|\right)(x).$$

Proposition 1. *Let $0<\alpha<n$.*

(a) If $1\leq p<\frac{n}{\alpha}$ and $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$, then $I_{\alpha,\lambda}:L_p(\mu)\to L_q(\mu)$.

(b) If $\frac{1}{q}=1-\frac{\alpha}{n}$, then $I_{\alpha,\lambda}:L_1(\mu)\to L_q(\mu)$.

Proof. For the (a) part of the proposition, if we consider the (1.1) inequality, then we have that

$$\left(\int_{\mathbb{R}^d}|I_{\alpha,\lambda}f(x)|^q\,d\mu(x)\right)^{\frac{1}{q}}\leq C\|f\|_{L_{p(\mu)}}^{\frac{p\alpha}{n}}\left(\int_{\mathbb{R}^d}Mf(x)^{q(1-\frac{p\alpha}{n})}d\mu(x)\right)^{\frac{1}{q}}$$
$$=C\|f\|_{L_{p(\mu)}}^{\frac{p\alpha}{n}}\left(\int_{\mathbb{R}^d}Mf(x)^p d\mu(x)\right)^{\frac{1}{q}}\leq C\|f\|_{L_{p(\mu)}}^{\frac{p\alpha}{n}}\|f\|_{L_{p(\mu)}}^{\frac{p}{q}}=C\|f\|_{L_{p(\mu)}}.$$

Let us see what happens in the case of $p=1$. In this case, the inequality (1.1) becomes

$$|I_{\alpha,\lambda}f(x)|\quad\leq\quad C\,\|f\|_{L_{1(\mu)}}^{\frac{\alpha}{n}}Mf(x)^{1-\frac{\alpha}{n}}=C\,\|f\|_{L_{1(\mu)}}^{\frac{\alpha}{n}}Mf(x)^{\frac{1}{q}}.$$

Use that M is of the weak type $(1,1)$, we conclude

$$\mu\{x\in\mathbb{R}^d:|I_{\alpha,\lambda}f(x)|>\varepsilon\}\leq\mu\left\{x\in\mathbb{R}^d:Mf(x)>\left(\frac{\varepsilon}{C\,\|f\|_{L_{1(\mu)}}^{\frac{\alpha}{n}}}\right)^q\right\}$$
$$\leq\left(\frac{C\,\|f\|_{L_{1(\mu)}}^{\frac{\alpha}{n}}}{\varepsilon}\right)^q\|f\|_{L_{1(\mu)}}=\left(\frac{C\,\|f\|_{L_{1(\mu)}}}{\varepsilon}\right)^q.$$

Theorem 2. *Let $0<\alpha<n$ and $0<\varepsilon<\min\{\alpha,(n-\alpha)\frac{2|\lambda|}{n}\}$. Then for any bounded function with bounded support f, we have*

$$|I_{\alpha,\lambda}f(x)|\leq C\left(M_{\alpha-\varepsilon}f(x)M_{\alpha+\varepsilon}f(x)\right)^{\frac{1}{2}}\ ,$$

where C only depends on n, α, λ and ε.

Proof. We can split $I_{\alpha,\lambda}f$ for $\rho > 0$

$$|I_{\alpha,\lambda}f(x)| = \int\limits_{|x-y|_\lambda<\rho} |x-y|_\lambda^{\frac{2|\lambda|}{n}(\alpha-n)} |f(y)|\, d\mu(y)$$
$$+ \int\limits_{|x-y|_\lambda\geq\rho} |x-y|_\lambda^{\frac{2|\lambda|}{n}(\alpha-n)} |f(y)|\, d\mu(y) = I + II.$$

For I we have

$$I = \sum_{k=0}^{\infty} \int\limits_{2^{(-k-1)}\rho\leq|x-y|_\lambda<2^{-k}\rho} |x-y|_\lambda^{\frac{2|\lambda|}{n}(\alpha-n)} |f(y)|\, d\mu(y)$$
$$\leq \sum_{k=0}^{\infty} \left(2^{-k-1}\rho\right)^{(\alpha-n)\frac{2|\lambda|}{n}} \int\limits_{Q_\lambda(x,2^{-k+1}\rho)} |f(y)|\, d\mu(y)$$
$$\leq C\rho^{\frac{2|\lambda|}{n}\varepsilon} M_{\alpha-\varepsilon}f(x) \sum_{k=0}^{\infty} 2^{-k\frac{2|\lambda|}{n}\varepsilon} \leq C\rho^{\frac{2|\lambda|}{n}\varepsilon} M_{\alpha-\varepsilon}f(x).$$

For II we obtain

$$II = \sum_{k=0}^{\infty} \int\limits_{2^k\rho\leq|x-y|_\lambda<2^{(k+1)}\rho} |x-y|_\lambda^{\frac{2|\lambda|}{n}(\alpha-n)} |f(y)|\, d\mu(y)$$
$$\leq \sum_{k=0}^{\infty} (2^k\rho)^{\frac{2|\lambda|}{n}(\alpha-n)} \int\limits_{2^k\rho\leq|x-y|_\lambda<2^{(k+1)}\rho} |f(y)|\, d\mu(y)$$
$$\leq C\sum_{k=0}^{\infty} (2^k\rho)^{(\alpha-n)\frac{2|\lambda|}{n}} \frac{(2^{k+2}\rho)^{(n-\alpha-\varepsilon)\frac{2|\lambda|}{n}}}{(2^{k+2}\rho)^{(n-\alpha-\varepsilon)\frac{2|\lambda|}{n}}} \int\limits_{|x-y|_\lambda<2^{(k+1)}\rho} |f(y)|\, d\mu(y)$$
$$\leq C\rho^{-\frac{2|\lambda|}{n}\varepsilon} M_{\alpha+\varepsilon}f(x) \sum_{k=0}^{\infty} 2^{-k\frac{2|\lambda|}{n}\varepsilon} \leq C\rho^{-\frac{2|\lambda|}{n}\varepsilon} M_{\alpha+\varepsilon}f(x).$$

Then for $0 < \alpha < n$ and $0 < \varepsilon < \min\{\alpha, (n-\alpha)\frac{2|\lambda|}{n}\}$ we have $0 < \alpha - \varepsilon < \alpha$ $< \alpha + \varepsilon < n$. Thus, collecting the both estimates,

$$|I_{\alpha,\lambda}f(x)| \leq C\left(\rho^{\frac{2|\lambda|}{n}\varepsilon} M_{\alpha-\varepsilon}f(x) + \rho^{-\frac{2|\lambda|}{n}\varepsilon} M_{\alpha+\varepsilon}f(x)\right)$$

for any $\rho > 0$. Minimizing the right-hand side with respect to ρ, we see that its minimum is reached at

$$\rho_{\min} = C_1 (M_{\alpha-\varepsilon} f(x)^{-1} M_{\alpha+\varepsilon} f(x))^{\frac{n}{4|\lambda|\varepsilon}}$$

and easy evaluations give

$$|I_{\alpha,\lambda} f(x)| \leq C \left(M_{\alpha-\varepsilon} f(x) M_{\alpha+\varepsilon} f(x) \right)^{\frac{1}{2}}. \qquad \square$$

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