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A NOTE ON ITERATIONS OF DARBOUX FUNCTIONS

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Abstract: We recently proved in a paper that if g is a continuous function that is nonconstant on every nonempty open interval, and f is a Darboux function such that, for every real number x, $f^n(x) = g(x)$ for some positive integer n and the set of all such n is bounded, then f is continuous. In this paper, we give an example to show that the above conclusion is not true if the condition "the set of all such n is bounded" is dropped. However, it is known that if g is the identity function, then f is continuous and, either f is the identity function or $f = f^{-1}$.

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1. Introduction

Definition 1. A function $f : R \to R$ is called a Darboux function if a and b are real numbers and $f(a) \neq f(b)$, then for any real number y between $f(a)$ and $f(b)$, there exists a real number x between a and b such that $y = f(x)$, that is, the image of every interval is an interval.

It is shown in [2, Proposition 4] that if f is a Darboux function, and for each real number x there exists a positive integer n such that $f^n(x) = x$, then f is continuous and, either $f(x) = x$ for every real number x or $f^2(x) = x$ for every real number x. We proved in [1] that if g is a continuous function that is nonconstant on every nonempty open interval, and f is a Darboux function such that, for every real number x, $f^n(x) = g(x)$ for some positive integer n and the set of all such n is bounded, then f is continuous.

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The following example shows that "the set of all such n is bounded" can not be dropped.

Example 1. There exist a continuous function g that is nonconstant on every nonempty open interval, and a discontinuous Darboux function f such that, for every real number x, $f^n(x) = g(x)$ for some positive integer n.

Proof. Define $f,\, g : [\frac{1}{2}, 1] \to [\frac{3}{4}, 1]$ by $f(x) = g(x) = \frac{1}{2}(x - \frac{1}{2}) + \frac{1}{2} + (\frac{1}{2})^2$. For every positive integer n, define $f : [-(\frac{1}{2})^n, -(\frac{1}{2})^{n+1}) \longrightarrow [\frac{1}{2}, 1]$ by

$$f(x) = 2^{n+1}(x + (\frac{1}{2})^n) + \frac{1}{2} \quad \text{if} \quad x \in [-(\frac{1}{2})^n, -3(\frac{1}{2})^{n+2}]\,,$$

and

$$f(x) = -2^{n+1}(x + (\frac{1}{2})^{n+1}) + \frac{1}{2} \quad \text{if} \quad x \in (-3(\frac{1}{2})^{n+2}, -(\frac{1}{2})^{n+1}).$$

It is not hard to verify that, for any positive integer k,

$$f^k(x) = 2^{n-k+2}(x + (\frac{1}{2})^n) + \frac{1}{2} + (\frac{1}{2})^2 + ... + (\frac{1}{2})^k \qquad \text{for} \quad x \in [-(\frac{1}{2})^n, -3(\frac{1}{2})^{n+2}], \quad (1)$$

and

$$f^k(x) = -2^{n-k+2}(x + (\frac{1}{2})^{n+1}) + \frac{1}{2} + (\frac{1}{2})^2 + ... + (\frac{1}{2})^k \qquad \text{for} \quad x \in (-3(\frac{1}{2})^{n+2}, -(\frac{1}{2})^{n+1}).$$

For $n \in N$, define $g : [-(\frac{1}{2})^n, -(\frac{1}{2})^{n+1}) \longrightarrow R$ by

$$g(x) = f^n(x) \;\text{ if }\; x \in [-(\frac{1}{2})^n, \quad -3(\frac{1}{2})^{n+2}], \qquad (*)$$

and

$$g(x) = f^{n+1}(x) \;\text{ if }\; x \in (-3(\frac{1}{2})^{n+2}, \quad -(\frac{1}{2})^{n+1}). \qquad (**)$$

First, we show that $g : [-\frac{1}{2}, 0) \longrightarrow R$ is continuous at $-3(\frac{1}{2})^{n+2}$. By (1), we have $f^n(-3(\frac{1}{2})^{n+2}) = 2^{n-n+2}(-3(\frac{1}{2})^{n+2} + (\frac{1}{2})^n) + \frac{1}{2} + (\frac{1}{2})^2 + ...+ (\frac{1}{2})^n = \frac{1}{2} + (\frac{1}{2})^2 + ...+ (\frac{1}{2})^n + (\frac{1}{2})^n$. Hence, by (*) and (1), $g(-3(\frac{1}{2})^{n+2}) = \frac{1}{2} + (\frac{1}{2})^2 + ...+ (\frac{1}{2})^n + (\frac{1}{2})^n$ and g is left continuous at $-3(\frac{1}{2})^{n+2}$. As x approaches to $-3(\frac{1}{2})^{n+2}$ from the right side, $f^{n+1}(x)$ approaches to the value obtained from

(*) by replacing k by $n+1$, and x by $-3(\frac{1}{2})^{n+2}$, which is $\frac{1}{2}+(\frac{1}{2})^2+...+(\frac{1}{2})^n+(\frac{1}{2})^{n+1}+(\frac{1}{2})^{n+1}=\frac{1}{2}+(\frac{1}{2})^2+...+(\frac{1}{2})^n+(\frac{1}{2})^n$. Hence, by (**), the right side limit of g at $-3(\frac{1}{2})^{n+2}$ is $\frac{1}{2}+(\frac{1}{2})^2+...+(\frac{1}{2})^n+(\frac{1}{2})^n=g(-3(\frac{1}{2})^{n+2})$. This shows that g is continuous at $-3(\frac{1}{2})^{n+2}$.

To see that g is continuous at $-(\frac{1}{2})^{n+1}$, note that, by (*) and (**), the left side limit of g at $-(\frac{1}{2})^{n+1}$ is $\frac{1}{2}+(\frac{1}{2})^2+...+(\frac{1}{2})^{n+1}$. By replacing n by $n+1$ in (*), we have $g(-(\frac{1}{2})^{n+1})=f^{n+1}(-(\frac{1}{2})^{n+1})$. By replacing k by $n+1$, n by $n+1$, and x by $-(\frac{1}{2})^{n+1}$ in (1), the right side limit of f^{n+1} at $-(\frac{1}{2})^{n+1}$ is $f^{n+1}(-(\frac{1}{2})^{n+1})=\frac{1}{2}+(\frac{1}{2})^2+...+(\frac{1}{2})^{n+1}$. Hence, the right side limit of g at $-(\frac{1}{2})^{n+1}$ is $\frac{1}{2}+(\frac{1}{2})^2+...+(\frac{1}{2})^{n+1}=g(-(\frac{1}{2})^{n+1})$. This shows that g is continuous at $-(\frac{1}{2})^{n+1}$. Clearly, $g:[-\frac{1}{2},\,0)\longrightarrow R$ is continuous at other points. Thus, $g:[-\frac{1}{2},\,0)\longrightarrow R$ is continuous.

Let $f(0)=1$ and $g(0)=1$. In (1) and (2), $0<\ f^k(x)\leq 2^{-k}+\frac{1}{2}+(\frac{1}{2})^2+...+(\frac{1}{2})^k$. Note that $\frac{1}{2}+(\frac{1}{2})^2+...+(\frac{1}{2})^k=1-(\frac{1}{2})^k\rightarrow 1$ as $k\rightarrow\infty$. This implies that as x approaches to 0 from the left side, $g(x)$ approaches to 1. Hence, g is left continuous at 0. Thus, we have defined functions $f,\,g:[-\frac{1}{2},\,0]\cup[\frac{1}{2},1]\rightarrow R$ such that:

(i) $f(x)=g(x)$ for $x\in[\frac{1}{2},1]$,

(ii) for each $x\in[-\frac{1}{2},\,0]$, there exists $n\in N$ such that $f^n(x)=g(x)$,

(iii) $g:[-\frac{1}{2},\,0]\cup[\frac{1}{2},1]\rightarrow R$ is continuous that is nonconstant on every nonempty open interval, and

(iv) $f:[-\frac{1}{2},0]\rightarrow R$ maps intervals into intervals, and $f:[\frac{1}{2},1]\rightarrow R$ is continuous.

We extend the continuous function $g:[-\frac{1}{2},\,0]\cup[\frac{1}{2},1]\rightarrow R$ to a continuous function on R that is nonconstant on every nonempty open interval. Extend $f:[-\frac{1}{2},\,0]\cup[\frac{1}{2},1]\rightarrow R$ as $f(x)=g(x)$ for $x\in R\backslash[-\frac{1}{2},\,0]\cup[\frac{1}{2},1]$. Since $f((-\frac{1}{2})^n)=\frac{1}{2}$ for every n, f is discontinuous at 0. The function f is Darboux because it maps intervals into intervals. Thus, $g:R\rightarrow R$ is continuous that is nonconstant on every nonempty open interval, and f is a discontinuous Darboux function such that, for every real number x, $f^n(x)=g(x)$ for some $n\in N$. □

References

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