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SEMILOCALLY B-E-CONVEX FUNCTIONS

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Abstract: In this paper, based on the E-convexity, semilocal convexity and B-vexity, a new class of generalized convexity called semilocal B-E-convexity is presented. Then some basic and important properties of the proposed convexity are discussed. At the same time, some optimality conditions for the relative generalized convex programming are discussed.

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1. Introduction

Convexity and generalized convexity play a key role in many aspects of optimization, such as optimality conditions, saddle-point theorems, duality theorems of alternatives, and convergence of optimization algorithms, so the research on convexity and generalized convexity is one of the important aspects in mathematical programming. During the past several decades, various significant generalizations of convexity have been presented. In 1999, Youness [7] brought forward the concept of E-convex programming by using a point to

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point map E. In 2002, Chen [2] introduced a class of semi-E-convex functions and discussed their basic properties. In 2003, with the help of two point-to-set maps E and F, Jian [4] introduced the concepts of (E,F)-convexity. More recently, Jian, Hu, Tang, and Zheng [5] further presented a kind of semi-(E,F)-convexity. In 1977, Ewing [3] introduced semilocally convexity. In 1992, Weir [6] provided some optimality conditions and duality results for semilocally convex programming by establishing a theorem of alternatives. In 1991, Bector and Singh [1] introduced a B-vex function, and discussed its basic properties.

In this paper, based on the E-convexity, semilocal convexity and B-vexity, we first bring forward a new type of generalized convexity – locally starshaped E-convex set and semilocally B-E-convex function, then discuss the important and basic properties of this kind of functions, establish the optimality conditions for the relative generalized convex programming.

This paper is organized as follows. The preliminary results and some known definitions which have relationships with this work are recalled in Section 2. In Section 3, we will introduce the locally starshaped E-convex set, semilocally B-E-convex functions and discuss their properties. In Section 4, we will discuss semilocally B-E-convex programming and some of their optimality conditions.

2. Preliminaries

In this section, we recall several known definitions which have some relationships with our work. Let M be a nonempty subset in R^n, and let R and R_+ denote the sets of real numbers and nonnegative real numbers, let maps f: $M \to R$, g: $M \to R$, b: $M \times M \times [0,1] \to [0,1]$.

Definition 2.1. (see [7]) A set $M \subseteq R^n$ is said to be an E-convex set if there is a point to point map $E : M \to R^n$ such that

$$\lambda E(x) + (1-\lambda)E(y) \in M, \quad \forall x, y \in M, \quad \forall \lambda \in [0,1]. \tag{2.1}$$

Definition 2.2. (see [7]) A function $f : M \to R$ is said to be E-convex on a set $M \subseteq R^n$ if there is a point to point map $E : M \to R^n$ such that M is an E-convex set and

$$f(\lambda E(x) + (1-\lambda)E(y)) \leq \lambda f(E(x)) + (1-\lambda) f(E(y)), \quad \forall x, y \in M, \quad \forall \lambda \in [0,1]. \tag{2.2}$$

Definition 2.3. (see [2]) f is said to be a semi-E-convex function if there is a point to point map $E : M \to R^n$ such that M is an E-convex set and

$$f(\lambda E(x) + (1-\lambda)E(y)) \leq \lambda f(x) + (1-\lambda)f(y), \quad \forall x, y \in M, \quad \forall \lambda \in [0,1]. \tag{2.3}$$

Definition 2.4. (see [6]) A set $M \subseteq R^n$ is said to be a locally starshaped set if corresponding to each pair of points $x, y \in M$, there is a maximal positive number $a(x,y) \leq 1$, such that

$$\lambda x + (1-\lambda)y \in M, \quad \forall \lambda \in (0, a(x,y)). \tag{2.4}$$

Definition 2.5. (see [6]) A function $f : R^n \to R$ is said to be a semilocally convex function on a locally starshaped set $M \subseteq R^n$ if corresponding to each pair of points $x, y \in M$, there is a maximal positive number $d(x,y) \leq a(x,y)$, such that

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y), \quad \forall \lambda \in (0, d(x,y)). \tag{2.5}$$

Definition 2.6. (see [1]) f is said to be a B-vex with respect to map b, if M is a convex set and

$$f(\lambda x + (1-\lambda)y) \leq \lambda b(x,y,\lambda)f(x) + (1-\lambda b(x,y,\lambda))f(y), \quad \forall \lambda \in [0,1], \quad \forall x, y \in M. \tag{2.6}$$

3. Semilocally B-E-Convex Functions

In this section, first of all, we introduce a new class of generalized convex function on a new generalized convex set called locally starshaped E-convex set. Then we discuss some important properties of the generalized convexity.

Definition 3.1. A set $M \subseteq R^n$ is said to be a locally starshaped E- convex set if there is a point to point map $E : M \to R^n$ such that corresponding to each pair of points $x, y \in M$, there is a maximal positive number $a(x,y) \leq 1$, such that

$$\lambda E(x) + (1-\lambda)E(y) \in M, \quad \forall \lambda \in [0, a(x,y)]. \tag{3.1}$$

Remark 3.1. If M is a locally starshaped E-convex set, then $E(M) \subseteq M$.

Remark 3.2. Each locally starshaped set must be locally starshaped E-convex with respect to map $E(x) \equiv x$. But the converse is not necessarily true. See the following example.

Example 3.1. Consider set M=[0,1]∪[2,3], define map E as

$$E(x) = \begin{cases} x, & x \in [0,1], \\ x-2, & x \in [2,3]. \end{cases}$$

Then M is locally starshaped E-convex, but not locally starshaped convex since there is no positive number $a(1,2) \leq 1$ such that $\lambda \times 2 + (1-\lambda) \times 1 \in M, \forall \lambda \in [0, a(1,2)]$.

Similar, each E-convex set must be locally starshaped E-convex by taking $a(x,y) = d(x,y) = 1$. But the converse is not necessarily true. See the following example.

Example 3.2. Consider set $M = (0,1) \cup (2,3)$, define map E as $E(x)$=x, then M is locally starshaped E-convex, but not E-convex, since $\frac{1}{2}E(\frac{4}{5})$+$\frac{1}{2}$ $E(\frac{5}{2})$=$\frac{33}{20}$$\notin M$.

Proposition 3.1. *If sets $M_i \subseteq R^n$ $(i = 1, 2, ..., m)$ are all locally starshaped E-convex for the same map E, then their intersection $M = \bigcap_{i=1}^{m} M_i$ is also locally starshaped E-convex with respect to the same map E.*

Proof. Since M_i is locally starshaped E-convex for the same map E, for each pair of points x, $y \in M_i$, there is a maximal positive number $a_i(x,y) \leq 1$ satisfying (3.1). Let $a(x,y)$=min{$a_i(x,y)$, i=1,2,...,m}. Then $\lambda E(x) + (1-\lambda)E(y) \in M, \forall \lambda \in [0, a(x,y)]$. Thus, M is a locally starshaped E-convex set with respect to the same map E. □

Definition 3.2. A function $f : R^n \to R$ is said to be a semilocally B-E-convex function on a locally starshaped E-convex set $M \subseteq R^n$ if corresponding to each pair of points $x, y \in M$ (with a maximal positive number $a(x,y) \leq 1$ satisfying (3.1)), there is a maximal positive number $d(x,y) \leq a(x,y)$, such that

$$f(\lambda E(x) + (1-\lambda)E(y)) \leq b(x,y,\lambda)f(x) + (1-b(x,y,\lambda))f(y), \quad \forall \lambda \in [0, d(x,y)]. \quad (3.2)$$

Definition 3.3. A function $f : R^n \to R$ is said to be a strictly semilocally B-E-convex on a locally starshaped E-convex set $M \subseteq R^n$ if corresponding to each pair of points $x, y \in M$ and $x \neq y$ (with a maximal positive number

$a(x,y) \leq 1$ satisfying (3.1)), there is a maximal positive number $d(x,y) \leq a(x,y)$, such that

$$f(\lambda E(x)+(1-\lambda)E(y)) < b(x,y,\lambda)f(x)+(1-b(x,y,\lambda))f(y), \quad \forall \lambda \in (0,d(x,y)). \quad (3.3)$$

Remark 3.3. If function f is semilocally B-E-convex, then $f(E(y)) \leq f(y), \forall y \in M$, by letting $x=y$ in (3.2).

Remark 3.4. Every semilocally convex function is semilocally B-E-convex, with maps $b(x,y,\lambda) \equiv \lambda$, $E(x)=x$. Every B-vex function is semilocally B-E-convex by taking $E(x) \equiv x$, $a(x,y)=d(x,y) \equiv 1$, and every E-convex function is semilocally B-E-convex by taking $b(x,y,\lambda) \equiv \lambda$, $a(x,y)=d(x,y) \equiv 1$. However, none of their converses is necessarily true. See the following example.

Example 3.3. (1) Let $M=R$ and define map f, b and E as:

$$f(x)=\begin{cases} -1, & x \notin (-2,2), \\ 0, & -2<x<2, \end{cases} \qquad b(x,y,\lambda) \equiv \lambda, \quad E(x)=-2-\mid x \mid .$$

Then f is semilocally B-E-convex with $a(x,y)=d(x,y) \equiv 1$, but not semilocally convex, since $f(\lambda \times 2+(1-\lambda)\times(-2)))=0>\lambda f(2)+(1-\lambda)f(-2)=-1, \forall \lambda \in (0,1)$.

(2) Consider the maps f, b and E defined as above, then f is semilocally B-E-convex, but not B-vex with respect to any map $b: M \times M \times [0,1] \to [0,1]$, since $f(\frac{1}{2}(2)+\frac{1}{2}(-2))=0> -1 =\frac{1}{2}b(2,-2,\frac{1}{2})f(2)+(1-\frac{1}{2}b(2,-2,\frac{1}{2}))f(-2)$.

(3) Define map f as:

$$f(x)=\begin{cases} -1, & x>0, \\ 1, & x \leq 0. \end{cases} \quad E(x)=x, \quad b(x,y,\lambda)=\begin{cases} 0, & y<0, \\ \lambda, & \text{otherwise.} \end{cases}$$

It is not difficult to show that f is semilocally B-E-convex, but not E-convex, since $f(\frac{1}{2}E(1)+\frac{1}{2}E(-2))=1> 0=\frac{1}{2}f(E(1))+\frac{1}{2}f(E(-2))$.

Proposition 3.2. *Suppose that $\{f_i, i \in I=\{1,2,..,m\}\}$ are semilocally B-E-convex functions, with respect to the same map b, on a locally starshaped E-convex set M. Then function $f=\sum_{i \in I} \alpha_i f_i$ ($\alpha_i \in R_+$) is semilocally B-E-convex with respect to maps b and E.*

This proof is elementary and omitted.

Proposition 3.3. *Suppose that $f: M \to R$ is a semilocally B-E-convex function defined on a locally starshaped E-convex set M with respect to maps E, b, and $\phi: R \to R$ is a convex function. Furthermore, suppose that ϕ is not decreasing. Then, the composition $\phi(f)$ is semilocally B-E-convex with respect to maps E, b.*

Proof. Let $x^1, x^2 \in M$. Then there exits a maximal positive number $a(x^1, x^2) \leq 1$ satisfying (3.1). Since f is semilocally B-E-convex, there is a positive number $d(x^1, x^2) \leq a(x^1, x^2)$ such that for all $\lambda \in [0, d(x^1, x^2)]$

$$f(\lambda E(x^1) + (1-\lambda)E(x^2)) \leq b(x^1, x^2, \lambda)f(x^1) + (1 - b(x^1, x^2, \lambda)f(x^2).$$

Notice that ϕ is convex and not decreasing, we get

$$\begin{aligned}\phi(f(\lambda E(x^1) + (1-\lambda)E(x^2))) &\leq \phi(b(x^1, x^2, \lambda)f(x^1) + (1 - b(x^1, x^2, \lambda))f(x^2)) \\ &\leq b(x^1, x^2, \lambda)\phi(f(x^1)) + (1 - b(x^1, x^2, \lambda))\phi(f(x^2)), \quad \forall \lambda \in [0, d(x^1, x^2)].\end{aligned}$$

So $\phi(f)$ is semilocally B-E-convex. □

Definition 3.4. The set $G = \{(x, \alpha) : x \in M \subseteq R^n, \alpha \in R\}$ is said to be a locally starshaped B-E-convex set corresponding to R^n if for each pair of points $(x^1, \alpha^1), (x^2, \alpha^2) \in G$, there is a map $E : R^n \to R^n$, and a maximal positive number $a((x^1, \alpha^1), (x^2, \alpha^2)) \leq 1$, such that

$$\begin{aligned}&(\lambda E(x^1) + (1-\lambda)E(x^2), b(x^1, x^2, \lambda)\alpha^1 + (1 - b(x^1, x^2, \lambda))\alpha^2) \in G, \\ &\qquad\qquad \forall \lambda \in [0, a((x^1, \alpha^1), (x^2, \alpha^2))].\end{aligned}$$

We now give a characterization of semilocally B-E-convex function f in term of its epigraph G_f by $G_f = \{(x, \alpha) : x \in M, f(x) \leq \alpha, \alpha \in R\}$.

Theorem 3.1. *A numerical function f defined on a locally starshaped E-convex set $M \subseteq R^n$ is semilocally B-E-convex iff G_f is a semilocally B-E-convex set corresponding to R^n with respect to maps E, b.*

Proof. Suppose that f is semilocally B-E-convex with respect to maps E, b. Let (x^1, α^1), $(x^2, \alpha^2) \in G_f$. Then $x^1, x^2 \in M$, $f(x^1) \leq \alpha^1$, $f(x^2) \leq \alpha^2$. Since M is locally starshaped E-convex, there is a maximal positive number $a(x^1, x^2) \leq 1$ satisfying (3.1). On the other hand, since f is semilocally B-E-convex, there is a positive number $d(x^1, x^2) \leq a(x^1, x^2)$ such that for all $\lambda \in [0, d(x^1, x^2)]$

$$\begin{aligned}f(\lambda E(x^1) + (1-\lambda)E(x^2) &\leq b(x^1, x^2, \lambda)f(x^1) + (1 - b(x^1, x^2, \lambda))f(x^2) \\ &\leq b(x^1, x^2, \lambda)\alpha^1 + (1 - b(x^1, x^2, \lambda))\alpha^2.\end{aligned}$$

Hence for all $\lambda \in [0, d(x^1, x^2)]$,

$$(\lambda E(x^1) + (1-\lambda)E(x^2), b(x^1, x^2, \lambda)\alpha^1 + (1 - b(x^1, x^2, \lambda))\alpha^2) \in G_f.$$

Thus G_f is a locally starshaped B-E-convex set corresponding to R^n with respect to maps E, b.

Conversely, if G_f is a locally starshaped B-E-convex set corresponding to R^n with respect to maps E, b, then for the points $(x^1, f(x^1)), (x^2, f(x^2)) \in G_f$, there is a maximal positive number $a((x^1, f(x^1)), (x^2, f(x^2))) \leq 1$, such that

$$(\lambda E(x^1) + (1-\lambda)E(x^2), b(x^1, x^2, \lambda)f(x^1) + (1 - b(x^1, x^2, \lambda))f(x^2)) \in G_f$$

for all $\lambda \in [0, a((x^1, f(x^1)), (x^2, f(x^2)))]$. That is $\lambda E(x^1) + (1-\lambda)E(x^2) \in M$, and

$$f(\lambda E(x^1) + (1-\lambda)E(x^2)) \leq b(x^1, x^2, \lambda)f(x^1) + (1 - b(x^1, x^2, \lambda)f(x^2)$$

holds for all $\lambda \in [0, a((x^1, f(x^1)), (x^2, f(x^2)))]$. Thus M is locally starshaped E-convex and f is semilocally B-E-convex on M with respect to maps E, b. □

Theorem 3.2. *If $\{S_i : i \in I = \{1, 2, ..., m\}\}$ is a family of locally starshaped B-E-convex sets corresponding to R^n, then their intersection $\bigcap_{i\in I} S_i$ is also a locally starshaped B-E-convex set corresponding to R^n.*

Proof. Let $(x^1, \alpha^1), (x^2, \alpha^2) \in \bigcap_{i\in I} S_i$. Then for each $i \in I, (x^1, \alpha^1), (x^2, \alpha^2) \in S_i$, since S_i is a locally starshaped B-E-convex set, there is a maximal positive number $a_i((x^1, \alpha^1), (x^2, \alpha^2)) \leq 1$ such that

$$(\lambda E(x^1) + (1-\lambda)E(x^2), b(x^1, x^2, \lambda)\alpha^1 + (1 - b(x^1, x^2, \lambda))\alpha^2) \in S_i$$

holds for all $\lambda \in [0, a_i((x^1, \alpha^1), (x^2, \alpha^2))]$.

Let $a((x^1, \alpha^1), (x^2, \alpha^2)) = \min_{i\in I}\{a_i((x^1, \alpha^1), (x^2, \alpha^2))\}$. Then for all $\lambda \in [0, a((x^1, \alpha^1), (x^2, \alpha^2))]$, we know that

$$(\lambda E(x^1) + (1-\lambda)E(x^2), b(x^1, x^2, \lambda)\alpha^1 + (1 - b(x^1, x^2, \lambda))\alpha^2) \in \bigcap_{i\in I} S_i,$$

Hence, the result follows. □

Theorem 3.3. *If $\{f_i : i \in I = \{1, 2, ..., m\}\}$ is a family of semilocally B-E-convex functions with respect to the same maps E and b on M, then function $f = \max_{i\in I} f_i$ is B-E-convex with respect to maps E and b.*

Proof. Since each f_i is semilocally B-E-convex on M, its epigragh

$$G_{f_i} = \{(x, \alpha) : x \in M, f_i(x) \leq \alpha, \alpha \in R\}$$

is a locally starshaped B-E-convex set with respect to maps E and b corresponding to R^n by Theorem 3.1. Therefore, their intersection

$$\begin{aligned} \bigcap_{i\in I} G_{f_i} &= \bigcap_{i\in I}\{(x, \alpha) : x \in M, \alpha \in R, f_i \leq \alpha, i \in I\} \\ &= \{(x, \alpha) : x \in M, \alpha \in R, f(x) \leq \alpha\} = G_f, \end{aligned}$$

is also a locally starshaped B-E-convex set corresponding to R^n by Theorem 3.2. Hence, by Theorem 3.1, f is semilocally B-E-convex on M. □

Theorem 3.4. *If f is a semilocally B-E-convex function on a locally starshaped E-convex set M, then the level set $S_\alpha=\{x : x \in M, f(x) \le \alpha\}$ is a locally starshaped E-convex set for each $\alpha \in R$.*

Proof. Let $x^1, x^2 \in S_\alpha$. Then $x^1, x^2 \in M$ and $f(x^1) \le \alpha$, $f(x^2) \le \alpha$. Taking into account M being a locally starshaped E-convex set, there is a maximal positive number $a(x^1, x^2) \le 1$ such that

$$\lambda E(x^1) + (1-\lambda)E(x^2) \in M, \quad \forall \lambda \in [0, a(x^1, x^2)].$$

In addition, in view of the semilocally B-E-convexity of f, there is a positive number $d(x^1, x^2) \le a(x^1, x^2)$ such that for all $\lambda \in [0, d(x^1, x^2)]$,

$$\begin{aligned} f(\lambda E(x^1) + (1-\lambda)E(x^2)) &\le b(x^1, x^2, \lambda) f(x^1) + (1 - b(x^1, x^2, \lambda)) f(x^2) \\ &\le b(x^1, x^2, \lambda)\alpha + (1 - b(x^1, x^2, \lambda))\alpha = \alpha. \end{aligned}$$

This implies that $\lambda E(x^1) + (1-\lambda)E(x^2) \in S_\alpha$, $\forall \lambda \in [0, d(x^1, x^2)]$. Thus S_α is locally starshaped E-convex set. □

However, the converse of Theorem 3.4 is not necessarily true. See the following example.

Example 3.4. Let functions f and E be defined on set M as:

$$f(x) = \begin{cases} -x, & x > 0, \\ 0, & x \le 0, \end{cases} \quad E(x) = x, \quad M = R.$$

Then

$$S_\alpha = \begin{cases} (-\infty, +\infty), & \text{if } \alpha \ge 0, \\ [-\alpha, +\infty), & \text{if } \alpha < 0 \end{cases}$$

is locally starshaped E-convex. However, for $\lambda > 0$ small enough,$x > 0$, $y < 0$, one has $f(\lambda E(x) + (1-\lambda)E(y)) = f(\lambda x + (1-\lambda)y) = 0$, $b(x, y, \lambda)f(x) + (1 - b(x, y, \lambda))f(y) = -xb(x, y, \lambda)$, this implies f is not semilocally B-E-convex for all maps b possessing the property: $b(x, y, \lambda) \not\equiv 0$ for $x > 0$, $y < 0$, and $\lambda > 0$ small enough.

Proposition 3.4. *If f is semilocally B-E-convex on the locally starshaped E-convex set M and f possesses directional derivative at point $E(x^2)$ with respect to the direction $E(x^1) - E(x^2)$ for $x^1, x^2 \in M$, $f(E(x^2)) = f(x^2)$, $b(x^1, x^2) = \lim_{\lambda \to 0^+} \frac{b(x^1, x^2, \lambda)}{\lambda}$, then*

$$f'(E(x^2); E(x^1) - E(x^2)) \le b(x^1, x^2)(f(x^1) - f(x^2)).$$

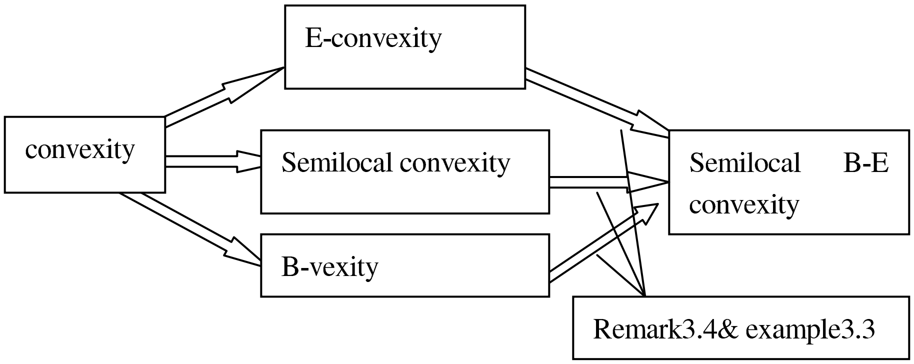

Figure 1:

Proof. Since f is semilocally B-E-convex on a locally starshaped E-convex set M, there is a maximal positive number $a(x^1, x^2) \leq 1$ such that

$$\lambda E(x^1) + (1-\lambda)E(x^2) \in M, \quad \forall \lambda \in [0, a(x^1, x^2)],$$

and a positive number $d(x^1, x^2) \leq a(x^1, x^2)$, such that for all $\lambda \in [0, d(x^1, x^2)]$,

$$f(\lambda E(x^1) + (1-\lambda)E(x^2)) \leq b(x^1, x^2, \lambda)f(x^1) + (1 - b(x^1, x^2, \lambda))f(x^2).$$

Since $f(E(x^2)) = f(x^2)$, we have

$$\frac{f(E(x^2) + \lambda(E(x^1) - E(x^2)) - f(E(x^2))}{\lambda} \leq \frac{b(x^1, x^2, \lambda)}{\lambda}(f(x^1) - f(x^2)).$$

Passing to the limit $\lambda \to 0^+$ in the inequality above, the result follows. □

At the end of this section, we summarize the relationships between different types of convexity discussed above in Figure 1.

4. Semilocally B-E-Convex Programming

Let us consider the following problem:

$$\text{(P)} \qquad \min \ \{f(x) : x \in M\}, \tag{4.1}$$

where $f : R^n \to R$ and there is a map $E : R^n \to R^n$ such that the feasible set M is a locally starshaped E-convex set and f is semilocally B-E-convex with respect to maps E and b.

Theorem 4.1. *For the semilocally B-E-convex programming (4.1) above, the following statements are true:*

(i) the optimal solution set Ω of P is locally starshaped E-convex;

(ii) if x^* is a local minimum for problem (4.1) and $E(x^*) = x^*$, $b(x, x^*, \lambda) > 0$ for any $x \neq x^*$ and $\lambda > 0$ small enough, then x^* is a global minimum.

(iii) if the objective function f is strictly semilocally B-E-convex on M, then the global optimal solution for problem (4.1) is unique.

Proof. (i) Suppose that $x^1, x^2 \in \Omega$. Then $f(x^1) = f(x^2)$. Taking into account M being a locally starshaped E-convex set, there is a maximal positive number $a(x^1, x^2) \leq 1$ such that

$$\lambda E(x^1) + (1-\lambda)E(x^2) \in M, \quad \forall \lambda \in [0, a(x^1, x^2)].$$

In addition, in view of the semilocally B-E-convexity of f, there is a positive number $d(x^1, x^2) \leq a(x^1, x^2)$ such that for all $\lambda \in [0, d(x^1, x^2)]$,

$$f(\lambda E(x^1) + (1-\lambda)E(x^2)) \leq b(x^1, x^2, \lambda)f(x^1) + (1 - b(x^1, x^2, \lambda))f(x^2) = f(x^1).$$

Again, from the optimality of x^1, we have $f(\lambda E(x^1) + (1-\lambda)E(x^2)) \geq f(x^1) = f(x^2)$. Thus $f(\lambda E(x^1) + (1-\lambda)E(x^2)) = f(x^1) = f(x^2)$, i.e., $\lambda E(x^1) + (1-\lambda)E(x^2) \in \Omega$, $\forall x \in [0, d(x^1, x^2)]$. This shows that Ω is a locally starshaped E-convex set.

(ii) Suppose that $N_\varepsilon(x^*)$ is a neighborhood of x^* with radius $\varepsilon > 0$, and f is minimized at x^* in $N_\varepsilon(x^*) \cap M$. For each $x \in M$ and $x \neq x^*$, there is a maximal positive number $a(x,x^*) \leq 1$ such that

$$\lambda E(x) + (1-\lambda)E(x^*) \in M, \quad \forall \lambda \in [0, a(x, x^*)]\,.$$

Furthermore, there is a positive number $d(x, x^*) \leq a(x, x^*)$ such that

$$f(\lambda E(x) + (1-\lambda)E(x^*)) \leq b(x, x^*, \lambda)f(x) + (1 - b(x, x^*, \lambda))f(x^*), \qquad \forall \lambda \in [0, d(x, x^*)].$$

In view of $E(x^*) = x^*$, so for $\lambda > 0$ and sufficiently small, $\lambda E(x) + (1-\lambda)E(x^*) \in N_\varepsilon(x^*) \cap M$. Then

$$f(x^*) \leq f(\lambda E(x) + (1-\lambda)E(x^*)) \leq b(x, x^*, \lambda)f(x) + (1 - b(x, x^*, \lambda))f(x^*).$$

This implies that $b(x, x^*, \lambda)f(x^*) \leq b(x, x^*, \lambda)f(x)$ holds for $\lambda > 0$ and sufficiently small. Since $b(x, x^*, \lambda) > 0$, we have $f(x^*) \leq f(x)$, i.e. x^* is a global minimum for problem (4.1).

(iii) By contradiction, suppose that $x^1, x^2 \in M$ are two different global optimal solutions for problem (4.1). Since x^1, $x^2 \in M$, there is a maximal positive number $a(x^1, x^2) \leq 1$ such that

$$\lambda E(x^1) + (1-\lambda)E(x^2) \in M, \quad \forall \lambda \in [0, a(x^1, x^2)].$$

Furthermore, since f is strictly semilocally B-E-convex on M, there is a positive number $d(x^1, x^2) \leq a(x^1, x^2)$ such that for all $\lambda \in (0, d(x^1, x^2))$,

$$f(\lambda E(x^1) + (1-\lambda)E(x^2)) < b(x^1, x^2, \lambda) f(x^1) + (1 - b(x^1, x^2, \lambda)) f(x^2) = f(x^1).$$

The last equality follows from $f(x^1) = f(x^2)$. This contradicts the fact that x^1 is a global optimal solution for (4.1), so the global optimal solution for problem (4.1) is unique. □

Theorem 4.2. *Suppose that function f is differentiable on M, and problem (4.1) is a semilocally B-E-convex programming.*

(i) (Necessary Optimality Conditions) *If $u \in M$ is a minimum for problem (4.1), then u satisfies the following inequality:*

$$\nabla f(E(u))^T (E(v) - E(u)) \geq 0, \quad \forall v \in M. \tag{4.2}$$

(ii) (Sufficient Optimality Conditions) *If $u \in M$ satisfies (4.2), $f(E(u)) = f(u)$ and $b(v,u) = \lim\limits_{\lambda \to 0^+} \frac{b(v,u,\lambda)}{\lambda} > 0$ hold for all $v \in M$ and $v \neq u$, then u is a minimum for problem (4.1).*

Proof. (i) Suppose that u is a minimum for problem (4.1). Since M is a locally starshaped E-convex set, for each $v \in M$, there is a maximal positive number $a(v,u) \leq 1$ such that

$$\lambda E(v) + (1-\lambda)E(u) \in M, \quad \forall \lambda \in [0, a(v,u)].$$

So, from the optimality of u and the differential property of f, we have

$$\begin{aligned} f(u) &\leq f(\lambda E(v) + (1-\lambda)E(u)) = f(E(u) + \lambda(E(v) - E(u))) \\ &= f(E(u)) + \lambda \nabla f(E(u))^T (E(v) - E(u)) + o(\lambda). \end{aligned}$$

That is,

$$\lambda \nabla f(E(u))^T (E(v) - E(u)) \geq f(u) - f(E(u)) + o(\lambda) \geq o(\lambda).$$

The last inequality follows since $f(E(u)) \leq f(u)$ (see Remark 3.3). Dividing the inequality above by λ and passing to the limit $\lambda \to 0^+$, we can conclude that the relationship (4.2) holds.

(ii) Let $v \in M$ and $v \neq u$, since f is semilocally B-E-convex on M, there is a positive number $d(v,u) \leq a(v,u)$ such that

$$f(\lambda E(v)+(1-\lambda)E(u)) \leq b(v,u,\lambda)f(v)+(1-b(v,u,\lambda))f(u), \quad \forall \lambda \in [0,d(v,u)],$$

$$\begin{aligned} f(\lambda E(v)+(1-\lambda)E(u)) &= f(E(u)+\lambda(E(v)-E(u))) \\ &= f(E(u))+\lambda \nabla f(E(u))^T(E(v)-E(u))+o(\lambda), \end{aligned}$$

which together with $f(E(u)) = f(u)$ implies that

$$\begin{aligned} f(v)-f(u) &\geq \frac{\lambda \nabla f(u)^T(E(v)-E(u))+o(\lambda)}{b(v,u,\lambda)} \\ &= \frac{\nabla f(u)^T(E(v)-E(u))+o(\lambda)/\lambda}{b(v,u,\lambda)/\lambda}. \end{aligned}$$

Passing to limit $\lambda \to 0^+$ in the inequality above, we have

$$f(v)-f(u) \geq \frac{\nabla f(E(u))^T(E(v)-E(u))}{b(v,u)} \geq 0 \quad \text{for} \quad v \in M \quad \text{and} \quad v \neq u,$$

i.e. $f(v) \geq f(u)$ holds for $v \in M$ and $v \neq u$. This shows that u is a minimum for problem (4.1). □

In the subsequent discussion, we consider the following inequality constrained optimization problem:

$$(P_g) \qquad \begin{array}{ll} \min & f(x) \\ \text{s.t.} & g_i(x) \leq 0, \quad i \in I \triangleq \{1,\ldots,m\}. \end{array} \tag{4.3}$$

Denote the feasible set of (P_g) by $M_g = \{x \in R^n : g_i(x) \leq 0, i \in I\}$.

Theorem 4.3. *For the problem (4.3), f and $g_i: \ R^n \to R$ are all semilocally B-E-convex functions on R^n, the following statements are all true:*

(i) The feasible set M_g of problem (4.3) is locally starshaped E-convex. Furthermore, problem (4.3) is a semilocally B-E-convex programming.

(ii) The optimal solution set Ω_g of (4.3) is locally starshaped E-convex.

(iii) If x^* is a local minimum for (4.3) and $E(x^*) = x^*$, $b(x,x^*,\lambda) > 0$ for any $x \neq x^*$ and $\lambda > 0$ small enough, then x^* is a global minimum.

(iv) If the objective function f is strictly semilocally B-E-convex on R^n, then the global optimal solution for (4.3) is unique.

Proof. (i) Noting that $M_g = \bigcap_{i=1}^{m} M_i$, $M_i \triangleq \{x \in R^n : g_i(x) \leq 0\}$, by Theorem 3.4 and Proposition 3.1, one knows that the feasible set M_g is locally starshaped E-convex. Therefore, problem (4.3) is a semilocally B-E-convex programming, and result (i) follows. Therefore, results (ii)-(iv) follow immediately from Theorem 4.1. □

Theorem 4.4. (Sufficient Optimality Conditions for the Constrained Problem (4.3)) *Let $x^* \in M_g$ and $I^* = \{i \in I : g_i(x^*) = 0\}$. Assume that $f(E(x^*)) = f(x^*)$, $g_i(E(x^*)) = g_i(x^*)$, $(i \in I^*)$. Suppose that function f is semilocally b_0-E-convex and $g_i : i \in I^*$ is semilocally b_i-E-convex on R^n, f and g_i $(i \in I^*)$ all possess directional derivative along direction $E(x) - E(x^*)$ at x^*, for each $x \in M_g$. Assume that there is a multiplier $\lambda^* = (\lambda_i^* : i \in I^*)$ such that $(x,^* \lambda^*)$ satisfies the following system: for each $x \in M_g$,*

$$f'(E(x^*); E(x) - E(x^*)) + \sum_{i \in I^*} \lambda_i^{*T} g_i'(E(x^*); E(x) - E(x^*)) = 0,$$
$$\lambda^* \geq 0, \quad i \in I^*. \quad (4.4)$$

If the maps b_0, b_i satisfy

$$b_0(x, x^*) = \lim_{\lambda \to 0^+} \frac{b_0(x, x^*, \lambda)}{\lambda} > 0, \quad b_i(x, x^*) = \lim_{\lambda \to 0^+} \frac{b_i(x, x^*, \lambda)}{\lambda},$$

$\forall x \in M$, $\forall i \in I^*$, then x^* is an optimal solution of problem (4.3).

Proof. Since f is semilocally b_0-E-convex and g_i is semilocally b_i-E-convex, by Proposition 3.4, we have

$$f'(E(x^*); E(x) - E(x^*)) \leq b_0(x, x^*)(f(x) - f(x^*)),$$
$$g_i'(E(x^*); E(x) - E(x^*)) \leq b_i(x, x^*)(g_i(x) - g_i(x^*)) = b_i(x, x^*) g_i(x), \quad i \in I^*.$$

The last inequalities and (4.4) yield:

$$b_0(x, x^*)(f(x) - f(x^*)) \geq -\sum_{i \in I^*} \lambda_i^* g_i'(E(x^*); E(x) - E(x^*))$$
$$\geq -\sum_{i \in I^*} \lambda_i^* b_i(x, x^*) g_i(x).$$

The last inequality above follows from $\lambda^* \geq 0$, $b_i(x, x^*) \geq 0$, $g_i(x) \leq 0$, $i \in I^*$. So $f(x) \geq f(x^*)$ follows from $b_0(x, x^*) > 0$. Hence, x^* is an optimal solution of (4.3). □

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