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A PERISHABLE INVENTORY SYSTEM WITH SERVICE FACILITIES AND BATCH MARKOVIAN DEMANDS

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Abstract: This article considers a continuous review $(s,\ S)$ inventory system at a service facility in which the waiting hall for customers is of fixed size. We assume batch arrivals of customers. The time points of arrivals of each batch is assumed to form a Poisson process and the number of customers in each batch is a random variable. Each customer undergoes a random time of service at the end of which he/she is issued their order. The service time follows a state-dependent negative exponential distribution. The items of inventory are perishable in nature with exponential life time. It is also assumed that lead time for the reorders is assumed to be distributed as exponential independent of the service time distribution. The joint probability distribution of the number of customers in the system and the inventory level is obtained in both transient and steady state cases. The measures of system performance in the steady state are derived. The results are illustrated with numerical examples.

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1. Introduction

In most of the inventory models considered in the literature, demanded items are directly delivered from the stock (if available). The demands occurring during the stock-out period are either lost (lost sales) or satisfied only after arrival of ordered items (backlogging). In the latter case, it is assumed that either all (full backlogging) or any prefixed number of demands (partial backlogging) occurring during the stock-out period are satisfied. See Nahmias [14], Raafat [16], Kalpakam and Arivarignan [11, 12], Elango and Arivarignan [8], Liu and Yang [13] and Yadavalli et al [19] for a review.

However, in the case of inventories maintained with service facilities, a demanded item is delivered to the customer after some service time. In this case the items are delivered *not* at the time of a demand but after a random time of service causing the formation of queues. This policy necessitates the study of both the inventory level and queue length (joint) distributions.

Recently Berman et al [3] have considered an inventory management system with a service facility using one item of inventory for each service. They assumed that both the demand and service are deterministic. Queues can occur only during stock-outs. They determined the optimal order quantity that minimizes the total cost rate.

Berman and Kim [5] analyzed a similar problem in a stochastic environment where arrival time points of customers form a Poisson process and the service times are exponentially distributed. A logically related model has been studied by He et al [10], who analyzed a Markovian inventory – production system, where customer demands are processed by a single machine in a batch of size one. Berman and Sapna [6] studied an inventory control problem at a service facility requiring one item of the inventory. They assumed Poisson arrivals, arbitrarily distributed service times and zero lead times. They analyzed the system with finite waiting room. Under a specified cost structure, the optimal ordering quantity that minimizes the long-run expected cost rate has been derived.

Elango [7] has considered a Markovian inventory system with instantaneous supply of reorders at a service facility. The service time is assumed to have an exponential distribution with parameter depending on the number of waiting customers. Arivarignan et al [1] have extended this model to include exponential lead time. Perumal and Arivarignan [15] have considered a Markovian inventory system with infinite waiting room. Arivarignan and Sivakumar [2] have considered an inventory system with arbitrarily distributed demand, exponential service time and exponential lead time.

In this paper we have considered a perishable inventory management system at a service facility. We assume that the arrival time points of a batch of customers form a Poisson process with parameter λ (> 0) and that the number of customers arriving at an arrival instant is a random variable Y with probability function $p_k = \Pr\{Y = k\}$, $k = 1, 2, 3, \ldots$. Individual customer demands single item and it is delivered after completing service. The service times are assumed to be distributed as negative exponential with rate depending on the number of customers in the system. The maximum capacity of the inventory is fixed as S and the waiting space is limited to accommodate a maximum of N customers including one at service point. The life time of each item is assumed to have negative exponential distribution with rate γ (> 0). The ordering policy is to place an order as and when the inventory level drops to a prefixed level s ($0 < s < S$) and it is assumed that the lead time is distributed as exponential with parameter β (> 0).

This paper is organized as follows. Section 2 presents the assumptions and notations followed in the rest of the paper. The transient and steady state analysis of the joint probability distribution for the inventory level and the number of customers in the system are presented in Section 3. In Section 4, the different measures of system performance in steady state are derived and the total expected cost rate is calculated in Section 5. Section 6 presents the cost analysis of the model using numerical examples.

2. Model Description

Consider a service facility in which perishable items are stocked and the items are delivered to the demanding customers. The demand is for single item per customer. The maximum capacity of the inventory is S and the maximum number of customers permitted in the waiting space, including the one in the service, is N. The following assumptions are made:

— The arrival times of batches of customers form a Poisson process with parameter λ and the number of customers arriving in any batch is a random variable Y with probability function $p_k = \Pr\{\ Y = k\}$, $k = 1, 2, \ldots$; $p_k \geq 0$ and $\sum p_k = 1$.

— After the stock is depleted, any arriving customer is permitted to enter as long as the number of customers in the system does not exceed N.

— If a batch has more customers than the available space, the customers will be allowed according to the available space and other customers are turned down.

— Life time of each item has negative exponential distribution with parameter γ (> 0).

— The service time for each customer follows a negative exponential distribution with parameter μ_n (> 0), where n denotes the number of customers in the system $(0 < n \le N)$.

— An (s, S) ordering policy is adopted with positive lead time. The ordering quantity is Q $(= S - s > s + 1)$. The requirement $Q > s + 1$, ensures that after a replenishment the inventory level will be always above the reorder level. Otherwise it may not be possible to place reorder which leads to perpetual shortage.

— The lead time is exponentially distributed with parameter β (> 0) which is independent of the distribution of service.

Notations.

— A^*_α- Laplace transform of any arbitrary matrix $A(t)$, for $Re\ \alpha > 0$.
— $E_1 = \{0, 1, 2, \ldots, S\}$.
— $E_2 = \{0, 1, 2, \ldots, N\}$.
— $E = E_1 \times E_2$.
— $\mathbf{0}$ = Zero matrix.
— $\boldsymbol{e} = (1, 1, 1, \ldots, 1)'$.
— $\boldsymbol{I}$ = Identity matrix.

3. Analysis

Let $L(t)$ denote the inventory level and $X(t)$ denote the number of customers (waiting and being served) in the system, at time $t+$. From the assumptions made on the input and output processes, it can be shown that $(L, X) = \{(L(t), X(t)); t \ge 0\}$, on the state space E, is a Markov process. The infinitesimal generator of this process,

$$A = ((\ a((i, q), (j, r))\ )), \quad (i, q), (j, r) \in E$$

can be obtained by using the following arguments:

— The arrival of a batch of k customers make a transition from (i, q) to $(i, q + k)$ $k = 1, 2, \ldots, N - q - 1$ with intensity of transition λp_k, $i = 0, 1, 2, \ldots, S$, $q = 0, 1, 2, \ldots, N-2$. When $k \ge N-q$ the state (i, q) goes to (i, N) with intensity of transition $\lambda p'_q$, where $p'_q = \sum\limits_{m=N-q}^{\infty} p_m$, $i = 0, 1, 2, \ldots, S$; $q = 0, 1, 2, \ldots, N - 1$.

— The completion of service makes one customer leave the system and decrease the inventory level by 1. Thus a transition takes place from (i, q) to $(i-1, q-1)$ with intensity of transition μ_q, $i = 1, 2, \ldots, S,\ q = 1, 2, \ldots, N$.

— A transition from (i, q) to $(i-1, q)$ will take place when any one of i items fails at a rate of γ; thus intensity of transition is $i\gamma$, $i = 1, 2, \ldots, S,\ q = 0, 1, 2, \ldots, N$.

— A transition from (i, q) to $(i+Q, q)$, for $i = 0, 1, 2, \ldots, s$; $q = 0, 1, 2, \ldots, N$ will take place with the intensity of transition β when a replenishment for Q items occurs.

Hence we get $a((i, q), (j, r)) =$

$$\begin{cases}
\lambda p_k & j = i; & r = q+k;\quad k = 1, \ldots, N-q-1, \\
 & i = 0, 1, 2, \ldots, S; & q = 0, 1, 2, \ldots, N-1, \\
\lambda p'_q & j = i; & r = N, \\
 & i = 0, 1, 2, \ldots, S; & q = 0, 1, 2, \ldots, N-2, N-1, \\
i\gamma & j = i-1; & r = q, \\
 & i = 1, 2, \ldots, S; & q = 0, 1, 2, \ldots, N, \\
\mu_q & j = i-1; & r = q-1, \\
 & i = 1, 2, \ldots, S; & q = 1, 2, \ldots, N, \\
-(\lambda + i\gamma) & j = i; & r = q, \\
 & i = s+1, s+2, \ldots, S; & q = 0, \\
-(\mu_q + i\gamma) & j = i; & r = q, \\
 & i = s+1, s+1, \ldots, S; & q = N, \\
-(\beta + \lambda + i\gamma) & j = i; & r = q, \\
 & i = 1, 2, \ldots, s; & q = 0, \\
-(\beta + \mu_q + i\gamma) & j = i; & r = q, \\
 & i = 1, 2, \ldots, s; & q = N \\
-(\lambda + \mu_q + i\gamma) & j = i; & r = q, \\
 & i = s+1, s+2, \ldots, S; & q = 1, 2, \ldots, N-1, \\
-(\beta + \lambda + \mu_q + i\gamma) & j = i; & r = q, \\
 & i = 1, 2, \ldots, s; & q = 1, 2, \ldots, N-1, \\
-(\lambda + \beta) & j = i; & r = q, \\
 & i = 0; & q = 0, 1, 2, \ldots, N-1, \\
-\beta & j = i; & r = q, \\
 & i = 0; & q = N, \\
\beta & j = i+Q; & r = q, \\
 & i = 0, 1, 2, \ldots, s; & q = 0, 1, 2, \ldots, N, \\
0 & \text{otherwise}, &
\end{cases}$$

where $p'_q = \sum_{m=N-q}^{\infty} p_m$.

Define

$$A_{ij} = ((a((i, q), (j, r))))_{q, r \in E_2}, \quad i, j \in E_1.$$

The infinitesimal generator A can be written in terms of sub matrices A_{ij}, namely

$$A = ((A_{ij}))$$

where

$$A_{ij} = \begin{cases} A_i & j = i,\ \ i = 0, 1, \ldots, S\,, \\ B_i & j = i - 1,\ i = 1, 2, \ldots, S\,, \\ C & j = i + Q,\ i = 0, 1, \ldots, s\,, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

The sub matrices are given by,:

$$A_i = \begin{pmatrix} -d_i & \lambda p_1 & \lambda p_2 & \lambda p_3 & \cdots & \lambda p_{N-1} & \lambda p'_0 \\ 0 & -(d_i + \mu_1) & \lambda p_1 & \lambda p_2 & \cdots & \lambda p_{N-2} & \lambda p'_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -(d_i + \mu_{N-1}) & \lambda p'_{N-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & -(i\gamma + \mu_N + \beta) \end{pmatrix},$$

with $d_i = (\lambda + i\gamma + \beta)$, $i = 1, 2, \ldots, s$,

$$A_i = \begin{pmatrix} -g_i & \lambda p_1 & \lambda p_2 & \lambda p_3 & \cdots & \lambda p_{N-1} & \lambda p'_0 \\ 0 & -(g_i + \mu_1) & \lambda p_1 & \lambda p_2 & \cdots & \lambda p_{N-2} & \lambda p'_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -(g_i + \mu_{N-1}) & \lambda p'_{N-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & -(i\gamma + \mu_N) \end{pmatrix}$$

with $g_i = (\lambda + i\gamma)$, $i = s + 1, s + 2, \ldots, S$,

$$A_0 = \begin{pmatrix} -(\beta + \lambda) & \lambda p_1 & \lambda p_2 & \lambda p_3 & \cdots & \lambda p_{N-1} & \lambda p'_0 \\ 0 & -(\beta + \lambda) & \lambda p_1 & \lambda p_2 & \cdots & \lambda p_{N-2} & \lambda p'_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -(\beta + \lambda) & \lambda p'_{N-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\beta \end{pmatrix},$$

$$B_i = \begin{pmatrix} i\gamma & 0 & \cdots & \cdots & 0 & 0 \\ \mu_1 & i\gamma & \cdots & \cdots & & \cdots \\ 0 & \mu_2 & i\gamma & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & & \mu_{N-1} & i\gamma & 0 \\ 0 & \cdots & & \cdots & \mu_N & i\gamma \end{pmatrix}, \quad i = 1, 2, \ldots, S-1, S,$$

and

$$C = \begin{pmatrix} \beta & 0 & 0 & \cdots & \cdots & 0 \\ 0 & \beta & 0 & 0 & \cdots & 0 \\ \vdots & 0 & \beta & 0 & \cdots & \cdots \\ \vdots & \vdots & \vdots & & \vdots & 0 \\ \cdots & \cdots & \cdots & 0 & \beta & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & \beta \end{pmatrix}.$$

3.1. Transient Analysis

Define

$$\phi((i,q),(j,r);t) = \Pr[L(t) = j, X(t) = r | L(0) = i, X(0) = q], \qquad (i,q),(j,r) \in E.$$

Let $\phi_{i,j}(t)$ denote a matrix whose $(q,r)^{\text{th}}$ element is $\phi((i,q),(j,r);t)$ and $\Phi(t)$ denote a block partitioned matrix with the sub matrix $\phi_{i,j}(t)$ at (i,j)-th position. The Kolmogorov's differential equation can be written as

$$\Phi'(t) = \Phi(t)A,$$

the solution of which is given by

$$\Phi(t) = e^{At},$$

where e^{At} represents

$$I + \frac{At}{1!} + \frac{A^2t^2}{2!} + \cdots .$$

Alternatively, we have

$$\Phi^*_\alpha = (\alpha I - A)^{-1}, \quad \text{where} \quad \Phi^*_\alpha = ((\ \phi^*_\alpha(i,j)\ )),$$

$$\phi^*_\alpha(i,j) = ((\ \phi^*_\alpha((i,q)(j,r))\ ))_{q,r\in E_2}, \quad \text{and}$$

$$\phi^*_\alpha(i,q),(j,r)) = \int_0^\infty e^{-\alpha t}\phi((i,q),(j,r);t)dt\,.$$

The matrix $(\alpha I - A)$ has the following block partitioned form

$$(\alpha I - A) = \begin{array}{c} S \\ S-1 \\ \cdots \\ s \\ s-1 \\ \cdots \\ 1 \\ 0 \end{array} \begin{pmatrix} D_S & -B_S & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & D_{S-1} & -B_{S-1} & \cdots & \mathbf{0} & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -C & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -C & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & D_1 & -B_1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & D_0 \end{pmatrix},$$

where

$$D_i = \ \alpha I - A_i, \quad i = 0, 1, \ldots, S$$

(note that we have numbered the rows and columns in the decreasing order of magnitude).

It may be observed that $(\alpha I - A)$ is an almost lower triangular matrix in block partitioned form. That is, if we denote the (i,j)-th sub matrix of $(\alpha I - A)$ by P_{ij}, then we have

$$P_{ij} = \mathbf{0} \qquad i = 1, 2, \ldots, S; \qquad j > i - 1\,.$$

To compute $P^{-1} = (\alpha I - A)^{-1}$ we proceed as described below:

Consider a lower triangular matrix

$$U = \begin{array}{c} S \\ S-1 \\ \cdots \\ 1 \\ 0 \end{array} \begin{pmatrix} U_{SS} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ U_{(S-1)S} & U_{(S-1)(S-1)} & \cdots & \mathbf{0} & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots & \\ U_{1S} & U_{1(S-1)} & \cdots & U_{11} & \mathbf{0} \\ U_{0S} & U_{0(S-1)} & \cdots & U_{01} & U_{00} \end{pmatrix},$$

with $U_{ii} = 1$, $i = 0, 1, 2, \ldots, S$ and an almost lower triangular matrix

$$R = \begin{array}{c} S \\ S-1 \\ \cdots \\ 1 \\ 0 \end{array} \begin{pmatrix} \mathbf{0} & -B_S & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -B_{(S-1)} & \cdots & \mathbf{0} & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & -B_1 \\ R_{0S} & R_{0(S-1)} & R_{0(S-2)} & \cdots & R_{01} & R_{00} \end{pmatrix},$$

such that $PU = R$. We find the sub matrices U_{ij} and R_{0j} by computing the product PU and equating it to R. The (i,j)-th sub matrix of PU, denoted by $[PU]_{ij}$ is given by

$$\begin{array}{lll}
-B_i U_{jj}, & i=1,2,\dots,S; & j=i-1, \\
D_i U_{ij} - B_i U_{(i-1)j}, & i=s+1,\dots,S; & j=i,i+1,\dots,S, \\
 & \text{or} & \\
 & i=1,2,\dots,s; & j=i,i+1,\dots,Q-1, \\
D_0 U_{0j}, & i=0; & j=0,1,2,\dots,Q-1, \\
-CU_{Qj} + D_0 U_{0j}, & i=0; & j=Q,Q+1,\dots,S, \\
-CU_{(Q+1)j} + D_i U_{ij} - B_i U_{(i-1)j}, & i=1,2,\dots,s; & j=Q+1,Q+2\dots,S.
\end{array}$$

By equating the sub matrices of PU to the corresponding elements of R, we get

$$U_{ij} = \begin{cases}
(B_{i+1}^{-1} D_{i+1})(B_{i+2}^{-1} D_{i+2})\dots(B_j^{-1} D_j), & i=0,1,2,\dots,j+1\,, \\
 & j=1,2,\dots,Q\,, \\
 & \text{or} \\
 & i=j-Q, j-Q+1,\dots,S\,, \\
 & j=Q+1,Q+2,\dots,S\,, \\
-B_{i+1}^{-1} C U_{(i+Q+1)j} + B_{i+1}^{-1} D_{i+1} U_{(i+1)j}, & i=0,1,\dots,j-Q-1\,, \\
 & j=Q+1,Q+2,\dots,S\,,
\end{cases}$$

and

$$R_{0j} = \begin{cases}
D_0, & j=0, \\
D_0 U_{0j}, & j=1,2,\dots,Q-1, \\
-CU_{Qj} + D_0 U_{0j}, & j=Q,Q+1,\dots,S.
\end{cases}$$

The equation $PU = R$ implies

$$(PU)^{-1} = R^{-1}\,, \quad U^{-1}P^{-1} = R^{-1}\,, \quad P^{-1} = UR^{-1}.$$

It can be verified that the inverse of R is given by

$$R^{-1} = \begin{pmatrix}
R_{0S}^{-1} R_{0(S-1)} B_S^{-1} & R_{0S}^{-1} R_{0(S-2)} B_{(S-1)}^{-1} & \cdots & R_{0S}^{-1} R_{00} B_1^{-1} & R_{0S}^{-1} \\
-B_S^{-1} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & -B_{(S-1)}^{-1} & \cdots & \mathbf{0} & \mathbf{0} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\mathbf{0} & \mathbf{0} & \cdots & -B_1^{-1} & \mathbf{0}
\end{pmatrix}.$$

Since the expression for R^{-1} involves the term R_{0S}^{-1}, we shall show that the latter exists. From $PU = R$, we get

$$\begin{aligned} \det(PU) &= \det(R)\,, \\ \det(P)\det(U) &= \det(R_{0S})\det(-B_1)\det(-B_2)\cdots\det(-B_S). \end{aligned}$$

Since U and B's are lower triangular matrices their determinant values are not equal to zero. Hence $\det(R_{0S})$ is not equal to zero. This proves the existence of R_{0S}^{-1}.

From $P^{-1} = UR^{-1}$, we can compute the (i,j)-th sub-matrix (denoted by P^{ij}) of $P^{-1} = (\alpha I - A)^{-1}$ and it is given by,

$$P^{ij} = \begin{cases} U_{iS}R_{0S}^{-1}R_{0(j-1)}B_j^{-1}, & i = j, j+1, \ldots, S; \quad j = 1, 2, \ldots, S\,, \\ U_{iS}R_{0S}^{-1}, & i = 0, 1, 2, \ldots, S; \quad j = 0\,, \\ U_{iS}R_{0S}^{-1}R_{0(j-1)} - U_{i(j-1)}B_j^{-1}, & i = 0, 1, \ldots, S-1; \quad j = i+1, \ldots, S. \end{cases}$$

3.2. Steady State Analysis

It can be seen from the structure of the infinitesimal matrix A defined on the finite state space that E is irreducible. Let the limiting distribution be defined by

$$\pi^{(i,q)} = \lim_{t\to\infty} \Pr\left[(L(t), X(t)) = (i,q)\right], \quad (i,q) \in E\,,$$

where $\pi^{(i,q)}$ is the steady state probability for the state (i,q). Denote

$$\pi = (\pi^{(S)}, \pi^{(S-1)}, \ldots, \pi^{(2)}, \pi^{(1)}, \pi^{(0)})\,,$$

with $\pi^{(k)} = (\pi^{(k,0)}, \pi^{(k,1)}, \ldots \pi^{(k,N)})$ for $k = 0, 1, 2, \ldots, S$. The limiting distribution exists, and it satisfies the following equations:

$$\pi A = \mathbf{0} \text{ and } \sum_{(i,q)}\sum_{\in E} \pi^{(i,q)} = 1\,.$$

The first equation of the above yields the following set of equations

$$\begin{aligned} &\pi^{(i)}A_i + \pi^{(i-Q)}C = \mathbf{0}, && i = S\,, \\ &\pi^{(i+1)}B_{i+1} + \pi^{(i)}A_i + \pi^{(i-Q)}C = \mathbf{0}, && i = Q, Q+1, \ldots, S-1\,, \\ &\pi^{(i+1)}B_{i+1} + \pi^{(i)}A_i = \mathbf{0}, && i = 0, 1, \ldots, Q-1\,. \end{aligned}$$

After tedious simplifications, we obtain

$$
\begin{aligned}
&\pi^{(i)} = (-1)^i \pi^{(0)} A_0 B_1^{-1} A_1 B_2^{-1} \cdots A_{i-1} B_i^{-1} && i = 1, 2, \ldots, Q,\\
&= (-1)^i \pi^{(0)} A_0 B_1^{-1} A_1 B_2^{-1} \cdots A_{i-1} B_i^{-1} - \pi^{(0)} C B_i^{-1} && i = Q+1,\\
&= (-1)^i \pi^{(0)} A_0 B_1^{-1} A_1 B_2^{-1} \cdots A_{i-1} B_i^{-1}\\
&+(-1)^{i-Q} \pi^{(0)} C B_{Q+1}^{-1} A_{Q+1} \cdots A_{i-1} B_i^{-1}\\
&+(-1)^{i-Q} \pi^{(0)} \left\{ \sum_{l=1}^{i-Q-1} \left(A_0 B_1^{-1} A_1 B_2^{-1} \cdots A_{l-1} B_l^{-1} \right) C \right.\\
&\left. \left(B_{Q+l+1}^{-1} A_{Q+l+1} \cdots B_i^{-1} \right) \right\}, \quad i = Q+2, Q+3, \ldots, S,
\end{aligned}
$$

where $\pi^{(0)}$ can be obtained by solving,

$$
\pi^{(S)} A_S + \pi^{(s)} C = \mathbf{0} \ \text{ and } \ \sum_{i=0}^{S} \pi^{(i)} \mathbf{e} = 1,
$$

that is

$$
\begin{aligned}
\pi^{(0)} \Bigg[(-1)^S \left(\prod_{i=0}^{S-1} \psi_i \right) A_S + (-1)^S C B_{Q+1}^{-1} \left(\prod_{i=0}^{S-1} \psi_i \right) A_S \\
+(-1)^S \left\{ \sum_{l=1}^{s_1-1} \left(\prod_{i=0}^{l-1} \psi_i \right) C \left(B_{Q+l+1}^{-1} \prod_{i=Q+l+1}^{S-1} \psi_i \right) \right\} A_S \\
+ (-1)^{s_1} \left(\prod_{i=0}^{S-1} \psi_i \right) C \Bigg] = \mathbf{0},
\end{aligned}
$$

and

$$
\begin{aligned}
\pi^{(0)} \Bigg[I + \sum_{i=1}^{Q+1} (-1)^i \prod_{i=0}^{i-1} \psi_i - C B_{Q+1}^{-1} \\
+ \sum_{i=Q+2}^{S1} \Bigg\{ (-1)^i \left(\prod_{k=0}^{i-1} \psi_k \right) + (-1)^{i-Q} C B_{Q+1}^{-1} \left(\prod_{k=Q+1}^{i-1} \psi_k \right) \\
+(-1)^{i-Q} \left\{ \sum_{l=1}^{i-Q-1} \left(\prod_{i=0}^{l-1} \psi_i \right) C B_{Q+l+1}^{-1} \left(\prod_{i=Q+l+1}^{i-1} \psi_i \right) \right\} \Bigg\} \Bigg] \mathbf{e} \ = \ 1,
\end{aligned}
$$

where $\psi_i = A_i B_{i+1}$.

4. Measures of System Performance

In this section we derive some stationary performance measures of the system.

4.1. Reorder Rate

Let β_R denote the mean reorder rate then we have,

$$\beta_R = \sum_{r=1}^{N} \mu_r \pi^{(s+1,r)} + (s+1)\gamma \sum_{r=0}^{N} \pi^{(s+1,r)}.$$

4.2. Mean Inventory Level

Let β_I denote the average inventory level in the steady state. Then we have

$$\beta_I = \sum_{j=1}^{S} j \sum_{r=0}^{N} \pi^{(j,r)} = \sum_{j=1}^{S} j \pi^{(j)} \boldsymbol{e}.$$

4.3. Mean Perishable Rate

The expected perishable rate β_P in the steady state is given by

$$\beta_P = \sum_{i=1}^{S} \sum_{j=0}^{N} i\gamma \pi^{(i,j)}.$$

4.4. Mean Balking Rate

Let β_B denote the mean balking rate of the customers in the steady state which is given by

$$\beta_B = \lambda \sum_{j=0}^{N} \sum_{i=0}^{S} \pi^{(i,\,j)} \sum_{k=N+1-j}^{\infty} (k-N+j) p_k.$$

4.5. Mean Waiting Time

Let $\overline{W}$ denote the mean waiting time of the customers in the steady state which is given by

$$\overline{W} = \frac{L_1}{\lambda_a},$$

where

$$L_1 = \sum_{n=1}^{N} n \sum_{i=0}^{S} \pi^{(i,n)},$$

and the effective arrival rate (Ross [18]), λ_a is given by

$$\lambda_a = \lambda \sum_{i=0}^{S} \left[\sum_{j=0}^{N-1} \pi^{(i,j)} \sum_{k=1}^{N-j} p_k \right].$$

5. Cost Analysis

The long run total expected cost rate $TC(S, s, N)$, is given by

$$TC(S, s, N) = h\beta_I + K\beta_R + b\beta_B + c_1\beta_P + c_2\overline{W},$$

where h is the holding cost per unit per unit time, K is the fixed cost per order, b is the balking cost per unit per unit time, c_1 is the perishable cost per unit per unit time and c_2 is the cost per unit time wait in the system. Then we have,

$$\begin{aligned} TC(S,s,N) = & h \sum_{j=1}^{S} j \sum_{r=0}^{N} \pi^{(j,r)} + K \left\{ \sum_{r=1}^{N} \mu_r \pi^{(s+1,r)} + (s+1)\gamma \sum_{r=0}^{N} \pi^{(s+1,r)} \right\} \\ & + b\lambda \sum_{j=0}^{N} \sum_{i=0}^{S} \pi^{(i,j)} \sum_{k=N+1-j}^{\infty} (k - N + j) p_k + c_1 \sum_{i=1}^{S} \sum_{j=0}^{N} i\gamma\pi^{(i,j)} \\ & + c_2 \left\{ \frac{\sum_{n=0}^{N} n \sum_{i=0}^{S} \pi^{(i,n)}}{\lambda \sum_{i=0}^{S} \left[\sum_{j=0}^{N-1} \pi^{(i,j)} \sum_{k=1}^{N-j} p_k \right]} \right\}. \end{aligned}$$

Since the computation of the π's are recursive, it is quite difficult to show the convexity of the total expected cost rate analytically. However we present the following example to demonstrate the computability of the results derived in our work, and to illustrate the existence of local optima when the total cost function is treated as a function of only two variables.

	Total Expected Cost Rate					
s / S	9	10	11	12	13	14
30	247.5594	247.3898	247.4808	247.8201	248.4077	249.2557
31	247.3414	247.1004	247.1002	247.3251	247.7707	248.4427
32	**247.2420**	246.9441	246.8704	247.0027	247.3330	247.8624
33	247.2463	**246.9433**	**246.7707**	**246.8280**	247.0645	247.4779
34	247.3416	246.9635	246.7838	246.8305	**246.9408**	247.2596
35	247.5171	247.1122	246.8954	246.8433	246.9417	**247.1834**
36	247.7639	247.3391	247.0933	247.0024	247.0508	247.2296

Table 1: Total expected cost rate as a function of S and s

	Total Expected Cost Rate						
N / S	2	3	4	5	6	7	8
18	20.0585	19.7932	20.2191	21.0615	22.1874	23.5253	25.0406
19	20.0298	19.7033	20.0774	20.8749	21.9627	23.2688	24.7573
20	**20.0178**	19.6374	19.9664	20.7245	21.7782	23.0560	24.5208
21	20.0195	**19.5910**	**19.8803**	**20.6038**	21.6271	22.8792	24.3228
22	20.0325	19.6109	19.8948	20.7076	**21.5037**	**22.7326**	**24.1570**
23	20.0549	19.6444	19.9666	20.8319	21.6257	22.8115	24.2183

Table 2: Total expected cost rate as a function of S and N

6. Numerical Illustrations

Table 1 gives the total expected cost rate for various combinations of S and s and by keeping fixed values for other parameters and costs. Namely $N = 5, \lambda = 2, \beta = 0.7, \gamma = 0.8, h = 0.2, K = 20, b = 1, c_1 = 5, c_2 = 25, \mu_1 = 4, \mu_i = \mu_1 + 0.2(i-1)$ for $i = 2, 3, \ldots N$, and $p_j = 0.6(0.4)^j \quad j = 1, 2, \ldots$.

Table 2 gives the total expected cost rate for various combinations of S and N and by keeping fixed values for other parameters and costs. Namely $s = 6, \lambda = 2, \beta = 0.5, \gamma = 0.8, h = 0.01, K = 1$,
$b = 8, c_1 = 0.5, c_2 = 0.5, \mu_1) = 8, \mu_i = \mu_1 - 0.4(i-1)$, for $i = 2, 3, \ldots, N$, and $p_j = 0.4(0.6)^j \quad j = 1, 2, \ldots$.

Table 3 gives the total expected cost rate for various combinations of N and s and by keeping fixed values for other parameters and costs. Namely $S = 30, \lambda = 20, \beta = 0.8, \gamma = 0.8, h = 0.005, K = 4$,
$b = 0.35, c_1 = 0.64, c_2 = 0.15, \mu_1 = 40, \mu_i = \mu_1 - 2 \times (i-1)$, for $i = 2, 3, \ldots, N$, and $p_j = 0.7(0.3)^j \quad j = 1, 2, \ldots$.

In all the tables, the minimum cost rate for each row is underlined and the minimum cost rate for each column is shown in bold. Since the value which

	Total Expected Cost Rate				
N / s	2	3	4	5	6
9	19.8450	19.7100	19.7279	19.8256	19.9705
10	**19.8440**	19.7017	19.7167	19.8143	19.9609
11	19.8480	**19.6932**	**19.7088**	19.8077	19.9548
12	19.8531	19.6966	19.7264	**19.8040**	**19.9483**
13	19.8609	19.7068	19.7467	19.8132	19.9513

Table 3: Total expected cost rate as a function of s and N

is both underlined and in bold, is smaller than the row minima and column minima, the total cost function is convex with respect to the variables indicated in the respective tables. Thus we obtain (local) optima in each of the tables.

7. Conclusions

In this paper we have described a perishable inventory system with service facility and batch Markovian demands. This model is suitable for the cases where the demanded item is delivered only after a random service time. We have derived the joint probability distribution of the inventory level and the number of customers both in the transient and steady state. We have also derived the stationary measures of system performances. We have provided a numerical example to illustrate the convexity of the total expected cost rate in steady state.

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