International Journal of Pure and Applied Mathematics
Volume 31 No. 3 2006, 379-400

NUMERICAL SOLUTION OF RIEMANN PROBLEM VIA BOUNDARY INTEGRAL EQUATION WITH CORNERS

Munira Ismail[1 §], Bahrom Sanugi[2], Ali Hassan Mohd Murid[3]
[1,2,3]Department of Mathematics
Faculty of Science
University of Technology of Malaysia
81310 UTM Skudai, Johor Darul Ta'zim, MALAYSIA
[1]e-mail: mi@mel.fs.utm.my
[2]e-mail: bs@mel.fs.utm.my
[3]e-mail: ahmm@mel.fs.utm.my

Abstract: This paper offers an effective technique for seeking numerical solution of the interior Riemann problem on a simply connected region with a finite number of corners by Picard iteration method. Previously we have obtained an integral equation associated with the problem and constructed an iterative formula that facilitated numerical integrations. Numerical examples presented in this paper reveal that solutions obtained by applying Gaussian quadrature are excellent, however it can only provide solution values at off-corner points. In this paper, we constructed a new iterative technique that interpolate solution at each corner point using the values we obtained at off-corner points. By this technique the problem of finding solution at the corners is resolve since we are able to maintain excellent accuracy of solutions everywhere on the boundary including the corners. Proofs of the solvability and uniqueness of the integral equation and its equivalence to the problem are also included.

AMS Subject Classification: 30E25, 45B05, 30C30
Key Words: interior Riemann problem, Fredholm integral equation, Picard iteration method

Received: August 11, 2006

[§]Correspondence author

1. Introduction

We are considering the Riemann problem, a class of boundary value problems for analytic functions on a region with corners, i.e., an arbitrary simply connected region bounded by a curve having a continuously turning tangent except possibly at a finite number of corners whose interior angles are well defined, where there may occur a jump discontinuity of the first derivative in the complex plane. Until now there is no known boundary integral equation for this problem since the most recent investigation that was done for the problem involved smooth arbitrary simply connected region by Murid et al [13], [14] and Wegmann et al [19].

The complex Dirichlet problem, which is a well-known classical boundary value problem is a particular case of the Riemann problem. Swarztrauber [18] derived an integral equation related to the Dirichlet problem for the same type of region and previously we extended his result to obtain an integral equation related to the Riemann problem in Munira et al [12].

In this paper, we prove that the integral equation derived in Munira et al [12] is a generalization of the integral equation derived in Murid et al [13], [14]. Showing their equivalence everywhere excepts at the corners does this and the fact that both integral equations have the same kernel known as generalized Neumann kernel. Here, we establish the solvability of the integral equation derived in Munira et al [12] based on Murid et al [14] and Wegmann et al [19] since the corner points can be ignored as in the view of Delves et al [4] and Stenger et al [17].

The organization of this paper is as follows: After the presentation of some auxillary material in Section 2, we recall in Section 3 the integral equation related to the problem to established the solvability of the integral equation and its equivalence to the problem. In Section 4 we choose Picard iteration method based on a statement by Wegmann [20] that this method always converge for the Neumann kernel and the proof can be found in Gaier [5]. Our problem deals with an integration contour that has corner points, a difficulty that needed treating when seeking numerical solution as mentioned in Baker [1] and Kress [11] and in this paper, we illustrate through several numerical techniques that the method is successfully employed for the problem. Using a technique we have developed in Munira et al [12], we show through numerical examples in Section 6 that the equidistant Simpson's rule or trapezium rule that calculates solution on the boundary including the corner points sometimes yield poor result and are inaccurate and Kress [11] also reported this when he was solving for boundary integral equation in domains with corners.

Further numerical experiments in Section 6 reveal that Gaussian quadrature yields excellent results such that solution is correct up to at least fourteen decimal places, however it does not provide solutions at the corners. In Section 5, we overcome the problem of finding solutions at the corner points by providing a new iterative technique that uses solutions at off-corner points to interpolate solutions at the corners and we are able to maintain excellent accuracy of solutions everywhere on the boundary including the corners.

2. Definitions and Preliminaries

Throughout this paper unless explicitly specified, we define Γ to be at least a piecewise smooth Jordan curve with a counterclockwise parameterization Γ : $t = t(\tau)$, $0 \leq \tau \leq \rho$; having a continuously turning tangent except possibly at a finite number of corners $t_1, t_2, \ldots, t_n$ in the complex plane where the inclinations of the positive semi-tangents have jumps of $\alpha_1, \alpha_2, \ldots, \alpha_n$, respectively. The angel α_i is positive if the jump is in the counterclockwise direction and negative otherwise.

Then define the local interior angle $\beta(t)$ as π at any point t where Γ has a unique tangent, and as $\beta_k = \pi - \alpha_k$ at $t = t_k$. Cusps will not be considered, and hence it will be assumed that $0 < \beta(t) < 2\pi$.

Suppose also that Ω^+ and Ω^- are the interior and exterior of Γ respectively such that the origin of the coordinate systems belong to Ω^+ and ∞ belongs to Ω^-. In this paper we define a region with corners to mean a simply connected region Ω^+ whose boundary is Γ.

According to Gakhov [6], in investigating the limiting values of a Cauchy type integral, the integration contour may have corners. Recall that, if $f(z)$ is an analytic function in Ω^+ and continuous in $\Omega^+ \cup \Gamma$, then the Cauchy integral formula is

$$\frac{1}{2\pi i}\int_\Gamma \frac{f(\omega)}{\omega - z}d\omega = \begin{cases} f(z), & z \in \Omega^+, \\ 0, & z \in \Omega^-. \end{cases} \tag{1}$$

The integral of the form

$$\Phi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\phi(\omega)}{\omega - z}d\omega\,, \tag{2}$$

where $\phi(\omega)$ satisfies the Hölder condition is called the Cauchy type integral and $\phi(\omega)$ is its density and it can be shown that

$$\int_\Gamma \frac{d\omega}{\omega - t} = i\beta(t), \quad t \in \Gamma\,, \tag{3}$$

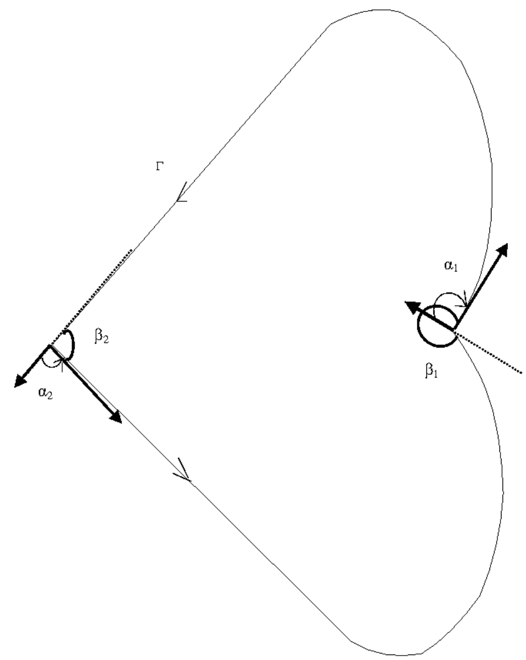

Figure 1: β_1 is obtuse, β_2 is acute

from Henrici [8], or more generally from Hille [7] that

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{f(\omega)}{\omega - t}d\omega = \frac{1}{2\pi}\beta(t)f(t), \quad t \in \Gamma, \tag{4}$$

where these singular integrals are being understood in the sense of the principal value.

The Sokhotskyi formulas, especially for an integration contour having corner points are the limiting values $\Phi^+(t), \Phi^-(t)$ at all $t \in \Gamma$ for the Cauchy type integral (2), i.e.,

$$\Phi^+(t) = \left(1 - \frac{\beta(t)}{2\pi}\right)\phi(t) + \frac{1}{2\pi i}\int_{\Gamma}\frac{\phi(\omega)}{\omega - t}d\omega, \tag{5}$$

$$\Phi^-(t) = -\frac{\beta(t)}{2\pi}\phi(t) + \frac{1}{2\pi i}\int_{\Gamma}\frac{\phi(\omega)}{\omega - t}d\omega. \tag{6}$$

For off-corner points when $\beta(t) = \pi$, (5) and (6) become the ordinary Sokhotskyi formulas which can be found in Gakhov [6].

Two useful formulas related to (5) and (6) which we will use in a later section are the jump relations

$$\Phi^+(t) - \Phi^-(t) = \phi(t), \tag{7}$$

$$\Phi^+(t) + \Phi^-(t) = \left(1 - \frac{\beta(t)}{\pi}\right)\phi(t) + \frac{1}{\pi i}\int_\Gamma \frac{\phi(\omega)}{\omega - t} d\omega. \tag{8}$$

Suppose that $a(t), b(t)$ and $\gamma(t)$ are real functions on the boundary Γ, all satisfying Hölder conditon with $a^2 + b^2 \neq 0$ for all $t \in \Gamma$. In this paper we shall assume that the functions $a(t)$ and $b(t)$ have continuous first order derivative so that $c(t) = a(t) + ib(t)$ is a continuous non-vanishing function given on Γ. Then the index of a function $c(t)$, the interior and exterior Riemann problems are defined as follows.

Index of a Function. The index of the function $c(t)$ with respect to the contour Γ denoted by $\chi = \mathrm{Ind}_\Gamma[c(t)]$ is the increment of its argument in traversing the curve in the positive direction, divided by 2π; i.e., $\chi = \frac{1}{2\pi}[\arg(c(t))]_\Gamma = \frac{1}{2\pi}[\arg(c(t(\tau)))]_0^\rho$. This can be expressed by the integral

$$\chi = \frac{1}{2\pi}\int_\Gamma d\arg c(t) = \frac{1}{2\pi}\int_\Gamma d\ln c(t)\,,$$

which is understood in the Stieltjes sense.

Gakhov [6] also presents a method to calculate the index numerically.

Interior Riemann Problem. The interior Riemann problem consists of finding all functions $f(z)$ analytic in Ω^+ and continuous in $\Omega^+ \cup \Gamma$ and satisfying boundary conditions $a(t)u(t)^+ - b(t)v(t)^+ = \gamma(t)$, for $t \in \Gamma$ which can be written as $\mathrm{Re}[(a(t) + ib(t))(u(t)^+ + iv(t)^+)] = \gamma(t)$.

If $f^+(t) = u(t) + iv(t)$, where f^+ denotes the limit of $f(z)$ when $z \in \Omega^+$ approaches $t \in \Gamma$, we have as boundary condition

$$\mathrm{Re}[c(t)f^+(t)] = \gamma(t). \tag{9}$$

If $\gamma(t) \neq 0$ then (9) is non-homogeneous, while if $\gamma(t) = 0$ we have a homogeneous problem

$$\mathrm{Re}[c(t)f^+(t)] = 0. \tag{10}$$

Exterior Riemann Problem. The exterior Riemann problem consists of finding all functions $f(z)$ analytic in Ω^-, continuous in $\Omega^- \cup \Gamma$, and satisfying boundary condition

$$\mathrm{Re}[c(t)f^-(t)] = \gamma(t). \tag{11}$$

If $\gamma(t) \neq 0$ then (11) is non-homogeneous, while if $\gamma(t) = 0$ we have a homogeneous problem

$$\mathrm{Re}[c(t)f^{-}(t)] = 0. \tag{12}$$

The following lemma from Gakhov [6] shows the relation between the solvability of the Riemann problem and $\chi = \mathrm{Ind}_{\Gamma}[c(t)]$ which is the index of the function $c(t)$.

Lemma 2.1. *In the case $\chi \leq 0$, the homogeneous interior Riemann problem (10), has $-2\chi+1$ linearly independent solution and the non-homogeneous problem (9), is solvable and its solution depends linearly on $-2\chi+1$ arbitrary real constants. In the case $\chi > 0$, the homogeneous problem has only the trivial solution and the non-homogeneous problem is solvable only if $2\chi-1$ conditions are satisfied. If the latter conditions are satisfied the non-homogeneous solution has a unique solution.*

Lemma 2.2. *In the case $\chi > 0$, the homogeneous exterior Riemann problem (12), has $2\chi - 1$ linearly independent solution and the non-homogeneous problem (11), is solvable and its solution depends linearly on $2\chi - 1$ arbitrary real constants. In the case $\chi \leq 0$, the homogeneous problem has only the trivial solution and the non-homogeneous problem is solvable only if $-2\chi + 1$ conditions are satisfied. If the latter conditions are satisfied the non-homogeneous solution has a unique solution.*

For convenience we review the integral equation in the following theorem derived in Murid et al [13], [14] related to the Riemann problem (9) on the region Ω^{+} such that its boundary Γ is specifically smooth without corners and that the unit tangent to Γ given by $T(\omega) = \frac{d\omega}{|d\omega|}$ exists everywhere.

Theorem 2.1. *Suppose $f(z)$ is any solution analytic in G^{+} of the problem with smooth boundary Γ satisfying boundary conditions (9). If $\mu(t) = \mathrm{Im}[c(t)f^{+}(t)]$, $t \in \Gamma$, then $\mu(t)$ satisfies the Fredholm integral equation*

$$\mu(t) - \int_{\Gamma} N(c)(t,\omega)\mu(\omega)|dw| = \mathrm{Im}[c(t)L^{-}(t)], \quad t \in \Gamma, \tag{13}$$

where $N(c)(t,\omega)$ is the generalized Neumann kernel

$$N(c)(t,\omega) = \begin{cases} \dfrac{1}{\pi}\mathrm{Im}[\dfrac{c(t)}{c(\omega)}\dfrac{T(\omega)}{\omega - t}], & \omega \neq t \in \Gamma, \\ \dfrac{1}{2\pi|t'(s)|}\mathrm{Im}[\dfrac{t''(s)}{t'(s)} - \dfrac{2c'(t(s))t'(s)}{c(t(s))}], & \omega = t \in \Gamma, \end{cases} \tag{14}$$

and

$$L(z) = \frac{1}{2\pi i}\int_{\Gamma}\frac{2\gamma(t)}{c(t)}\frac{dt}{t-z}, \tag{15}$$

$t \in \Gamma$ *and* $z \in \Omega^+$.

The proof can be found in Murid et al [13].

3. Integral Equation for the Interior Riemann Problem

In this section, we shall give an extension of (13) to the case where Γ is allowed to have corners. By Lemma 2.1, the Riemann problem (9) may or may not be solvable and the following theorem identifies its solution when it is solvable.

Theorem 3.1. *Suppose $f(z)$ is the solution of the interior Riemann problem (9), then $f(z)$ is given by*

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma}\frac{\gamma(\omega)+i\mu(\omega)}{c(\omega)(\omega-z)}d\omega, \quad z \in \Omega^+, \tag{16}$$

whose boundary values is given by

$$f^+(t) = \frac{\gamma(t)+i\mu(t)}{c(t)}, \quad t \in \Gamma, \tag{17}$$

where $t \in \Gamma$, and

$$\mu(t) = \operatorname{Im}[c(t)f^+(t)]. \tag{18}$$

Furthermore,

$$\frac{\gamma(t)+i\mu(t)}{c(t)} = \frac{1}{\beta(t)i}\int_{\Gamma}\frac{\gamma(\omega)+i\mu(\omega)}{c(\omega)(\omega-t)}d\omega, \quad t \in \Gamma, \tag{19}$$

and $\mu(t)$ satisfies

$$\mu(t) - \frac{1}{\beta(t)}\int_{\Gamma}\mu(\omega)\operatorname{Im}\left[\frac{c(t)}{c(\omega)}\frac{d\omega}{\omega-t}\right] = \frac{-1}{\beta(t)}\int_{\Gamma}\gamma(\omega)\operatorname{Re}\left[\frac{c(t)}{c(\omega)}\frac{d\omega}{\omega-t}\right]. \tag{20}$$

Proof. If $u(t)$ satisfies (18) then

$$c(t)f^+(t) \quad = \quad \operatorname{Re}[c(t)f^+(t)] + i\operatorname{Im}[c(t)f^+(t)] = \gamma(t) + i\mu(t), \tag{21}$$

which implies that the boundary values $f^+(t)$ is given by (17) and in accordance with the Cauchy integral formula, $f(z)$ can be represented by (16).

Furthermore, $f(z) = 0$ when $z \in \Omega^-$ implying that $f^- = 0$. Hence by the Sokhotskyi formula (6) we have

$$0 = -\frac{\beta(t)}{2\pi}\frac{\gamma(t) + i\mu(t)}{c(t)} + \frac{1}{2\pi i}\int_\Gamma \frac{\gamma(\omega) + i\mu(\omega)}{c(\omega)(\omega - t)}d\omega, \quad t \in \Gamma. \tag{22}$$

This leads to (19). Now, multiplying both sides of (19) by $c(t)$ and taking its imaginary part gives

$$\mu(t) = \frac{-1}{\beta(t)}\int_\Gamma \gamma(\omega)\ \mathrm{Re}\left[\frac{c(t)}{c(\omega)}\frac{d\omega}{\omega - t}\right] + \frac{1}{\beta(t)}\int_\Gamma \mu(\omega)\ \mathrm{Im}\left[\frac{c(t)}{c(\omega)}\frac{d\omega}{\omega - t}\right], \tag{23}$$

the integral equation for the Riemann problem (9) derived previously in Munira et al [12] which can be rewritten as (20). □

The solvability and uniqueness of the integral equation (20) is established next.

3.1. Solvability and Uniqueness of Integral Equation

To define the solvability and uniqueness of the integral equation (20), we illustrate here that it is a generalization of (13). Consider our integral equation (20), where $\beta(t)$ is the interior angle. If Γ is smooth everywhere then $\beta(t)$ is always π. In this case it is obvious that the kernel of integral equation (20) is also the generalized Neumann kernel $N(c)(t, \omega)$ of (13) except that it is no longer compact. However, it may be written as a summation of a compact part and a bounded part so that the theory of second kind integral equations can be applied as was done for the Neumann kernel by Ratsfield [16] and Cryer [2].

What is left to do is to show that the right hand side of (20) is equivalent with the right hand side of (13). Applying the Sokhotskyi formula (6) to (15) and multiplying the result with $c(t)$, we have

$$c(t)L^-(t) = -\gamma(t) - i\frac{1}{\pi}\int_\Gamma \frac{c(t)}{c(\omega)}\frac{\gamma(\omega)}{\omega - t}d\omega. \tag{24}$$

Since γ is a real function then the imaginary part of (24) gives

$$\mathrm{Im}[c(t)L^-(t)] = -\frac{1}{\pi}\int_\Gamma \gamma(\omega)\ \mathrm{Re}\left[\frac{c(t)}{c(\omega)}\frac{d\omega}{\omega - t}\right], \tag{25}$$

which is equivalent to the right hand side of (20) when $\beta(t)$ is always π. And this equivalence everywhere except at the corners proves that (20) is a generalization of (13).

Here, we establish the solvability of the integral equation (20) based on Murid et al [14] and Wegmann et al [19] since we can ignore a finite number of corner points as described in Delves [4] and Stenger et al [17]. By requiring a minimal regularity assumptions on the functions and kernel involved in the integral equation to be in the function space $L^2[0, 2\pi]$, a finite number of corners are sets of measure zero that have no significance although we shall have to consider them during numerical integrations.

We now can classify equation (20) as a Fredholm integral equation of the second kind whose kernel is the generalized Neumann kernel $N(c)(t, \omega)$ and the function on the right-hand side of (20) is the forcing function while its parameter λ has value 1. According to Fredholm alternative theorem, if λ is a value that is not an eigenvalue of its kernel then the integral equation has unique solution but when it is an eigenvalue of its kernel then the integral equation is solvable only if the forcing function satisfies certain conditions Jerri [10].

The following Lemma 3.1 and Lemma 3.2 summarized some results obtained by Murid et al [14] and Wegmann et al [19], where the proofs for Γ being the unit disc and for Γ being a general smooth Jordan curve respectively can be found.

Lemma 3.1. *If $\chi > 0$ then $\lambda = 1$ is not an eigenvalue of N.*

Lemma 3.2. *If $\chi \leq 0$ then $\lambda = 1$ is an eigenvalue of N with $-2\chi + 1$ corresponding eigenfunctions.*

Theorem 3.2. *If $\chi > 0$, then integral equation (20) is uniquely solvable. If $\chi \leq 0$, then integral equation (20) is non-uniquely solvable.*

Proof. For the case $\chi > 0$, Lemma 3.1 implies that in accordance with the Fredholm alternative theorem, integral equation (20) is uniquely solvable. For the case $\chi \leq 0$, Lemma 3.2 implies that in accordance with the Fredholm alternative theorem, integral equation (20) is always solvable and its general solution contains $-2\chi + 1$ arbitrary constants; hence it is non-uniquely solvable. □

Theorem 3.1 states that if $f(z)$ given by (16) is a solution of the problem (9), then the real function $\mu(t)$ given by (18) is a solution of the integral equation (20). In the next section, we shall prove the converse, to show the relevance of the solution μ of the integral equation (20) for the problem (9). This enable us to establish equivalence.

3.2. Equivalence of the Interior Riemann Problem and

the Integral Equation

In this section we would like to prove the following theorem that if $\mu(t)$ is a solution of the integral equation (20), then the function $f(z)$ defined by (16) is a solution of the Riemann problem (9).

Theorem 3.3. *Suppose $\gamma(t)$ and $c(t)$ are specified by (9), and $\mu(t)$ is a real-valued function defined and satisfies Hölder condition for all $t \in \Gamma$. If $\mu(t)$ is a solution of the integral equation (20) then it satisfies (18) and $f(z)$ given by (16) is the solution of (9).*

Proof. Suppose the function $\mu(t)$ is a solution of the integral equation (20) and $f(z)$ be defined by (16) satisfying the Sokhotskyi formulas (5)-(6). By Sokhotskyi formula (5)

$$f^+(t) = \left(1 - \frac{\beta(t)}{2\pi}\right)\left(\frac{\gamma(t) + i\mu(t}{c(t)}\right) + \frac{1}{2\pi i}\int_\Gamma \frac{\gamma(\omega) + i\mu(\omega)}{c(\omega)}\frac{d\omega}{\omega - t}. \tag{26}$$

Multiplying both sides of (26) by $c(t)$ gives

$$c(t)f^+(t) = \left(1 - \frac{\beta(t)}{2\pi}\right)(\gamma(t) + i\mu(t)) + \frac{1}{2\pi i}\int_\Gamma (\gamma(\omega) + i\mu(\omega))\frac{c(t)}{c(\omega)}\frac{d\omega}{\omega - t}. \tag{27}$$

Rewriting $\frac{c(t)}{c(\omega)}\frac{d\omega}{\omega - t} = A + iB$ the integral equation (20) becomes

$$\mu(t) = -\frac{1}{\beta(t)}\int_\Gamma (\gamma(\omega)A - \mu(\omega)B) \Rightarrow \int_\Gamma (\gamma(\omega)A - \mu(\omega)B) = -\beta(t)\mu(t). \tag{28}$$

Equation (27) can also be written in terms of A and B as

$$\begin{aligned} c(t)f^+(t) &= \left(1 - \frac{\beta(t)}{2\pi}\right)(\gamma(t) + i\mu(t)) - \frac{i}{2\pi}\int_\Gamma (\gamma(\omega)A - \mu(\omega)B) + i(\mu(\omega)A \\ &\quad + \gamma(\omega)B) = \left(1 - \frac{\beta(t)}{2\pi}\right)(\gamma(t) + i\mu(t)) - \frac{i}{2\pi}\int_\Gamma (\gamma(\omega)A - \mu(\omega)B) \\ &\quad + \frac{1}{2\pi}\int_\Gamma (\mu(\omega)A + \gamma(\omega)B). \end{aligned} \tag{29}$$

Substituting (28) into (29) and simplifying gives

$$c(t)f^{+}(t) = \gamma(t) + i\mu(t) - \frac{\beta(t)}{2\pi}\gamma(t) + \frac{1}{2\pi}\int_{\Gamma}(\mu(\omega)A + \gamma(\omega)B). \qquad (30)$$

The imaginary part of (30) proves that $\mu(t)$ also satisfies

$$\mathrm{Im}[c(t)f^{+}(t)] = \mu(t). \qquad (31)$$

By formula (7), we have

$$f^{+}(t) - f^{-}(t) = \frac{\gamma(t) + i\mu(t)}{c(t)}, \qquad (32)$$

which upon multiplying both sides by $c(t)$ gives

$$c(t)f^{+}(t) - c(t)f^{-}(t) = \gamma(t) + i\mu(t). \qquad (33)$$

The imaginary part of (33) is

$$\mathrm{Im}[c(t)f^{+}(t) - c(t)f^{-}(t)] = \mu(t)\,, \qquad (34)$$

and due to (31) implies that the integral equation (20) is equivalent to the exterior Riemann problem

$$\mathrm{Im}[c(t)f^{-}(t)] = 0\,, \qquad (35)$$

which can be transformed into (12) by replacing $c(t)$ by $\tilde{c}(t) = ic(t)$. Equations (33) and (35) imply that the function $f(z)$ defined by (16) satisfies

$$\mathrm{Re}[c(t)f^{+}(t)] - c(t)f^{-}(t) = \gamma(t). \qquad (36)$$

If $\chi \leq 0$ then by Lemma 2.2, (35) has only the trivial solution hence $f^{-}(t) = 0$ and substitute this into (36) implies that $f(z)$ satisfies the problem (9) and its boundary values are given by (17).

However in the case of $\chi > 0$, by Lemma 2.1 the Riemann problem (9) may be unsolvable and it follows from Theorem 3.1 that the integral equation (20) has a solution when the problem is solvable, this implies that $f(z)$ defined by (16) satisfies (36) with $c(t)f^{-}(t) = 0$, hence $f(z)$ satisfies the Riemann problem (9) and its boundary values are given by (17). □

With this proof, we have established the equivalence of the Riemann problem (9) and integral equation (20).

Hence, by Lemma 2.1, and Theorems 3.1, 3.2 and 3.3, we can conclude that when the Riemann problem (9) is solvable and if $\chi > 0$, its solution $f(z)$ can always be calculated via the integral equation (20) and they are equivalently uniquely solvable. However if $\chi \leq 0$ the Riemann problem (9) and the integral equation (20) are equivalently non-uniquely solvable. In this paper, we will only solve the Riemann problem (9) for the case $\chi > 0$ when it is uniquely solvable.

4. Implementing Picard Iteration Method

Functional analysis allows us to apply Picard iteration method directly to integral equations Rall [15] and the numerical solution $\mu(t)$ of (20) may be obtained by

$$\mu_{n+1}(t) = \frac{-1}{\beta(t)} \int_\Gamma \gamma(\omega) \text{ Re}\left[\frac{c(t)}{c(\omega)}\frac{d\omega}{\omega - t}\right] + \frac{1}{\beta(t)} \int_\Gamma \mu_n(\omega) \text{ Im}\left[\frac{c(t)}{c(\omega)}\frac{d\omega}{\omega - t}\right]. \quad (37)$$

Our choice of Picard iteration method was based on a statement by Wegmann [20] that this method always converge for the Neumann kernel and the proof can be found in Gaier [5]. Our problem deals with an integration contour that has corner points, a difficulty that needed special treatment when seeking numerical solution as mentioned in Baker [1] and Kress [11] and in this paper, we illustrate through several numerical techniques that the method is successfully employed for the problem. In our examples, starting with $\mu_0(t) = 0$ the iterates $\mu_n(t)$ converges to solution $\mu(t)$.

In previous work, Munira et al [12] managed to eliminate singularities existing in (37) to facilitate numerical integration and we have done this by applying subtractions as was shown in Davis [3]. Letting $h(t) = 1/c(t)$, the iterative formula is

$$\begin{aligned}\mu_{n+1}(t) = &\frac{-1}{\beta(t)} \int_\Gamma [\gamma(\omega) - \gamma(t)] \text{ Re}\left[\frac{c(t)}{c(\omega)}\frac{d\omega}{\omega - t}\right] \\ &+ \frac{1}{\beta(t)} \int_\Gamma [\mu_n(\omega) - \mu_n(t)] \text{ Im}\left[\frac{c(t)}{c(\omega)}\frac{d\omega}{\omega - t}\right] - \frac{\gamma(t)}{\beta(t)} \int_\Gamma \text{ Re}\left[c(t)\frac{h(\omega) - h(t)}{\omega - t}d\omega\right] \\ &+ \mu_n(t) \cdot \left(\frac{1}{\beta(t)} \int_\Gamma \text{ Im}\left[c(t)\frac{h(\omega) - h(t)}{\omega - t}d\omega\right] + 1\right). \quad (38)\end{aligned}$$

Each integral in (38) is of the form $\int_\Gamma \mathcal{G}(t,\omega)d\omega$.

We anticipate difficulties in the numerical evaluations of the integration on contour with corners due to the non-existence of derivatives there, but according to Issacson et al [9] they are mere finite jumps and can be easily remedied by treating Γ as a piecewise smooth Jordan curve consists of a finite sequence of directed smooth curves $(\Gamma_1, \Gamma_2, \cdots, \Gamma_m)$. Then the contour integral of $\mathcal{G}(t,\omega)$ can be represented by

$$\int_{\Gamma} \mathcal{G}(t,\omega)d\omega = \sum_{k=1}^{m} \int_{\Gamma_k} \mathcal{G}(t,\omega)d\omega. \tag{39}$$

Suppose Γ_k is parametrized by $\Gamma_k : \omega = \omega(s), \tau_k < s < \tau_{k+1}$. Therefore,

$$\int_{\Gamma_k} \mathcal{G}(t,\omega)d\omega = \int_{\tau_k}^{\tau_{k+1}} \mathcal{G}(r,s)\omega'(s)ds \quad (\text{for } k = 1, 2, \cdots, m)$$

where $\omega'(s)$ exist at the endpoints as one-sided limits. This integral is restricted to the k-th subinterval $[\tau_k, \tau_{k+1}]$ for Γ_k, where the integrand $\mathcal{G}(r,s)$ is smooth in $[\tau_k, \tau_{k+1}]$ and the corner points are now the endpoints $s = \tau_k$ and $s = \tau_{k+1}$ defining the interval $\tau_k < s < \tau_{k+1}$. Thus

$$\int_{\Gamma} \mathcal{G}(t,\omega)d\omega = \sum_{k=1}^{m} \int_{\tau_k}^{\tau_{k+1}} \mathcal{G}(r,s)\omega'(s)ds. \tag{40}$$

Applying a suitable quadrature rule involving $n+1$ points s_j with corresponding weights w_j we have

$$\sum_{k=1}^{m} \int_{\tau_k}^{\tau_{k+1}} \mathcal{G}(r,s)\omega'(s)ds \approx \sum_{k=1}^{m} \sum_{j=1}^{n+1} w_j G(r,s_j),$$

where $G(r,s) = \mathcal{G}(r,s)\omega'(s)$. Applying this to (38) we have for every $r = r_i, i = 1, 2, \ldots, n+1$, the system of $n+1$ linear algebraic equations

$$\begin{aligned} \mu_{n+1}(r_i) = &-\frac{1}{\beta(r_i)} \sum_{k=1}^{m} \sum_{j=1}^{n+1} w_j A(r_i, s_j) + \frac{1}{\beta(r_i)} \sum_{k=1}^{m} \sum_{j=1}^{n+1} w_j B(r_i, s_j) \\ &- \frac{\gamma(r)}{\beta(r_i)} \sum_{k=1}^{m} \sum_{j=1}^{n+1} w_j C(r_i, s_j) + \frac{\mu_n(r_i)}{\beta(r_i)} \sum_{k=1}^{m} \sum_{j=1}^{n+1} w_j D(r_i, s_j) + \mu_n(r_i), \end{aligned} \tag{41}$$

where:

$$\begin{aligned}
A(r_i, s_j) &= [\gamma(s_j) - \gamma(r_i)] \text{ Re}\left[\frac{c(r_i)}{c(s_j)} \frac{\omega'(s_j)}{(\omega(s_j) - t(r_i))}\right], \\
B(r_i, s_j) &= [\mu_n(s_j) - \mu_n(r_i)] \text{ Im}\left[\frac{c(r_i)}{c(s_j)} \frac{\omega'(s_j)}{(\omega(s_j) - t(r_i))}\right], \\
C(r_i, s_j) &= \text{ Re}\left[c(r_i) \frac{h(s_j) - h(r_i)}{(\omega(s_j) - t(r_i))} \omega'(s_j)\right], \\
D(r_i, s_j) &= \text{ Im}\left[c(r_i) \frac{h(s_j) - h(r_i)}{(\omega(s_j) - t(r_i))} \omega'(s_j)\right],
\end{aligned} \tag{42}$$

when $r_i \neq s_j$, but when $r_i = s_j$ we have

$$\begin{aligned}
&A(r_i, s_j) = \gamma'(s_j),\ B(r_i, s_j) = 0,\ C(r_i, s_j) = \text{ Re}\left[c(r_i) h'(s_j)\right], \\
&D(r_i, s_j) = \text{ Im}\left[c(r_i) h'(s_j)\right].
\end{aligned} \tag{43}$$

Swarztrauber [18] used Simpson's rule in his work and the advantage of this method is that it provide solutions on the boundary that include the corner points. Here we investigated Simpson's rule for the Riemann problem in the evaluation of (41) and found that results obtained are not always accurate.

In this paper, numerical examples reveal that Gaussian quadrature always gives results that are accurate up to at least fourteen decimal places for solutions at off-corner points, however it does not provide solutions at the corners. We will present in Section 5 to overcome this problem. By this technique we are able to maintain excellent accuracy of solutions everywhere on the boundary including the corners.

Rather then taking equally spaced points for approximating an integral, Gaussian quadrature is concerned with choosing the points for evaluation in an optimal manner that leads to increase accuracy of approximation. It present a procedure for choosing the values of the Gauss abscissas and weights, $\hat{v}_j, \hat{w}_j,\ j = 1, \cdots, n+1$ to perform the approximation for the integral

$$\int_{\tau_k}^{\tau_{k+1}} G(r, s) ds \approx \sum_{j=1}^{n+1} w_j G(r_i, s_j),$$

where

$$s_j = \frac{(\tau_{k+1} - \tau_k)\hat{v}_j + \tau_{k+1} + \tau_k}{2}, \quad w_j = \frac{\tau_{k+1} - \tau_k}{2}\ \hat{w}_j.$$

These are obtained by a simple linear transformation

$$\hat{v}_j = \frac{1}{\tau_{k+1} - \tau_k}(2s - \tau_k - \tau_{k+1}), \quad ds = \frac{\tau_{k+1} - \tau_k}{2} d\hat{v}_j,$$

so that the interval of integration is translated from $[\tau_k, \tau_{k+1}]$ to $[-1, 1]$ in order to use the Legendre polynomials for the abscissas and weights $\hat{v}_j, \hat{w}_j$. Davies [3] provided a subroutine *GRULE* to compute these values at $n+1$ points.

We apply this procedure to (41) when we solve Examples 1-4 in Section 6 and obtained solutions at off-corner points. The difficulty of finding solutions at the corners after this is resolved next.

5. Iterative Formula for Interpolating Solution at a Corner

We formulate a new iterative technique that will interpolate solution at a corner using the off-corner solutions obtained through formula (41) with Gaussian quadrature. According to Theorem 3.1, solutions at the boundary $f^+(t)$ are given by equation (17) and should satisfy (19). Letting $\frac{\gamma(t)+i\mu(t)}{c(t)}$ in (19) be $g(t)$ which represent solution at the boundary $f^+(t)$ and let t_* be a corner point while $\beta(t_*)$ be the interior angle at t_*, we have

$$g(t_*) = \frac{1}{\beta(t_*)i} \int_\Gamma \frac{g(\omega)}{(\omega - t_*)} d\omega. \tag{44}$$

Results obtained when we use (44) to interpolate solution at a corner using solutions we obtained for off-corner points improve dramatically when we introduce an iterative technique defined as

$$g_{n+1}(t_*) = \frac{1}{\beta(t_*)i} \int_\Gamma \frac{g_n(\omega) - g_n(t_*)}{(\omega - t_*)} d\omega + g_n(t_*). \tag{45}$$

In practice, (45) converges rapidly as shown in the results presented in Table 1 of Section 6. Formula (45) is constructed from (44) by the following subtraction

$$g_{n+1}(t_*) = \frac{1}{\beta(t_*)i} \int_\Gamma \frac{g_n(\omega) - g_n(t_*)}{(\omega - t_*)} d\omega + g_n(t_*) \frac{1}{\beta(t_*)i} \int_\Gamma \frac{1}{(\omega - t_*)} d\omega$$

and it is due to (3) that we arrive at (45).

If we denote $M(t_*, \omega) = \frac{g_n(\omega)-g_n(t_*)}{(\omega - t_*)}$, then (45) becomes

$$\begin{aligned} g_{n+1}(t_*) &= \frac{1}{\beta(t_*)i} \int_\Gamma M(t_*, \omega) d\omega + g_n(t_*) \\ &= \frac{1}{\beta(t_*)i} \sum_{k=1}^{m} \int_{\Gamma_k} M(t_*, \omega) d\omega + g_n(t_*). \end{aligned}$$

We then apply the Gaussian quadrature rule to each of the integrals $\int_{\Gamma_k}$ as describe earlier.

6. Numerical Examples

In this section, we present examples of Riemann problems whose boundaries include one, two, three or four corners as shown in Figure 1(a), 1(b), 2(a) and 2(b) respectively. For these figures, the respective contours $\Gamma^1, \Gamma^2, \Gamma^3$ and Γ^4 are given by $t(s) = x(s) + iy(s)$ such that:

$$\Gamma^1 \quad : \quad t(s) = 1 - 2\sin\frac{s}{2} + i\sin s, \quad 0 \le s \le 2\pi\,,$$

$$\Gamma^2 \quad : \quad t(s) = \begin{cases} \frac{\sqrt{10}}{5}\cos s + \frac{3\sqrt{10}}{20}\sin s + i\frac{\sqrt{10}}{4}\sin s & 0 \le s \le \pi\,, \\ \frac{\sqrt{10}}{5}\cos s - \frac{3\sqrt{10}}{20}\sin s + i\frac{\sqrt{10}}{4}\sin s, & \pi \le s \le 2\pi. \end{cases},$$

$$\Gamma^3 \quad : \quad t(s) = \begin{cases} 2 - 2s + (4s-2)i, & 0 \le s \le 1\,, \\ 2 - 2s + (6-4s)i, & 1 \le s \le 2\,, \\ 4s - 10 - 2i, & 3 \le s \le 4\,, \end{cases}$$

$$\Gamma^4 \quad : \quad t(s) = \begin{cases} 1 + 3\sin 2s - i\cos 2s, & 0 \le s \le \frac{\pi}{4}\,, \\ -\cos 2s + i(1 - 3\sin 2s), & \frac{\pi}{4} \le s \le \frac{\pi}{2}\,, \\ -1 - 3\sin 2s + i\cos 2s, & \frac{\pi}{2} \le s \le \frac{3\pi}{4}\,, \\ -1 + 3\sin 2s + i\cos 2s, & \frac{3\pi}{4} \le s \le 2\pi\,. \end{cases}$$

When investigating the Simpson's rule, results yield are excellent when the analytic functions that we are seeking for the unique interior Riemann problem on region with corners are complex constants but not when they are non-constant analytic functions as for Examples 6.1-6.4 of Section 6 whose boundaries are given by Γ^1-Γ^4 respectively. Results were poor when we compare our numerical solutions f_n of formula (41) using Simpson's rule with their exact solutions f by their maximum error norm denoted by $\| f_n - f \|_\infty$ at the n-th iteration. The maximum error norm for Examples 6.1-6.4 were only $3.2e^{-}2, 1.0e^{-}3, 2.9e^{-}1, 6.2e^{-}1$ respectively even after the 100-th iterations. Not satisfied with these results, we present results obtained using Gaussian quadrature to (41) for solutions at off-corner points and then using these values in formula (45) from Section 5, we interpolate solutions for each corner points.

Example 6.1. The contour for the boundary is given by $t(s) = x(s)+iy(s)$ of Γ^1 and has a corner as shown in Figure 1, $c(t(s)) = t(s)$, hence $\chi = 1$ and

$$\gamma(t(s)) = ((x(s))^2 - (y(s))^2)x - 2xy^2.$$

The Riemann problem on region with corners has the unique solution $f(z) = z^2$.

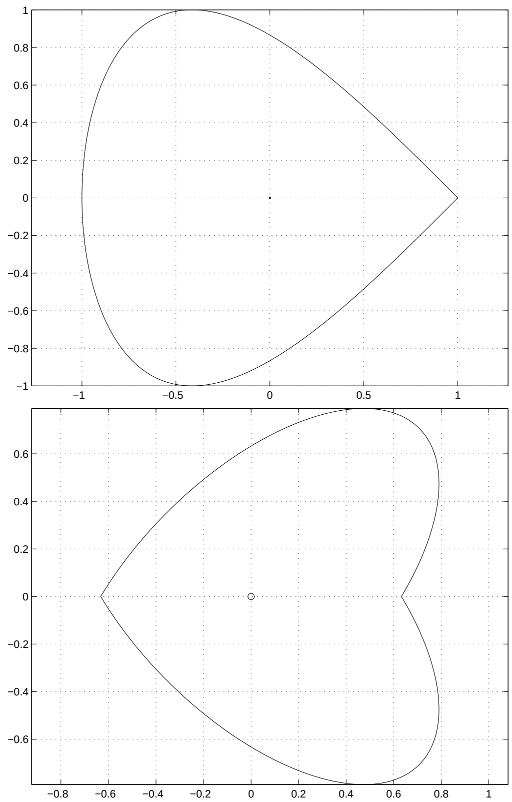

Figure 2: (a) $\Gamma^1 : t(s)$ with one corner; (b) $\Gamma^2 : t(s)$ with two corners

Example 6.2. The contour for the boundary is given by $t(s) = x(s)+iy(s)$ of Γ^2 and has a corner as shown in Figure 2, $c(t(s)) = t(s)$, hence $\chi = 1$ and

$$\gamma(t(s)) = y(s)\cos(x(s))(e^{-y(s)} - e^{y(s)})/2 + x(s)\sin(x(s))(e^{-y(s)} + e^{y(s)})/2.$$

The Riemann problem on region with corners has the unique solution $f(z) = sinz$.

Example 6.3. The contour for the boundary is given by $t(s) = x(s)+iy(s)$ of Γ^3 and has a corner as shown in Figure 3, $c(t(s)) = t(s)$, hence $\chi = 1$ and

$$\gamma(t(s)) = e^x(s)[(2x(s) - y(s))\cos(y(s)) - (x(s) + 2y(s))\sin(y(s))].$$

The Riemann problem on region with corners has the unique solution $f(z) = (2+i)e^z$.

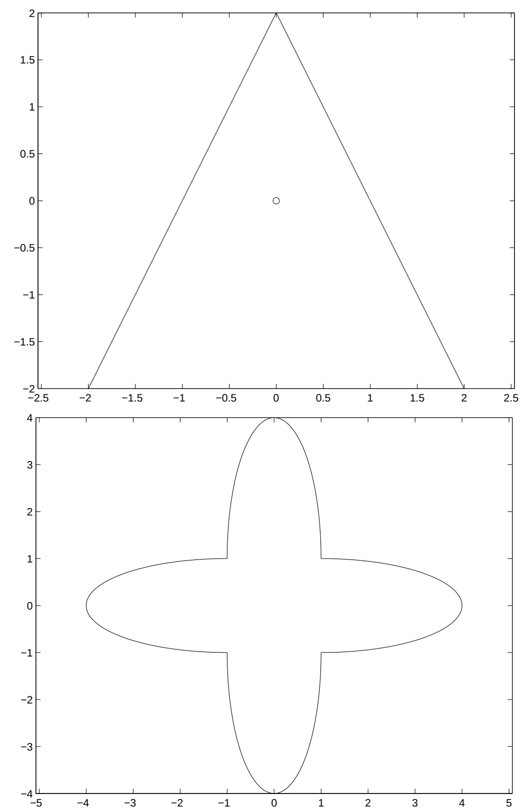

Figure 3: (a) $\Gamma^3 : t(s)$ with three corners; (b) $\Gamma^4 : t(s)$ with four corners

Example 6.4. The contour for the boundary is given by $t(s) = x(s)+iy(s)$ of Γ^4 and has a corner as shown in Figure 3, $c(t(s)) = t(s)$, hence $\chi = 1$ and

$$\gamma(t(s)) = 3((x(s))^2 - (y(s))^2) - 10x(s)y(s).$$

The Riemann problem on region with corners has the unique solution $f(z) = (3 + 5i)z$.

Respective results for Examples 6.1-6.4 presented in Table 1 show the maximum error norms for off-corner points followed by the maximum error norm at each corner. We also show in brackets, the approximations using cubic spline interpolation provided by Matlab at each corner for comparison.

The results reveals that our method can provide solutions on the boundary including the corner points that are very accurate and correct up to at least fourteen decimal places for the interior Riemann problem on region with corners.

FORMULA (41) WITH GAUSSIAN QUADRATURE TABLE 1: $\parallel f - f_n \parallel_\infty$ for Example 6.1-6.4				
n	Example 6.1: Γ^1 ,$f(z) = z^2$	Example 6.2: Γ^2 $f(z) = sinz$	Example 6.3: Γ^3 $f(z) = (2+i)e^z$	Example 6.4: Γ^4 $f(z) = (3+5i)z$
10	$2.4e-05$	$1.2e-05$	$4.7e-02$	$1.9e-01$
20	$1.1e-08$	$1.2e-09$	$4.7e-04$	$3.4e-03$
30	$4.9e-12$	$1.2e-13$	$8.1e-06$	$7.0e-05$
40	$2.6e-15$	$2.5e-15$	$2.0e-07$	$1.5e-06$
50	$1.6e-15$	$2.3e-15$	$5.6e-09$	$3.2e-08$
60		$2.3e-15$	$1.6e-10$	$6.9e-10$
70			$5.1e-12$	$1.4e-11$
80			$1.6e-13$	$3.9e-13$
90			$4.5e-14$	$8.6e-14$
$\parallel f - f_n \parallel_\infty$ for Corner 1				
30 (Spline Approx)	$1.7e-15$ $(2.5e-10)$	$1.1e-16$ $(6.1e-12)$	$3.6e-14$ $(9.1e-11)$	$1.7e-14$ $(7.8e-12)$
$\parallel f - f_n \parallel_\infty$ for Corner 2				
30 (*Spline Approx*)		$8.4e-16$ $(1.2e-05)$	$1.1e-14$ $(4.7e-02)$	$1.5e-14$ $(1.9e-01)$
$\parallel f - f_n \parallel_\infty$ for Corner 3				
30 (*Spline Approx*)			$4.0e-15$ $(1.1e-04)$	$3.5e-14$ $(1.6e+01)$
$\parallel f - f_n \parallel_\infty$ for Corner 4				
30 (*Spline Approx*)				$3.5e-14$ $(4.3e-03)$

Table 1:

In practice we take $\mu_0(t) = 0$ as the initial value to begin iterations. Convergence to the solution $\mu(t)$ may be attained monotonically or alternatingly. In the case of alternating convergence to the solution, the process can sometimes be observed to be very slow. Example 6.3 is of this case and with a speeding up process using midpoint value at every 10-th iteration proves helpful in attaining convergence.

7. Conclusion

This paper offers an effective technique for seeking numerical solutions of the interior Riemann problem on region with corners by Picard iteration method. In this paper, numerical results in Table 1 reveal that Gaussian quadrature gives results that are accurate up to least fourteen decimal places for solutions at off-corner points, but however it does not provide solutions at the corners. This is because the corner points are the end-points interval of an integral which the rule avoids since they are not considered as "optimum points" for evaluation of the integral.

Here, we have overcome the problem of finding solutions at the corners by providing a new iterative technique that uses solution values at off-corner points to interpolate solutions at the corners. By this technique we are able to maintain the same accuracy throughout the boundary including the corners as shown in Table 1. We also presented in Table 1, results at the corners by cubic spline interpolation and we conclude that our technique gives more accurate solutions. Here, we presented proofs of the solvability and uniqueness of the integral equation and established its equivalence to the problem.

Acknowledgements

This work was supported in part by the Research Management Center, University of Technology of Malaysia, Project Vote: 75150.

We also would like to acknowledge the contribution of Dr Mohamed M. S. Nasser for his ideas and assistance throughout the research.

References

[1] C.T.H. Baker, *The Numerical Treatment of Integral Equations*, Clarendon Press, Oxford (1977).

[2] C. Cryer, *Numerical Functional Analysis*, Pws-Kent Publishing Company, Boston (1989).

[3] P.J. Davis, P. Rabinowitz, *Methods of Numerical Integration*, Academic Press, Inc., Orlando (1984).

[4] L.M. Delves, J.L. Mohamed, *Computational Methods for Integral Equations*, Cambridge University Press, Cambridge (1985).

[5] D. Gaier, Konstruktive Methoden der konformen Abbildung, *Springer Tracts in Natural Philosophy Ergebnisse der Angewandten Mathematik*, Springer-Verlag, Berlin, **3** (1964).

[6] F.D. Gakhov, *Boundary Value Problem* (Ed. I.N. Sneddon), English translation of Russian Edition (1963), Pergamon Press, Oxford (1966).

[7] E. Hille, *Analytic Function Theory*, Blaisdell, New York (1959).

[8] P. Henrici, *Applied and Computational Complex Analysis*, John Wiley, New York (1986).

[9] E. Issacson, H.B. Keller, *Analysis of Numerical Methods*, Wiley, New York (1966).

[10] A.J. Jerri, *Introduction to Integral Equations with Applications*, Marcel Dekker,Inc., New York and Basel (1985).

[11] R.A. Kress, Nystrom Method for boundary integral equations in domains with corners, *Numerische Mathematik*, Springer-Verlag, **58** (1990), 145-161.

[12] I. Munira, A.H.M. Murid, B. Sanugi, Integral equation approach for the numerical solution of Riemann problem for region with corners, *Int. J. Simulation and Process Modelling*, **2**, No. 1,2 (2006), 25-32.

[13] A.H.M. Murid, M.R.M. Razali, M.M.S. Nasser, Solving Riemann problem using Fredholm integral equation of the second kind, *Proceeding of Simposium Kebangsaan Sains Matematik Ke-10,* Universiti Teknologi Malaysia, Malaysia (2002), 171-178.

[14] A.H.M. Murid, M.R.M. Razali, Nasser Eigenproblem of the Generalized Neumann Kernel, *Bulletin of the Malaysian Mathematical Sciences Society*, Second Series, **26** (2003), 13-33.

[15] L.B. Rall, Theory of nonlinear integral equations, In: *Numerical Solution of Integral Equations*, Clarendon Press, Oxford (1974), 207-222.

[16] A. Ratsfield, Iterative solution of linear systems arising from the Nystrom method for the double layer potential equation over curves with corners, *Mathematical Mehtods in the Applied Sciences*, John Wiley, **16** (1993), 443-455.

[17] F. Stenger, R. Schmidtlein, Conformal maps via sinc methods, *World Scientific Series in Approximation and Decompositions*, October, 13-17 (1997), 505-552.

[18] P.N. Swarztrauber, On the numerical solution of the Dirichlet problem for a region of general shape, *SIAM Journal of Numerical Analysis*, **9**, No. 2 (June 1972), 300-306.

[19] R. Wegmann, A.H.M. Murid, M.M.S. Nasser, The Riemann-Hilbert problem and the generalized Neumann kernel, *J. Comp. Appl. Math*, **182** (2005), 388-415.

[20] R. Wegmann, Methods for numerical conformal mapping, *Handbook of Complex Analysis: Geometric Function Theory*, Elsevier, Chapter 9, **2** (2005), 353-432.