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MONOTONE BOUNDARY CONDITIONS FOR A CLASS OF NONLINEAR HYPERBOLIC SYSTEMS

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Abstract: We study the existence, uniqueness and asymptotic behaviour of the strong and weak solutions to a nonlinear boundary value problem associated to the telegraph system, on the positive semi-axis of spatial variable. This problem has applications in integrated circuits modelling. For the proof of our theorems we use some results from the theory of monotone operators and of nonlinear evolution equations of monotone type in Hilbert spaces.

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1. Introduction

We shall investigate the nonlinear hyperbolic system

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) + \dfrac{\partial v}{\partial x}(t,x) + \alpha(x,u) = f(t,x)\,, \\ \dfrac{\partial v}{\partial t}(t,x) + \dfrac{\partial u}{\partial x}(t,x) + \beta(x,v) = g(t,x), \end{cases} \quad t>0\ x>0\,, \tag{S}$$

with the boundary condition

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$$\begin{pmatrix} u(t,0) \\ S(w'(t)) \end{pmatrix} \in -G \begin{pmatrix} v(t,0) \\ w(t) \end{pmatrix} + B(t), \quad t > 0\,, \tag{BC}$$

and the initial data

$$\begin{cases} u(0,x) = u_0(x), \;\; v(0,x) = v_0(x), \;\; x > 0\,, \\ w(0) = w_0. \end{cases} \tag{IC}$$

Here the unknown functions u, v and also the functions f, g are the vectorial ones depending on $(t,x) \in I\!R_+ \times I\!R_+$ with values in $I\!R^n$, and the unknown function w is a vectorial one depending on $t \in I\!R_+$ with values in $I\!R^m$. The functions α and β are of the form $\alpha(x,u) = \mathrm{col}(\alpha_1(x,u_1),\ldots,\alpha_n(x,u_n))$, $\beta(x,v) = \mathrm{col}(\beta_1(x,v_1),\ldots,\beta_n(x,v_n))$, S is a positive diagonal matrix, G is an operator in the space $I\!R^{n+m}$ which satisfy some assumptions and $B(t) = \mathrm{col}(b_1(t),\ldots,b_{n+m}(t)) \in I\!R^{n+m}$, $\forall\, t > 0$.

This problem has applications in the theory of integrated circuits (see Marinov et al [9], Moroşanu [10], Moroşanu et al [11], for further references). The system (S) for $x \in (0,1)$ and $t > 0$ with the boundary condition

$$\begin{pmatrix} u(t,0) \\ -u(t,1) \\ S(w'(t)) \end{pmatrix} \in -G \begin{pmatrix} v(t,0) \\ v(t,1) \\ w(t) \end{pmatrix} + B(t), \quad t > 0\,,$$

and the initial data

$$u(0,x) = u_0(x), \;\; v(0,x) = v_0(x), \;\; x \in (0,1), \;\; w(0) = w_0,$$

has been investigated in Luca [4], Luca et al [8] for existence, uniqueness and regularity of the solutions, and in Luca [5] for asymptotic behaviour of the solutions. We also mention the papers Luca [6], Luca [7] where we studied a nonlinear hyperbolic system of higher order PDEs.

In the present paper we shall prove the existence, uniqueness and asymptotic behaviour of the strong and weak solutions for the problem (S) + (BC) + (IC) in the cases $B(t) \equiv$ const. and $B(t) \not\equiv$ const. For the basic notations, concepts and results in the theory of monotone operators and nonlinear evolution equations of monotone type we refer the reader to Barbu [1], Brezis [2], Moroşanu [10].

We introduce the assumptions that we shall use in the paper:

(A1) a) The functions $x \to \alpha_k(x,p)$ and $x \to \beta_k(x,p)$ are measurable on $I\!R_+$, for any fixed $p \in I\!R$. Besides, the functions $p \to \alpha_k(x,p)$ and $p \to \beta_k(x,p)$ are continuous and nondecreasing from $I\!R$ into $I\!R$, for a.a. $x \in I\!R_+$, $k = \overline{1,n}$.

b) There exist $a_k > 0$, $b_k > 0$, $k = \overline{1,n}$ and the functions $c_k^1,\ c_k^2 \in L^2(I\!R_+)$ such that

$$|\alpha_k(x,p)| \le a_k|p| + c_k^1(x), \quad |\beta_k(x,p)| \le b_k|p| + c_k^2(x),$$

for a.a. $x \in I\!R_+$, $\forall\, p \in I\!R$, $k = \overline{1,n}$.

c) There exist $\chi_1 > 0$, $\chi_2 > 0$ such that

$$(\alpha_k(x,p_1) - \alpha_k(x,p_2))(p_1 - p_2) \ge \chi_1(p_1 - p_2)^2,$$

$$(\beta_k(x,p_1) - \beta_k(x,p_2))(p_1 - p_2) \ge \chi_2(p_1 - p_2)^2,$$

for a.a. $x \in I\!R_+$, $\forall\, p_1,\, p_2 \in I\!R$.

(A2) a) $G : D(G) \subset I\!R^{n+m} \to I\!R^{n+m}$ is a maximal monotone operator (possibly multivalued), $D(G) \neq \emptyset$. Moreover, $G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}$ with

$$G_{11} : D(G_{11}) \subset I\!R^n \to I\!R^n, \quad G_{12} : D(G_{12}) \subset I\!R^m \to I\!R^n,$$

$$G_{21} : D(G_{21}) \subset I\!R^n \to I\!R^m, \quad G_{22} : D(G_{22}) \subset I\!R^m \to I\!R^m.$$

b) There exists $\zeta_1 > 0$ such that $\forall\, x,\, y \in D(G)$, $x = \mathrm{col}\,(x^a, x^b)$, $y = \mathrm{col}\,(y^a, y^b) \in I\!R^n \times I\!R^m$ and $\forall\, w_1 \in G(x)$, $w_2 \in G(y)$ we have

$$< w_1 - w_2, x - y >_{I\!R^{n+m}} \ge \zeta_1 \|x^b - y^b\|^2_{I\!R^m}.$$

c) There exists $\zeta_2 > 0$ such that $\forall\, x, y \in D(G)$ and $\forall\, w_1 \in G(x)$, $w_2 \in G(y)$ we have

$$< w_1 - w_2, x - y >_{I\!R^{n+m}} \ge \zeta_2 \|x - y\|^2_{I\!R^{n+m}}.$$

(A3) $S = \mathrm{diag}(s_1, \ldots, s_m)$ with $s_j > 0$, $j = \overline{1,m}$.

The above assumption (A2)a is a technical one and it is automatically satisfied if G is a matrix.

2. The Case $B(t) \equiv b_0$ (a Constant Vector)

In this case we replace G by $\widetilde{G}$, defined by $\widetilde{G}w = Gw - b_0$, which is also, under the assumption (A2)a, a maximal monotone operator. So, we can suppose without loss of generality that $B(t) \equiv 0$.

We shall express our problem as a Cauchy problem in a certain Hilbert space. For, we consider the Hilbert spaces $X = (L^2(\mathbb{R}_+; \mathbb{R}^n))^2$, $\mathbb{R}^m$ and $Y = X \times \mathbb{R}^m$ with the following scalar products

$$\langle f, g\rangle_X =< f_1, g_1 >_{L^2(\mathbb{R}_+;\mathbb{R}^n)} + < f_2, g_2 >_{L^2(\mathbb{R}_+;\mathbb{R}^n)},$$

$$f = \mathrm{col}(f_1, f_2), \quad g = \mathrm{col}(g_1, g_2) \in X,$$

$$\langle x, y\rangle_s = \sum_{i=1}^{m} s_i x_i y_i, \quad x,\ y \in \mathbb{R}^m,$$

$$\langle \begin{pmatrix} f \\ x \end{pmatrix}, \begin{pmatrix} g \\ y \end{pmatrix} \rangle_Y = \langle f, g\rangle_X + \langle x, y\rangle_s, \quad \begin{pmatrix} f \\ x \end{pmatrix}, \begin{pmatrix} g \\ y \end{pmatrix} \in Y.$$

We define the operator $\mathcal{A} : D(\mathcal{A}) \subset Y \to Y$,

$$D(\mathcal{A}) = \{y = \mathrm{col}\,(u, v, w) \in Y;\ \ u,\ v \in H^1(\mathbb{R}_+; \mathbb{R}^n),\ \ \mathrm{col}(v(0), w) \in D(G),$$

$$u(0) \in -G_{11}(v(0)) - G_{12}(w)\},$$

$$\mathcal{A}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} v' \\ u' \\ S^{-1}G_{21}(v(0)) + S^{-1}G_{22}(w) \end{pmatrix}, \quad \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in D(\mathcal{A})$$

and the operator $\mathcal{B} : D(\mathcal{B}) \subset Y \to Y$, $D(\mathcal{B}) = \{y = \mathrm{col}\,(u, v, w) \in Y,\ \ \mathcal{B}(y) \in Y\}$,

$$\mathcal{B}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \alpha(\cdot, u) \\ \beta(\cdot, v) \\ 0 \end{pmatrix}.$$

Remark 1. Under the assumption (A2)a we have $D(\mathcal{A}) \neq \emptyset$ and $\overline{D(\mathcal{A})} = X \times \overline{D(G_{12}) \cap D(G_{22})}$. Indeed, because $D(G) \neq \emptyset$ it follows that there exists $\mathrm{col}(a, c) \in D(G)$ and $\mathrm{col}(b, d) \in G(\mathrm{col}(a, c))$, where $a,\ b \in \mathbb{R}^n$, $c,\ d \in \mathbb{R}^m$. Then the functions $u(x) = -be^{-x}$, $v(x) = ae^{-x}$, $x \in (0, \infty)$ and $w = c$ satisfy the conditions of $D(\mathcal{A})$. Moreover, if $y = \mathrm{col}(u, v, w) \in D(\mathcal{A})$ and $h,\ k \in C_0^\infty(\mathbb{R}_+; \mathbb{R}^n)$ then $\mathrm{col}(u + h, v + k, w) \in D(\mathcal{A})$. Because $\overline{C_0^\infty(\mathbb{R}_+; \mathbb{R}^n)} = L^2(\mathbb{R}_+; \mathbb{R}^n)$ we deduce that $\overline{D(\mathcal{A})} = X \times \overline{D(G_{12}) \cap D(G_{22})}$.

Remark 2. Under the assumptions (A1)ab we have $D(\mathcal{B}) = Y$. Indeed, for $y = \mathrm{col}(u, v, w) \in Y$ we have

$$\|\mathcal{B}(y)\|_Y^2 = \sum_{k=1}^{n} \left(\int_0^\infty |\alpha_k(x, u_k(x))|^2 dx + \int_0^\infty |\beta_k(x, v_k(x))|^2 dx \right)$$

$$\leq 2\sum_{k=1}^{n}\left\{\int_0^\infty [a_k^2|u_k(x)|^2+(c_k^1(x))^2]dx+\int_0^\infty [b_k^2|v_k(x)|^2+(c_k^2(x))^2]dx\right\} < \infty.$$

Lemma 1. *If the assumptions (A2)a and (A3) hold, then the operator $\mathcal{A}$ is maximal monotone in the space Y.*

Proof. We suppose without loss of the generality that the operator G is single-valued. The operator $\mathcal{A}$ is monotone; indeed we have

$$\begin{aligned}
< \mathcal{A}(y)-\mathcal{A}(\overline{y}), y-\overline{y} >_Y &= \int_0^\infty < u(x)-\overline{u}(x), v'(x)-\overline{v}'(x) >_{I\!R^n} dx \\
&+ \int_0^\infty < v(x)-\overline{v}(x), u'(x)-\overline{u}'(x) >_{I\!R^n} dx \\
&+ \underbrace{< G_{21}(v(0))-G_{21}(\overline{v}(0))+G_{22}(w)-G_{22}(\overline{w}), w-\overline{w} >_{I\!R^m}}_{M}
\end{aligned}$$

$$\begin{aligned}
&= -< u(0)-\overline{u}(0), v(0)-\overline{v}(0) >_{I\!R^n} + M = < G_{11}(v(0))-G_{11}(\overline{v}(0))+G_{12}(w) \\
&- G_{12}(\overline{w}), v(0)-\overline{v}(0) >_{I\!R^n} + M = < G\begin{pmatrix} v(0) \\ w \end{pmatrix} - G\begin{pmatrix} \overline{v}(0) \\ \overline{w} \end{pmatrix}, \begin{pmatrix} v(0) \\ w \end{pmatrix} \\
&- \begin{pmatrix} \overline{v}(0) \\ \overline{w} \end{pmatrix} >_{I\!R^{n+m}} \geq 0, \\
&\forall\, y=\mathrm{col}(u,v,w),\ \overline{y}=\mathrm{col}(\overline{u},\overline{v},\overline{w}) \in D(\mathcal{A})
\end{aligned}$$

(because $u,\, v \in H^1(I\!R_+; I\!R^n)$, $\lim_{x\to\infty} u(x) = \lim_{x\to\infty} v(x) = 0$, $u(0) = -G_{11}(v(0))$ $-G_{12}(w)$, $\overline{u}(0) = -G_{11}(\overline{v}(0)) - G_{12}(\overline{w})$).

We shall prove in what follows that the operator $\mathcal{A}$ is maximal monotone, that is we shall show, by Brezis [2, Proposition 2.2], that for any $\gamma = \mathrm{col}(p,q,r) \in Y$ there exists $y = \mathrm{col}(u,v,w) \in D(\mathcal{A})$ such that

$$y + \mathcal{A}(y) = \gamma. \tag{1}$$

Let $\gamma \in Y$ be arbitrary, but fixed for the moment. The equation (1) is equivalent to

$$\begin{cases} u + v' = p\,, \\ v + u' = q\,, \\ w + S^{-1}G_{21}(v(0)) + S^{-1}G_{22}(w) = r\,, \\ u(0) = -G_{11}(v(0)) - G_{12}(w). \end{cases} \tag{2}$$

By the equations $(2)_{1,2}$ we obtain

$$(u+v)+(u+v)' = p+q, \quad (u-v)-(u-v)' = p-q.$$

We denote by $h = u + v$ and $k = u - v$; then $u = \frac{h+k}{2}$, $v = \frac{h-k}{2}$. Therefore the above equations become

$$h + h' = p + q, \quad k - k' = p - q\,,$$

with the solutions

$$h(x) = e^{-x}\left(C_1 + \int_0^x e^{\tau}(p(\tau) + q(\tau))d\tau\right),$$

$$k(x) = e^{x}\left(C_2 + \int_0^x e^{-\tau}(q(\tau) - p(\tau))d\tau\right), \;\; C_1,\, C_2 \in I\!R^n.$$

Because $\lim\limits_{x\to\infty} k(x) = 0$, ($k \in H^1(I\!R_+; I\!R^n)$) we deduce

$$C_2 = -\int_0^{\infty} e^{-\tau}(q(\tau) - p(\tau))d\tau.$$

So

$$h(x) = e^{-x}\left(C_1 + \int_0^x e^{\tau}(p(\tau) + q(\tau))d\tau\right),$$

$$k(x) = e^{x}\int_x^{\infty} e^{-\tau}(p(\tau) - q(\tau))d\tau.$$

It is easily to show that $h,\, k \in H^1(I\!R_+; I\!R^n)$.

We shall determine $C_1 \in I\!R^n$ and $w \in I\!R^m$ such that the relations $(2)_{3,4}$ be satisfied, that is

$$\begin{cases} G_{11}(v(0)) + G_{12}(w) = -u(0) \\ G_{21}(v(0)) + G_{22}(w) = S(r - w) \end{cases}$$

$$\Leftrightarrow \quad G\begin{pmatrix} v(0) \\ w \end{pmatrix} = \begin{pmatrix} -u(0) \\ S(r) - S(w) \end{pmatrix}. \qquad (3)$$

We have

$$u(0) = \frac{h(0) + k(0)}{2} = \frac{C_1 + k(0)}{2}, \;\; v(0) = \frac{h(0) - k(0)}{2} = \frac{C_1 - k(0)}{2}$$

and

$$k(0) = \int_0^{\infty} e^{-\tau}(p(\tau) - q(\tau))d\tau.$$

The relation (3) becomes

$$G\begin{pmatrix} \dfrac{C_1 - k(0)}{2} \\ w \end{pmatrix} = \begin{pmatrix} -\dfrac{C_1 + k(0)}{2} \\ S(r) - S(w) \end{pmatrix} \quad \Leftrightarrow$$

$$G\begin{pmatrix} \frac{C_1}{2}-\frac{k(0)}{2} \\ w \end{pmatrix}+\begin{pmatrix} \frac{C_1}{2}-\frac{k(0)}{2} \\ S(w) \end{pmatrix}=\begin{pmatrix} -k(0) \\ S(r) \end{pmatrix}$$

$$\Leftrightarrow \left(G+\begin{pmatrix} I & 0 \\ 0 & S \end{pmatrix}\right)\begin{pmatrix} \frac{C_1}{2}-\frac{k(0)}{2} \\ w \end{pmatrix}=\begin{pmatrix} -k(0) \\ S(r) \end{pmatrix}. \qquad (4)$$

Because G is a maximal monotone operator and the matrix operator $\begin{pmatrix} I & 0 \\ 0 & S \end{pmatrix}$ is positive defined, continuous (so it is coercive and monotone), we deduce by Barbu [1, Corollary 1.3, Chapter II] that $G+\begin{pmatrix} I & 0 \\ 0 & S \end{pmatrix}$ is maximal monotone and

$$R\left(G+\begin{pmatrix} I & 0 \\ 0 & S \end{pmatrix}\right)=\mathbb{R}^{n+m}.$$

Therefore there exists $\mathrm{col}(z,w)$ such that

$$\left(G+\begin{pmatrix} I & 0 \\ 0 & S \end{pmatrix}\right)\begin{pmatrix} z \\ w \end{pmatrix}=\begin{pmatrix} -k(0) \\ S(r) \end{pmatrix}.$$

Then w and $C_1 = 2z + k(0)$ verify (4). It follows that $u = \mathrm{col}\left(\frac{h+k}{2}, \frac{h-k}{2}, w\right) \in D(\mathcal{A})$ is the (unique) solution of the equation (1). □

Lemma 2. *If the assumptions (A1)ab hold, then the operator $\mathcal{B}$ is maximal monotone in Y.*

Proof. The operator $\mathcal{B}$ is monotone; indeed

$$\begin{aligned}
&< \mathcal{B}(y)-\mathcal{B}(\overline{y}), y-\overline{y} >_Y \\
&\quad = \sum_{k=1}^{n}\int_0^{\infty}[\alpha_k(x,u_k(x))-\alpha_k(x,\overline{u}_k(x))][u_k(x)-\overline{u}_k(x)]dx \\
&\quad + \sum_{k=1}^{n}\int_0^{\infty}[\beta_k(x,v_k(x))-\beta_k(x,\overline{v}_k(x))][v_k(x)-\overline{v}_k(x)]dx \geq 0, \\
&\qquad\qquad \forall\, y=\mathrm{col}(u,v,w),\ \overline{y}=\mathrm{col}(\overline{u},\overline{v},\overline{w}) \in D(\mathcal{B})=Y.
\end{aligned}$$

To prove that $\mathcal{B}$ is maximal monotone we shall show that for any $\gamma = \mathrm{col}(p,q,r) \in Y$ there exists $y = \mathrm{col}(u,v,w) \in Y$ such that $y + \mathcal{B}(y) = \gamma$ or equivalently

$$\begin{cases} u_k+\alpha_k(\cdot,u_k)=p_k\,, \\ v_k+\beta_k(\cdot,v_k)=q_k, \quad k=\overline{1,n}\,, \\ w=r. \end{cases} \qquad (5)$$

After some considerations involving the assumptions (A1)ab (see also [6, Lemma 2]) we deduce that the functions

$$u_k(x) = (I + \alpha_k(x,\cdot))^{-1} p_k(x) = J_1^{\alpha_k}(x, p_k(x)),$$

$$v_k(x) = (I + \beta_k(x,\cdot))^{-1} q_k(x) = J_1^{\beta_k}(x, q_k(x)), \quad x \in I\!R_+, \;\; k = \overline{1,n},$$

verify $(5)_{1,2}$, where J_λ^A is the resolvent of A ($J_\lambda^A = (I + \lambda A)^{-1}$). The above functions u_k, v_k, $k = \overline{1,n}$ belong to $L^2(I\!R_+)$ and together with $w = r$ are the unique solution of the system (5). □

Using the operators $\mathcal{A}$ and $\mathcal{B}$ our problem (S)+(BC)+(IC) can be equivalently expressed as the following Cauchy problem in the space Y

$$\begin{cases} \dfrac{dy}{dt}(t) + (\mathcal{A} + \mathcal{B})(y(t)) \ni F(t,\cdot), \quad t > 0\,, \\ y(0) = y_0\,, \end{cases} \tag{P}$$

where $y(t) = \operatorname{col}(u(t), v(t), w(t))$, $F(t,\cdot) = \operatorname{col}(f(t,\cdot), g(t,\cdot), 0)$, $y_0 = \operatorname{col}(u_0, v_0, w_0)$.

We shall say that $y = \operatorname{col}(u, v, w)$ is a strong (weak) solution of the problem (S)+(BC)+ (IC) if y is a strong (respectively weak) solution of the problem (P) (see Barbu [1, Chapter III, Section 2]).

Theorem 1. *Assume the assumptions (A1)ab, (A2)a and (A3) hold. If* $f,\, g \in W^{1,1}(0,T; L^2(I\!R_+; I\!R^n))$ *($T > 0$ fixed),* $u_0,\, v_0 \in H^1(I\!R_+; I\!R^n)$*,* $col(v_0(0), w_0) \in D(G)$*,* $u_0(0) \in -G_{11}(v_0(0)) - G_{12}(w_0)$*, then the problem (P)⇔(S)+(BC) +(IC) has a unique strong solution* $y = col(u, v, w) \in W^{1,\infty}(0,T; Y)$*. Moreover*

$$u,\, v \in L^\infty(0,T; H^1(I\!R_+; I\!R^n)). \tag{6}$$

Proof. By Lemma 1 and Lemma 2 the operators $\mathcal{A}$ and $\mathcal{B}$ are maximal monotone in Y and $D(\mathcal{B}) = Y$. Using Rockafellar's Theorem (see Barbu [1, Theorem 1.7, Chapter II]) it follows that $\mathcal{A} + \mathcal{B}$ is also maximal monotone operator. By Barbu [1, Theorem 2.2, Corollary 2.1, Chapter III] we deduce that for $y_0 = \operatorname{col}(u_0, v_0, w_0) \in D(\mathcal{A})$ and $F \in W^{1,1}(0,T; Y)$ the problem (P)⇔(S)+(BC)+(IC) has a unique strong solution $y = \operatorname{col}(u, v, w) \in W^{1,\infty}(0,T; Y)$, $y(t) \in D(\mathcal{A})$, $\forall\, t \in [0,T)$. By extending correspondingly the functions f and g in the interval $[0, T + \varepsilon]$ with $\varepsilon > 0$, we obtain $y(T) \in D(\mathcal{A})$. The solution y is everywhere differentiable from right and

$$\frac{d^+ y}{dt}(t) = (F(t) - \mathcal{A}(y(t)) - \mathcal{B}(y(t)))^0, \;\; \forall\, t \in [0,T).$$

In addition we have

$$\left\| \frac{d^+y}{dt}(t) \right\|_Y \le \|(F(0) - \mathcal{A}(y_0) - \mathcal{B}(y_0))^0\|_Y + \int_0^t \left\| \frac{dF}{ds}(s) \right\|_Y ds, \ \forall t \in [0,T).$$

Using the equations of (S) and (A1)ab we obtain (6). □

Remark 3. For all $t \in [0,T)$ the above functions $u(t,\cdot),\ v(t,\cdot)$ satisfy the system (S) for a.a. $t \in I\!R_+$ (with $\partial^+u/\partial t,\ \partial^+v/\partial t$ instead of $\partial u/\partial t,\ \partial v/\partial t$) and together with $w(t)$ verify the boundary condition (BC) (with d^+w/dt instead of dw/dt) and the initial data (IC).

Using now Barbu [1, Corollary 2.2, Chapter III] we obtain

Theorem 2. *Assume the assumptions (A1)ab, (A2)a and (A3) hold. If $f,\ g \in L^1(0,T;L^2(I\!R_+;I\!R^n))$ ($T > 0$ fixed), $u_0,\ v_0 \in L^2(I\!R_+;I\!R^n)$, $w_0 \in D(G_{12}) \cap D(G_{22})$, then the problem (S)+(BC)+(IC) has a unique weak solution $y = col(u,v,w) \in C([0,T];Y)$.*

In what follows we shall present an existence result for the stationary problem associated to (P).

Theorem 3. *If the assumptions* (A1)abc, (A2)ab *and* (A3) *hold, then the stationary problem*

$$\mathcal{A}(y) + \mathcal{B}(y) \ni 0 \tag{7}$$

has a unique solution $y = \mathrm{col}(u,v,w) \in D(\mathcal{A})$.

Proof. By the assumptions of the theorem, the operator $\mathcal{A} + \mathcal{B}$ is strongly monotone. Indeed, for all $y = \mathrm{col}(u,v,w)$, $\overline{y} = \mathrm{col}(\overline{u},\overline{v},\overline{w}) \in D(\mathcal{A})$ we have

$$\begin{aligned} &< \mathcal{A}(y) + \mathcal{B}(y) - \mathcal{A}(\overline{y}) - \mathcal{B}(\overline{y}), y - \overline{y} >_Y \\ &\qquad =< G\begin{pmatrix} v(0) \\ w \end{pmatrix} - G\begin{pmatrix} \overline{v}(0) \\ \overline{w} \end{pmatrix}, \begin{pmatrix} v(0) - \overline{v}(0) \\ w - \overline{w} \end{pmatrix} >_{I\!R^{n+m}} \\ &\qquad + \sum_{k=1}^n \int_0^\infty [\alpha_k(x,u_k(x)) - \alpha_k(x,\overline{u}_k(x))][u_k(x) - \overline{u}_k(x)]dx \\ &\qquad + \sum_{k=1}^n \int_0^\infty [\beta_k(x,v_k(x)) - \beta_k(x,\overline{v}_k(x))][v_k(x) - \overline{v}_k(x)]dx \ge \\ &\zeta_1\|w - \overline{w}\|^2_{I\!R^m} + \sum_{k=1}^n \chi_1\|u_k - \overline{u}_k\|^2_{L^2(I\!R_+)} + \sum_{k=1}^n \chi_2\|v_k - \overline{v}_k\|^2_{L^2(I\!R_+)} \ge \chi_0\|y - \overline{y}\|^2_Y, \end{aligned}$$

where $\chi_0 = \min\{\chi_1,\ \chi_2,\ \zeta_1/s_i,\ i = \overline{1,m}\}$.

Therefore this operator is coercive and then $R(\mathcal{A}+\mathcal{B}) = Y$. So we deduce that the equation (7) has a unique solution $y = \mathrm{col}(u,v,w) \in D(\mathcal{A})$. □

Now using Theorem 3 and Brezis [2, Theoreme 3.9] we deduce

Theorem 4. *Assume that* (A1)abc, (A2)ab *and* (A3) *hold,* f, $g \in L^1_{loc}(I\!R_+; L^2(I\!R_+; I\!R^n))$ *verify the conditions* $\lim\limits_{t\to\infty} f(t) = f^0$, $\lim\limits_{t\to\infty} g(t) = g^0$, *strongly in* $L^2(I\!R_+; I\!R^n)$, *and* $\gamma = \mathrm{col}(p, q, r)$ *is the unique solution of the equation* (7). *Then* $\lim\limits_{t\to\infty} y(t) = \gamma$, *strongly in* Y, *where* $y(t) = \mathrm{col}(u(t), v(t), w(t))$, $t \geq 0$ *is an arbitrary weak solution of the equation* $(\mathrm{P})_1$. *More precisely*

$$\|y(t) - \gamma\|_Y \leq e^{-\chi_0 t}\|y(0) - \gamma\|_Y + \int_0^t e^{\chi_0(s-t)}\|F(s) - F^0\|_Y ds, \ \ t \geq 0,$$

where $F^0 = \mathrm{col}(f^0, g^0, 0)$.

If $\frac{dF}{dt} \in L^1(I\!R_+; Y)$ *and* $y(0) \in D(\mathcal{A})$ *then* $\lim_{t\to\infty} \left\|\frac{d^+y}{dt}(t)\right\|_Y = 0$ *strongly in* Y *and*

$$\int_0^\infty \left\|\frac{d^+y}{dt}(t)\right\|_Y dt \leq \frac{1}{\chi_0}\|((\mathcal{A}+\mathcal{B})(y(0)) - F(0))^0\|_Y + \frac{1}{\chi_0}\int_0^\infty \left\|\frac{dF}{dt}(t)\right\|_Y dt.$$

3. The Case $B(t) \not\equiv$ const.

In this case we make a change of functions $u_k = \widetilde{u}_k + \widetilde{\widetilde{u}}_k$, where $\widetilde{\widetilde{u}}_k(t,x) = \frac{1}{1+x} b_k(t)$, $k = \overline{1,n}$. Then the problem (S)+(BC)+(IC) can be written as

$$\left\{\begin{array}{l} \dfrac{\partial \widetilde{u}}{\partial t}(t,x) + \dfrac{\partial v}{\partial x}(t,x) + \alpha(x, \widetilde{u} + \widetilde{\widetilde{u}}(t,x)) = \widetilde{f}(t,x)\,, \\ \dfrac{\partial v}{\partial t}(t,x) + \dfrac{\partial \widetilde{u}}{\partial x}(t,x) + \beta(x,v) = \widetilde{g}(t,x), \end{array}\right. \qquad (\widetilde{\mathrm{S}})$$

$t > 0$, $x > 0$, with the boundary condition

$$\begin{pmatrix} \widetilde{u}(t,0) \\ S(w'(t)) \end{pmatrix} \in -G\begin{pmatrix} v(t,0) \\ w(t) \end{pmatrix} + \begin{pmatrix} 0 \\ B_2(t) \end{pmatrix}, \ \ t > 0 \qquad (\widetilde{\mathrm{BC}})$$

and the initial data

$$\left\{\begin{array}{l} \widetilde{u}(0,x) = \widetilde{u}_0(x), \ \ v(0,x) = v_0(x), \ \ x > 0\,, \\ w(0) = w_0, \end{array}\right. \qquad (\widetilde{\mathrm{IC}})$$

where

$$\begin{cases} \widetilde{f}_k(t,x) = f_k(t,x) - \dfrac{1}{1+x} b_k'(t), \\ \widetilde{g}_k(t,x) = g_k(t,x) + \dfrac{1}{(1+x)^2} b_k(t), \quad x>0, \quad t>0, \quad k=\overline{1,n}, \\ \widetilde{u}_{k0}(x) = u_{k0}(x) - \dfrac{1}{1+x} b_k(0), \quad x>0, \quad k=\overline{1,n}, \\ B_2(t) = \mathrm{col}(b_{n+1}(t), \ldots, b_{n+m}(t)). \end{cases}$$

Using once again the operators $\mathcal{A}$ and $\mathcal{B}$, the problem $(\widetilde{\mathrm{S}})$+$(\widetilde{\mathrm{BC}})$+$(\widetilde{\mathrm{IC}})$ can be equivalently formulated as a time dependent Cauchy problem in the space Y

$$\begin{cases} \dfrac{d}{dt} \begin{pmatrix} \widetilde{u} \\ v \\ w \end{pmatrix} + \mathcal{A} \begin{pmatrix} \widetilde{u} \\ v \\ w \end{pmatrix} + \mathcal{B} \begin{pmatrix} \widetilde{u} + \widetilde{\widetilde{u}}(t) \\ v \\ w \end{pmatrix} \ni \begin{pmatrix} \widetilde{f}(t,\cdot) \\ \widetilde{g}(t,\cdot) \\ S^{-1}B_2(t) \end{pmatrix}, \\ \begin{pmatrix} \widetilde{u}(0) \\ v(0) \\ w(0) \end{pmatrix} = \begin{pmatrix} \widetilde{u}_0 \\ v_0 \\ w_0 \end{pmatrix}, \end{cases} \qquad (\widetilde{\mathrm{P}})$$

where $\widetilde{f} = \mathrm{col}(\widetilde{f}_1, \ldots, \widetilde{f}_n)$, $\widetilde{g} = \mathrm{col}(\widetilde{g}_1, \ldots, \widetilde{g}_n)$, $\widetilde{u}_0 = \mathrm{col}(\widetilde{u}_{10}, \ldots, \widetilde{u}_{n0})$.

Theorem 5. *Assume the assumptions* (A1)ab, (A2)ac, (A3) *hold*, $f,\, g \in W^{1,1}(0,T;\, L^2(I\!R_+; I\!R^n))$ $(T>0$ *fixed*$)$, $b_k \in W^{1,2}(0,T)$, $k=\overline{1,n+m}$, $u_0,\, v_0 \in H^1(I\!R_+; I\!R^n)$, $w_0 \in I\!R^m$, $\mathrm{col}(v_0(0), w_0) \in D(G)$ *and* $B_1(0) \in u_0(0) + G_{11}(v_0(0)) + G_{12}(w_0)$. *Then the problem* $(\widetilde{\mathrm{P}}) \Leftrightarrow (\widetilde{\mathrm{S}})$+$(\widetilde{\mathrm{BC}})$+$(\widetilde{\mathrm{IC}})$ *has a unique strong solution* $y = \mathrm{col}(u,v,w) \in W^{1,\infty}(0,T;Y)$. *Moreover* $u,\, v \in L^\infty(0,T; H^1(I\!R_+; I\!R^n))$.

Proof. We suppose again that G is single-valued. We shall use some similar arguments as that used in Luca et al [8], Luca [7]. In a first stage we assume that $f,\, g \in W^{1,\infty}(0,T; L^2(I\!R_+; I\!R^n))$, $k=\overline{1,n}$, $b_k \in W^{2,\infty}(0,T)$, $k=\overline{1,n}$, $b_j \in W^{1,\infty}(0,T)$, $j=\overline{n+1,n+m}$, and the functions $\alpha_k(x,\cdot)$, $k=\overline{1,n}$ are Lipschitz continuous with Lipschitz constant L independent of x. We consider the operator $\mathcal{L}(t)$, $t\in[0,T]$, defined by $D(\mathcal{L}(t)) = D(\mathcal{A})$ and

$$\mathcal{L}(t) \begin{pmatrix} \widetilde{u} \\ v \\ w \end{pmatrix} = \mathcal{A} \begin{pmatrix} \widetilde{u} \\ v \\ w \end{pmatrix} + \mathcal{B} \begin{pmatrix} \widetilde{u} + \widetilde{\widetilde{u}}(t) \\ v \\ w \end{pmatrix} - \begin{pmatrix} \widetilde{f}(t,\cdot) \\ \widetilde{g}(t,\cdot) \\ S^{-1}B_2(t) \end{pmatrix},$$

$$\begin{pmatrix} \widetilde{u} \\ v \\ w \end{pmatrix} \in D(\mathcal{A}), \quad t>0.$$

By Lemma 2 we deduce that the operators $\mathcal{L}(t)$, $t\in[0,T]$ are maximal monotone in Y. Using the above assumptions we have

$$|\alpha_k(x,\widetilde{u}_k+\widetilde{\widetilde{u}}_k(t,x)-\alpha_k(x,\widetilde{u}_k+\widetilde{\widetilde{u}}_k(s,x))\le L|\widetilde{\widetilde{u}}_k(t,x)-\widetilde{\widetilde{u}}_k(s,x)|$$
$$=\frac{L}{1+x}|b_k(t)-b_k(s)|,\ \ \forall\, t,\, s\in[0,T],\ \ \text{for a.a. } x>0,\ \ k=\overline{1,n}$$
$$\Rightarrow\|\alpha_k(x,\widetilde{u}_k+\widetilde{\widetilde{u}}_k(t,\cdot))-\alpha_k(x,\widetilde{u}_k+\widetilde{\widetilde{u}}_k(s,\cdot))\|^2_{L^2(I\!R_+)}\le L^2|b_k(t)-b_k(s)|^2,$$
$$\forall\, t,\, s\in[0,T]. \quad (8)$$

Also we get

$$|\widetilde{f}_k(t,x)-\widetilde{f}_k(s,x)|\le|f_k(t,x)-f_k(s,x)|+\frac{1}{1+x}|b'_k(t)-b'_k(s)|,$$
$$\forall t,\, s\in[0,T],\ \ x>0$$

$$\Rightarrow\|\widetilde{f}_k(t,\cdot)-\widetilde{f}_k(s,\cdot)\|^2_{L^2(I\!R_+)}\le 2\|f_k(t,\cdot)-f_k(s,\cdot)\|^2_{L^2(I\!R_+)}+2|b'_k(t)-b'_k(s)|^2,$$
$$\forall\, t,\, s\in[0,T]\,, \quad (9)$$

and

$$|\widetilde{g}_k(t,x)-\widetilde{g}_k(s,x)|\le|g_k(t,x)-g_k(s,x)|+\frac{1}{(1+x)^2}|b_k(t)-b_k(s)|,$$
$$\forall t,\, s\in[0,T],\ \ x>0$$

$$\Rightarrow\|\widetilde{g}_k(t,\cdot)-\widetilde{g}_k(s,\cdot)\|^2_{L^2(I\!R_+)}\le 2\|g_k(t,\cdot)-g_k(s,\cdot)\|^2_{L^2(I\!R_+)}+\frac{2}{3}|b_k(t)-b_k(s)|^2,$$
$$\forall\, t,\, s\in[0,T]. \quad (10)$$

Therefore by (8)-(10) and our assumptions we obtain that there exists $L_1>0$ such that

$$\|\mathcal{L}(t)(\widetilde{y})-\mathcal{L}(s)(\widetilde{y})\|_Y\le L_1|t-s|,\ \ \forall\, t,\, s\in[0,T],\ \ \forall\,\widetilde{y}\in D(\mathcal{A})$$

$(\widetilde{y}=\text{col}(\widetilde{u},v,w))$.

In this way, the operator family $\{\mathcal{L}(t);\ \ t\in[0,T]\}$ verifies the conditions of Kato's Theorem (see Kato [3]). By the assumptions of our theorem we deduce that $\widetilde{y}_0=\text{col}(\widetilde{u}_0,v_0,w_0)\in D(\mathcal{A})$. Therefore it follows that the problem $(\widetilde{\text{P}})$ has a unique strong solution $\widetilde{y}=\text{col}(\widetilde{u},v,w)\in W^{1,\infty}(0,T;Y)$, $\text{col}(\widetilde{u}(t),v(t),w(t))\in D(\mathcal{A})$, $\forall\, t\in[0,T]$. Moreover $\widetilde{y}$ is everywhere differentiable from right on $[0,T)$

and

$$\begin{cases} \dfrac{d^+}{dt}\begin{pmatrix} \widetilde{u}(t) \\ v(t) \\ w(t) \end{pmatrix} + \mathcal{A}\begin{pmatrix} \widetilde{u}(t) \\ v(t) \\ w(t) \end{pmatrix} + \mathcal{B}\begin{pmatrix} \widetilde{u}(t) + \widetilde{\widetilde{u}}(t) \\ v(t) \\ w(t) \end{pmatrix} = \begin{pmatrix} \widetilde{f}(t,\cdot) \\ \widetilde{g}(t,\cdot) \\ S^{-1}B_2(t) \end{pmatrix}, \\ \begin{pmatrix} \widetilde{u}(0) \\ v(0) \\ w(0) \end{pmatrix} = \begin{pmatrix} \widetilde{u}_0 \\ v_0 \\ w_0 \end{pmatrix}. \end{cases}$$

Hence $y(t) = \mathrm{col}(u(t), v(t), w(t))$ verifies the problem

$$\begin{cases} \dfrac{d^+y}{dt}(t) + \mathcal{A}(y(t)) + \mathcal{B}(y(t)) = F_1(t,\cdot), \ \ 0 \le t < T, \ \text{ in } Y, \\ u(t,0) = -G_{11}(v(t,0)) - G_{12}(w(t)) + B_1(t), \ \ 0 \le t < T, \\ y(0) = y_0, \end{cases}$$

where $F_1(t,\cdot) = \mathrm{col}(f(t,\cdot), g(t,\cdot), S^{-1}B_2(t))$. So $y = \mathrm{col}(u,v,w)$ is a solution of the problem (S)+(BC)+(IC).

In a second stage we suppose that $\alpha_k(x,\cdot)$, $k = \overline{1,n}$ are not Lipschitz continuous and we replace the functions $\alpha_k(x,\cdot)$ by the Yosida approximations $\alpha_k^\lambda(x,\cdot)$, $k = \overline{1,n}$, $\lambda > 0$. By the above reasoning we deduce that the problem $(\widetilde{\mathrm{P}})$ with α_k^λ instead of α_k has a unique strong solution $\mathrm{col}(\widetilde{u}^\lambda, v^\lambda, w^\lambda) \in W^{1,\infty}(0,T;Y)$. Then $y^\lambda = \mathrm{col}(u^\lambda, v^\lambda, w^\lambda)$ verifies the following problem

$$\begin{cases} \dfrac{d^+y^\lambda}{dt}(t) + \mathcal{A}(y^\lambda(t)) + \mathcal{B}_\lambda(y^\lambda(t)) = F_1(t,\cdot), \ \ 0 \le t < T, \ \text{ in } Y, \\ u^\lambda(t,0) = -G_{11}(v^\lambda(t,0)) - G_{12}(w^\lambda(t)) + B_1(t), \ \ 0 \le t < T, \\ y^\lambda(0) = y_0, \end{cases} \tag{11}$$

with

$$\mathcal{B}_\lambda\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \mathrm{col}(\alpha_1^\lambda(\cdot,u_1), \dots, \alpha_n^\lambda(\cdot,u_n)) \\ \mathrm{col}(\beta_1(\cdot,v_1), \dots, \beta_n(\cdot,v_n)) \\ 0 \end{pmatrix}, \quad \lambda > 0.$$

We write the relation $(11)_1$ for $t+h$ and t, we substract the equations and we multiply the obtained relation by $y^\lambda(t+h) - y^\lambda(t)$ in the space Y. We obtain

$$\begin{aligned} &\frac{1}{2}\frac{d^+}{dt}\|y^\lambda(t+h) - y^\lambda(t)\|_Y^2 - < u^\lambda(t+h,0) - u^\lambda(t,0), v^\lambda(t+h,0) - v^\lambda(t,0) >_{\mathbb{R}^n} \\ &+ < G_{21}(v^\lambda(t+h,0)) - G_{21}(v^\lambda(t,0)) + G_{22}(w^\lambda(t+h)) - G_{22}(w^\lambda(t)), w^\lambda(t+h) \\ &- w^\lambda(t) >_{\mathbb{R}^m} \le < f(t+h,\cdot) - f(t,\cdot), u^\lambda(t+h) - u^\lambda(t) >_{L^2(\mathbb{R}_+;\mathbb{R}^n)} + < g(t+h,\cdot) \\ &- g(t,\cdot), v^\lambda(t+h) - v^\lambda(t) >_{L^2(\mathbb{R}_+;\mathbb{R}^n)} + < B_2(t+h) - B_2(t), w^\lambda(t+h) - w^\lambda(t) >_{\mathbb{R}^m} \end{aligned}$$

$$\Rightarrow \quad \frac{1}{2}\frac{d^+}{dt}\|y^\lambda(t+h)-y^\lambda(t)\|_Y^2 + < G_{11}(v^\lambda(t+h,0)) - G_{11}(v^\lambda(t,0)) + G_{12}(w^\lambda(t+h)) - G_{12}(w^\lambda(t)), v^\lambda(t+h,0) - v^\lambda(t,0) >_{I\!R^n} - < B_1(t+h) - B_1(t), v^\lambda(t+h,0) - v^\lambda(t,0) >_{I\!R^n} + < G_{21}(v^\lambda(t+h,0)) - G_{21}(v^\lambda(t,0)) + G_{22}(w^\lambda(t+h)) - G_{22}(w^\lambda(t)), w^\lambda(t+h) - w^\lambda(t) >_{I\!R^m} \leq < f(t+h,\cdot) - f(t,\cdot), u^\lambda(t+h) - u^\lambda(t) >_{L^2(I\!R_+;I\!R^n)} + < g(t+h,\cdot) - g(t,\cdot), v^\lambda(t+h) - v^\lambda(t) >_{L^2(I\!R_+;I\!R^n)} + < B_2(t+h) - B_2(t), w^\lambda(t+h) - w^\lambda(t) >_{I\!R^m} .$$

We obtain

$$\frac{1}{2}\frac{d^+}{dt}\|y^\lambda(t+h)-y^\lambda(t)\|_Y^2 + < G\begin{pmatrix} v^\lambda(t+h,0) \\ w^\lambda(t+h) \end{pmatrix} - G\begin{pmatrix} v^\lambda(t,0) \\ w^\lambda(t) \end{pmatrix}, \begin{pmatrix} v^\lambda(t+h,0) - v^\lambda(t,0) \\ w^\lambda(t+h) - w^\lambda(t) \end{pmatrix} >_{I\!R^{n+m}} - < B_1(t+h) - B_1(t), v^\lambda(t+h,0) - v^\lambda(t,0) >_{I\!R^n} \leq < f(t+h,\cdot) - f(t,\cdot), u^\lambda(t+h) - u^\lambda(t) >_{L^2(I\!R_+;I\!R^n)} + < g(t+h,\cdot) - g(t,\cdot), v^\lambda(t+h) - v^\lambda(t) >_{L^2(I\!R_+;I\!R^n)} + < B_2(t+h) - B_2(t), w^\lambda(t+h) - w^\lambda(t) >_{I\!R^m} .$$

By (A2)c the above inequality gives us

$$\frac{1}{2}\frac{d^+}{dt}\|y^\lambda(t+h)-y^\lambda(t)\|_Y^2 + \zeta_2\|v^\lambda(t+h,0) - v^\lambda(t,0)\|_{I\!R^n}^2 + \zeta_2\|w^\lambda(t+h) - w^\lambda(t)\|_{I\!R^m}^2 \leq \frac{1}{\zeta_0}\|B_1(t+h) - B_1(t)\|_{I\!R^n}^2 + \zeta_0\|v^\lambda(t+h,0) - v^\lambda(t,0)\|_{I\!R^n}^2 + \frac{1}{\zeta_0}\|B_2(t+h) - B_2(t)\|_{I\!R^m}^2 + \zeta_0\|w^\lambda(t+h) - w^\lambda(t)\|_{I\!R^m}^2 + \|F_0(t+h,\cdot) - F_0(t,\cdot)\|_X\|y^\lambda(t+h) - y^\lambda(t)\|_Y,$$
$$0 \leq t < t+h < T, \ \ \lambda > 0, \text{ where } F_0(t,\cdot) = \mathrm{col}(f(t,\cdot), g(t,\cdot)).$$

We choose $0 < \zeta_0 < \zeta_2$; then we obtain

$$\frac{1}{2}\frac{d^+}{dt}\|y^\lambda(t+h)-y^\lambda(t)\|_Y^2 \leq \frac{1}{\zeta_0}\|B(t+h) - B(t)\|_{I\!R^{n+m}}^2 + \|F_0(t+h,\cdot) - F_0(t,\cdot)\|_X\|y^\lambda(t+h) - y^\lambda(t)\|_Y, \quad 0 \leq t < t+h < T, \ \ \lambda > 0.$$

Integrating the above inequality over $[0,t]$ we deduce

$$\frac{1}{2}\|y^\lambda(t+h)-y^\lambda(t)\|_Y^2$$

$$\leq \frac{1}{2}\|y^\lambda(h) - y^\lambda(0)\|_Y^2 + \frac{1}{\zeta_0}\int_0^t \|B(s+h) - B(s)\|_{I\!R^{n+m}}^2 ds$$
$$+ \int_0^t \|F_0(s+h,\cdot) - F_0(s,\cdot)\|_X \|y^\lambda(s+h) - y^\lambda(s)\|_Y ds$$
$$\Rightarrow \frac{1}{2}\|y^\lambda(t+h) - y^\lambda(t)\|_Y^2 \leq \frac{1}{2}\Bigg(\|y^\lambda(h) - y^\lambda(0)\|_Y^2 + \frac{2}{\zeta_0}\int_0^T \|B(s+h)$$
$$- B(s)\|_{I\!R^{n+m}}^2 ds\Bigg) + \int_0^t \|F_0(s+h,\cdot) - F_0(s,\cdot)\|_X \|y^\lambda(s+h) - y^\lambda(s)\|_Y ds,$$
$$0 \leq t < t+h < T, \ \ \lambda > 0.$$

Using a variant of Gronwal's Lemma we obtain

$$\|y^\lambda(t+h) - y^\lambda(t)\|_Y \leq \|y^\lambda(h) - y^\lambda(0)\|_Y + \sqrt{\frac{2}{\zeta_0}}\Bigg(\int_0^T \|B(s+h)$$
$$- B(s)\|_{I\!R^{n+m}}^2 ds\Bigg)^{1/2} + \int_0^t \|F_0(s+h,\cdot) - F_0(s,\cdot)\|_X ds,$$
$$0 \leq t < t+h < T, \ \ \lambda > 0.$$

Therefore we get

$$\left\|\frac{d^+y^\lambda}{dt}(t)\right\|_Y \leq \left\|\frac{d^+y^\lambda}{dt}(0)\right\|_Y + \sqrt{\frac{2}{\zeta_0}}\left(\int_0^T \left\|\frac{dB}{ds}(s)\right\|_{I\!R^{n+m}}^2 ds\right)^{1/2}$$
$$+ \int_0^t \left\|\frac{df}{ds}(s,\cdot)\right\|_{L^2(I\!R_+;I\!R^n)} ds + \int_0^t \left\|\frac{dg}{ds}(s,\cdot)\right\|_{L^2(I\!R_+;I\!R^n)} ds \ \Rightarrow$$

$$\begin{gathered}\left\|\frac{d^+y^\lambda}{dt}(t)\right\|_Y \leq \left\|\frac{d^+y^\lambda}{dt}(0)\right\|_Y + \sqrt{\frac{2}{\zeta_0}}\left(\int_0^T \left\|\frac{dB}{ds}(s)\right\|_{I\!R^{n+m}}^2 ds\right)^{1/2} \\ + \int_0^T \left\|\frac{df}{ds}(s,\cdot)\right\|_{L^2(I\!R_+;I\!R^n)} ds + \int_0^T \left\|\frac{dg}{ds}(s,\cdot)\right\|_{L^2(I\!R_+;I\!R^n)} ds, \\ 0 \leq t < T, \ \ \lambda > 0.\end{gathered} \tag{12}$$

Because $\sup\left\{\left\|\frac{d^+y^\lambda}{dt}(0)\right\|_Y; \ \lambda > 0\right\} \leq$ const. (const. is a positive constant independent of λ), by the assumptions of the theorem, the inequality (12) gives us

$$\sup\left\{\left\|\frac{dy^\lambda}{dt}(t)\right\|_Y; \ \lambda > 0, \ 0 < t < T\right\} \leq \text{const.}$$

and then $\sup\{\|y^\lambda(t)\|_Y;\ \lambda>0,\ 0<t<T\}\le \text{const.}$

Therefore

$$\{u^\lambda;\ \lambda>0\},\ \{v^\lambda;\ \lambda>0\}\quad \text{are bounded in}\quad L^\infty(0,T;L^2(I\!R_+;I\!R^n)),$$

$$\{w^\lambda;\ \lambda>0\}\ \text{is bounded in}\ L^\infty(0,T;I\!R^m),$$

and using the assumption (A1)b we deduce that

$$\{\mathcal{B}_\lambda(y^\lambda(t));\ \lambda>0\}\ \text{is bounded in}\ L^\infty(0,T;Y). \tag{13}$$

By $(11)_1$ we have

$$\frac{1}{2}\frac{d}{dt}\|y^\lambda(t)-y^\mu(t)\|_Y^2\le -<\mathcal{B}_\lambda(y^\lambda(t))-\mathcal{B}_\mu(y^\mu(t)),y^\lambda(t)-y^\mu(t)>_Y,$$
$$0<t<T,\ \ \lambda>0. \tag{14}$$

Using now the relations (13) and (14) we get

$$\|y^\lambda(t)-y^\mu(t)\|_Y\le \text{const.}(\lambda+\mu)^{1/2},\ \ 0\le t\le T;\ \ \lambda,\ \mu>0.$$

Therefore we deduce that the sequence $\{y^\lambda;\ \lambda>0\}$ converges to some $y=\text{col}(u,v,w)$ in $C([0,T];Y)$ as $\lambda\to 0$.

By Lebesque's Dominated Convergence Theorem we obtain

$\mathcal{B}_\lambda(y^\lambda)\to\mathcal{B}(y)$, as $\lambda\to 0$, strongly in $L^2(0,T;Y)$.

By letting $\lambda\to 0$ in (11), $\mathcal{A}$ and G being demi-continuous operators, we obtain that y is a strong solution of the problem (S)+(BC)+(IC).

In the third stage (the general case) we approximate $f,\,g\in W^{1,1}(0,T;L^2(I\!R_+;I\!R^n))$ by $\{f^j\}_{j\ge1}$, $\{g^j\}_{j\ge1}\subset W^{1,\infty}(0,T;L^2(I\!R_+;I\!R^n))$ in the space $W^{1,1}(0,T;L^2(I\!R_+;I\!R^n))$, and $b_k\in W^{1,2}(0,T)$ by $\{b_k^j\}_{j\ge1}\subset W^{2,\infty}(0,T)$, $k=\overline{1,n}$, $b_i\in W^{1,2}(0,T)$ by $\{b_i^j\}_{j\ge1}\subset W^{1,\infty}(0,T)$, $i=\overline{n+1,n+m}$, in the space $W^{1,2}(0,T)$. Fixing $y_0=\text{col}(u_0,v_0,w_0)\in Y$ with $\widetilde{y}_0=\text{col}(\widetilde{u}_0,v_0,w_0)\in D(\mathcal{A})$ we deduce after some considerations (see also Luca et al [8], Luca [7]) that the sequence of the corresponding strong solutions $\{y^j=\text{col}(u^j,v^j,w^j)\}_{j\ge1}$ converges uniformly as $j\to\infty$ to $y=\text{col}(u,v,w)$ which is a strong solution of our problem. Then by (S) and (A2)b we obtain

$$u,\,v\in L^\infty(0,T;H^1(I\!R_+;I\!R^n)).\qquad\Box$$

Theorem 6. *Assume the assumptions* (A1)ab, (A2)ac *and* (A3) *hold. If* $f,\,g\in L^1(0,T;L^2(I\!R_+;I\!R^n))$ ($T>0$ *fixed*), $b_k\in L^2(0,T)$, $k=\overline{1,n+m}$,

$u_0,\, v_0 \in L^2(I\!\!R_+)$. $w_0 \in \overline{D(G_{12}) \cap D(G_{22})}$, *then the problem* (S)+(BC)+(IC) *has a unique weak solution* $y = \mathrm{col}(u, v, w) \in C([0, T]; Y)$.

Proof. By the assumptions of the theorem it follows that

$$y_0 = \mathrm{col}(u_0, v_0, w_0) \in \overline{D(\mathcal{A})}.$$

We consider $\{y_0^j\}_{j\geq 1} \subset Y$ such that $\widetilde{y}_0^j \in D(\mathcal{A})$ and $y_0^j \to y_0$, as $j \to \infty$, in Y. Also let the sequences $\{f^j\}_{j\geq 1}$, $\{g^j\}_{j\geq 1} \subset W^{1,1}(0, T; L^2(I\!\!R_+; I\!\!R^n))$ be such that $f^j \to f$, $g^j \to g$, as $j \to \infty$, in $L^1(0, T; L^2(I\!\!R_+; I\!\!R^n))$ and the sequences $\{b_k^j\}_{j\geq 1} \subset W^{1,2}(0, T)$ be such that $b_k^j \to b_k$, as $j \to \infty$, in $L^2(0, T)$, $k = \overline{1, n + m}$.

Then the corresponding strong solutions

$$y^j = \mathrm{col}(u^j, v^j, w^j) \in W^{1,\infty}(0, T; Y)$$

of the problem (S)+(BC)+(IC) satisfy the inequality

$$\|y^j(t) - y^l(t)\|_Y \leq \|y_0^j - y_0^l\|_Y + \sqrt{\frac{2}{\zeta_0}} \left(\int_0^T \|B^j(s) - B^l(s)\|^2_{I\!\!R^{n+m}} ds\right)^{1/2}$$
$$+ \int_0^t \|F_0^j(s, \cdot) - F_0^l(s, \cdot)\|^2_X ds, \;\; 0 \leq t \leq T, \;\; \forall\, j, l \in I\!\!N,$$

where $F_0^j = \mathrm{col}(f^j, g^j)$, $j \geq 1$, which leads us to the conclusion. □

Theorem 7. *Assume that* (A1)abc, (A2)ac *and* (A3) *hold. If* $f, g \in L^1_{loc}(I\!\!R_+; L^2(I\!\!R_+;\, I\!\!R^n))$, $b_k \in L^2(I\!\!R_+)$, $k = \overline{1, n + m}$ *such that* $\lim\limits_{t\to\infty} f(t) = f^0$, $\lim\limits_{t\to\infty} g(t) = g^0$, *strongly in* $L^2(I\!\!R_+; I\!\!R^n)$ *and* $\gamma = \mathrm{col}(p, q, r)$ *is the unique solution of the equation* (7). *Then* $\lim\limits_{t\to\infty} y(t) = \gamma$, *strongly in* Y, *where* $y(t) = \mathrm{col}(u(t), v(t), w(t))$, $t \geq 0$ *is an arbitrary weak solution of the equation* $(\widetilde{\mathrm{P}})_1$.

Proof. By Theorem 3 the operator $\mathcal{A} + \mathcal{B}$ is strongly monotone and the equation (7) has a unique solution $\gamma = \mathrm{col}(p, q, r) \in D(\mathcal{A})$. We define for any $l \in I\!\!N$ the function

$$B^l(t) = \begin{cases} B(t), & \text{for } 0 \leq t \leq l\,, \\ 0, & \text{for } t > l. \end{cases}$$

Let $y_0 = \mathrm{col}(u_0, v_0, w_0) \in \overline{D(\mathcal{A})}$; we denote by $y(t)$, $y^l(t)$, $t \geq 0$ the weak solutions of the problem (S)+(BC)+(IC) corresponding to data $\{B,\, f,\, g,\, y_0\}$, respectively $\{B^l,\, f,\, g,\, y_0\}$. Then we have

$$\|y^l(t) - y(t)\|_Y \leq \mathrm{const.} \left(\int_l^\infty \|B(s)\|^2_{I\!\!R^{n+m}} ds\right)^{1/2}, \;\; t > l.$$

Because for $t > l$, y^l is the weak solution corresponding to $B(t) \equiv 0$, by Theorem 4 we deduce that

$$y^l(t) \to \gamma, \text{ as } t \to \infty, \text{ in } Y \ (l \in I\!N). \tag{15}$$

Therefore this last conclusion with (15) and the inequality

$$\|y(t) - \gamma\|_Y \le \|y(t) - y^l(t)\| + \|y^l(t) - \gamma\|_Y$$

give us that $y(t) \to \gamma$, as $t \to \infty$, in Y. □

4. Some Considerations in the Case $x \in I\!R$

If the spatial variable x belongs to $I\!R$, then from the boundary condition (BC) it only remains

$$S(w'(t)) \in -G_{22}(w(t)) + B_2(t) \quad \Leftrightarrow \quad w'(t) \in -S^{-1}G_{22}(w(t)) + S^{-1}B_2(t).$$

This equation with the initial date $w(0) = w_0$ give by integration the function $w(t)$.

For u and v we obtain the problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) + \dfrac{\partial v}{\partial x}(t,x) + \alpha(x,u) = f(t,x)\,, \\ \dfrac{\partial v}{\partial t}(t,x) + \dfrac{\partial u}{\partial x}(t,x) + \beta(x,v) = g(t,x), \end{cases} \quad x \in I\!R, \ t > 0\,, \tag{$\overline{\text{S}}$}$$

with the initial data

$$u(0,x) = u_0(x), \ v(0,x) = v_0(x), \ x \in I\!R, \tag{$\overline{\text{IC}}$}$$

under the assumptions $(\widetilde{\text{A1}})$abc which are (A1)abc with $I\!R$ instead of $I\!R_+$.

We consider here the space $Z = (L^2(I\!R; I\!R^n))^2$ with the standard scalar product and the operators

$$\mathcal{C} : D(\mathcal{C}) \subset Z \to Z, \ D(\mathcal{C}) = (H^1(I\!R; I\!R^n))^2, \ \mathcal{C}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v' \\ u' \end{pmatrix},$$

$$\mathcal{D} : D(\mathcal{D}) \subset Z \to Z, \ \mathcal{D}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \alpha(\cdot,u) \\ \beta(\cdot,v) \end{pmatrix}.$$

Lemma 3. *The operator $\mathcal{C}$ is maximal monotone in Z.*

For the proof of Lemma 3 see Moroşanu [10].

Lemma 4. *If the assumptions* $(\widetilde{\text{A1}})$ab *hold, then the operator* $\mathcal{D}$ *is everywhere defined and maximal monotone.*

The proof of Lemma 4 is similar to that of Lemma 2 with $\mathbb{R}$ instead of $\mathbb{R}_+$.

Now using the operators $\mathcal{C}$ and $\mathcal{D}$ our problem $(\overline{\text{S}})$+$(\overline{\text{IC}})$ can be written as

$$\begin{cases} \dfrac{dz}{dt}(t) + \mathcal{C}(z(t)) + \mathcal{D}(z(t)) = \overline{F}(t,\cdot), \ \ t > 0, \ \text{ in } \ Z\,, \\ z(0) = z_0, \end{cases} \tag{$\overline{\text{P}}$}$$

where $z(t) = \text{col}(u(t), v(t))$, $z_0 = \text{col}(u_0, v_0)$.

Using similar arguments as in the case $x \in \mathbb{R}_+$ we obtain for the problem $(\overline{\text{P}})$ the results

Theorem 8. a) *Assume that* $(\widetilde{\text{A1}})$ab *hold. If* f, $g \in W^{1,1}(0,T; L^2(\mathbb{R}; \mathbb{R}^n))$ *(*$T > 0$ *fixed),* u_0, $v_0 \in H^1(\mathbb{R}; \mathbb{R}^n)$, *then the problem* $(\overline{\text{P}}) \Leftrightarrow (\overline{\text{S}}) + (\overline{\text{IC}})$ *has a unique strong solution* $z = \text{col}(u,v) \in W^{1,\infty}(0,T;Z)$. *Moreover* u, $v \in L^\infty(0,T;H^1(\mathbb{R};\mathbb{R}^n))$.

b) *Assume that* $(\widetilde{\text{A1}})$ab *hold. If* f, $g \in L^1(0,T;L^2(\mathbb{R};\mathbb{R}^n))$ *(*$T > 0$ *fixed), then the problem* $(\overline{\text{S}})$+$(\overline{\text{IC}})$ *has a unique weak solution* $z = \text{col}(u,v) \in C([0,T];Z)$.

Theorem 9. *Assume that* $(\widetilde{\text{A1}})$abc *hold. Then the operator* $\mathcal{C} + \mathcal{D}$ *is strongly monotone and the equation*

$$\mathcal{C}(z) + \mathcal{D}(z) \ni 0 \tag{16}$$

has a unique solution $z = \text{col}(u,v) \in Z$.

Theorem 10. *Assume that* $(\widetilde{\text{A1}})$abc *hold and* f, $g \in L^1_{loc}(\mathbb{R}_+; L^2(\mathbb{R}; \mathbb{R}^n))$ *verify the conditions* $\lim_{t\to\infty} f(t) = f^0$, $\lim_{t\to\infty} g(t) = g^0$, *strongly in* $L^2(\mathbb{R}; \mathbb{R}^n)$ *and* $\overline{\gamma}$ *is the unique solution of the equation* (16). *Then* $\lim_{t\to\infty} z(t) = \overline{\gamma}$, *strongly in* Z, *where* $z(t) = \text{col}(u(t), v(t))$, $t \geq 0$ *is an arbitrary weak solution of the equation* $(\overline{\text{P}})_1$.

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