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DIFFUSION PROCESSES HAVING GENERALIZED PARETO PROBABILITY DENSITY FUNCTIONS

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Abstract: Let $X(t)$ be a one-dimensional time-inhomogeneous diffusion process with infinitesimal mean $m(x,t)$ and variance $v(x,t)$. The initial state $X(0)$ of the process is random. We consider the Kolmogorov forward equation satisfied by the probability density function $f(x,t)$ of $X(t)$. We find particular cases for which f is the density function of a random variable having a generalized Pareto distribution (depending on t). We consider different possibilities for the infinitesimal parameters of the diffusion process, in particular the case when $m(x,t)$ is a constant and that when $v(x,t) \equiv v_0 > 0$. The solutions obtained correspond to the case when there is a reflecting boundary at the origin or at $-t$.

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1. Introduction

As is well known, the most important diffusion process, namely the Wiener process or Brownian motion $W(t)$, is a Gaussian process. In particular, we have:

$$W(t) \mid W(0) \sim \mathrm{N}(W(0) + \mu t, \sigma^2 t)\,, \tag{1}$$

where the drift μ is equal to 0 and the diffusion parameter σ^2 is equal to 1 in

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the case of the *standard* Brownian motion. Similarly, the Ornstein-Uhlenbeck process is a Gaussian process. In mathematical finance, the basic diffusion process is the geometric Brownian motion, defined by

$$X(t) = e^{W(t)}, \tag{2}$$

where $W(t)$ is a Wiener process. It follows that $X(t)$ has a lognormal density function. Finally, another process of interest is the Bessel process, whose transition density function involves a Bessel function (see Karlin and Taylor [3, p. 238]).

Now, a subject that has become quite popular in recent years is that of extreme value statistics (see Coles [1], for instance). Among the distributions used to model various quantities, such as the maximum and minimum temperatures, the generalized Pareto distribution is of special importance. Remember that we say that the random variable X has a generalized Pareto distribution (GPD) if

$$P[X \in (x, x+dx)] = \frac{\alpha}{\beta}\left(1 + \frac{(x-\gamma)}{\beta}\right)^{-\alpha-1} dx\,, \tag{3}$$

for $x \geq \gamma$, where we assume that α and β are positive parameters. We will denote this distribution by GPD(α, β, γ). In this note, we would like to find diffusion processes having generalized Pareto probability density functions. To do so, we must introduce versions of the density function above which depend on the time variable t. In Section 2, two particular t-dependent GPDs that lead to interesting processes will be presented. We will find the infinitesimal parameters of the stochastic processes $X(t)$ which possess such t-dependent GPDs. The paper will end with a few concluding remarks in Section 3.

2. Explicit Results

Let $X(t)$ be a one-dimensional diffusion process with infinitesimal mean $m(x,t)$ and variance $v(x,t)$, whose first-order density function $f(x,t)$ satisfies the Kolmogorov forward (or Fokker-Planck) equation

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}\{v(x,t)f(x,t)\} - \frac{\partial}{\partial x}\{m(x,t)f(x,t)\} = \frac{\partial}{\partial t}f(x,t)\,. \tag{4}$$

Remark. The functions $m(x,t)$ and $v(x,t)$ must be such that, for all $s \geq 0$, we have (see Lamberton and Lapeyre [4, p. 58], for instance):

$$\int_0^s |m(x,t)|\,dt < \infty \quad \text{and} \quad \int_0^s v(x,t)\,dt < \infty \tag{5}$$

The function $f(x,t)$ is defined by

$$f(x,t) = P[X(t) \in (x, x+dx)]/dx \tag{6}$$

and can also be expressed as follows:

$$f(x,t) = \int_{-\infty}^{\infty} p(x,t;x_0,t_0) f(x_0,t_0)\, dx_0 \tag{7}$$

for $t > t_0$, where

$$p(x,t;x_0,t_0) := P[X(t) \in (x, x+dx) \mid X(t_0) = x_0]/dx \tag{8}$$

is the transition density function of the stochastic process $X(t)$.

We want to find particular functions $m(x,t)$ and $v(x,t)$ for which $f(x,t)$ is the probability density function of a random variable having a GPD (depending on t).

2.1. First Particular Case

First, we consider the case when

$$f(x,t) = \frac{\alpha}{\beta+t}\left(1 + \frac{x}{\beta+t}\right)^{-\alpha-1} \tag{9}$$

for α, β, x and $t > 0$. That is, $X(t) \sim \text{GPD}(\alpha, \beta+t, 0)$.

Note that the initial state $X(0)$ of the process must be random and have a $\text{GPD}(\alpha, \beta, 0)$:

$$f(x,0) := \frac{P[X(0) \in (x, x+dx)]}{dx} = \frac{\alpha}{\beta}\left(1 + \frac{x}{\beta}\right)^{-\alpha-1}. \tag{10}$$

We substitute the function $f(x,t)$ above into the Kolmogorov equation (4) and we consider various possibilities.

Case Ia. First, we would like $X(t)$ to be, in fact, a time-homogeneous diffusion process. Unfortunately, there are no such processes for which $f(x,t)$ is given by the function in (9), as we will show.

Lemma 2.1. *If the infinitesimal mean and variance of $X(t)$ do not depend on t, then equation (4) has no solution of the form given in (9).*

Proof. If $m(x,t) = m(x)$ and $v(x,t) = v(x)$, where $v(x) > 0$ $\forall x > 0$, then equation (4) may be written as

$$\frac{v(x)}{2}(\alpha+1)(\alpha+2)(\beta+t+x)^{-2} + (\alpha+1)(\beta+t+x)^{-1}[m(x) - v_x(x)]$$

$$+\frac{v_{xx}(x)}{2} - m_x(x) = \alpha(\beta+t)^{-1} - (\alpha+1)(\beta+x+t)^{-1}\,. \tag{11}$$

There are indeed no functions $m(x)$ and $v(x)$ for which this equation is satisfied. □

Case Ib. Suppose next that the functions $m(x,t)$ and $v(x,t)$ do not depend on the variable x. That is, $m(x,t) = m(t)$ and $v(x,t) = v(t)$ (> 0). We find that again equation (4) has no solution. Therefore, the infinitesimal mean and/or the infinitesimal variance of $X(t)$ must depend on x.

Remark. Note that this case includes the one when $m(x,t)$ and $v(x,t)$ are both constants. Therefore, $X(t)$ cannot be a Wiener process.

Case Ic. Assume now that $v(x,t) = v(t) > 0$, but that $m(x,t)$ is general. Then (4) becomes

$$m_x - \frac{\alpha+1}{\beta+x+t}m = \frac{v}{2}\frac{(\alpha+1)(\alpha+2)}{(\beta+x+t)^2} + \frac{1}{\beta+t} - \frac{(\alpha+1)x}{(\beta+t)(\beta+x+t)}\,. \tag{12}$$

We can find the general solution of this first-order linear ordinary differential equation. This general solution involves of course an arbitrary constant. Choosing this constant to be zero, we obtain the following particular solution:

$$m(x,t) = \frac{x}{\beta+t} - \frac{(\alpha+1)v(t)}{2(\beta+x+t)} \tag{13}$$

which is relatively simple.

Case Id. Finally, if $m(x,t) = m(t)$ and $v(x,t)$ is general, then we must solve

$$\begin{aligned} v_{xx} - \frac{2(\alpha+1)}{\beta+x+t}v_x + \frac{(\alpha+1)(\alpha+2)}{(\beta+x+t)^2}v \\ = -\frac{2}{\beta+t} + \frac{2(\alpha+1)}{\beta+x+t}\left(\frac{x}{\beta+t} - m\right)\,. \end{aligned} \tag{14}$$

A particular solution of this second-order linear o.d.e. is (if $\alpha \neq 1$)

$$v(x,t) = \frac{2}{(\alpha-1)\alpha}(\beta+x+t)\left(1 - (\alpha-1)m(t) + \frac{\alpha x}{\beta+t}\right)\,, \tag{15}$$

where the function $m(t)$ must (at least) be such that $v(x,t) > 0$.

Remark. We could also try to find particular solutions of equation (4) in the case when both m and v depend explicitly on x and t. Such an example will be presented for a different function $f(x,t)$ in the next subsection.

Summing up, we may state the following proposition.

Proposition 2.1. *Suppose that the function $v(t)$ in (13) is such that the conditions in (5) are satisfied. Similarly, suppose that the function $m(t)$ in (15) is also such that the conditions in (5) are satisfied (and $v(x,t)$ is strictly positive). Then, in both cases, given that $X(0) \sim GPD(\alpha, \beta, 0)$, we may write that $X(t) \sim GPD(\alpha, \beta + t, 0)$.*

Remark. Notice that we have:

$$\int_0^\infty f(x,t)\,dx = \int_0^\infty \frac{\alpha}{\beta+t}\left(1+\frac{x}{\beta+t}\right)^{-\alpha-1} dx = 1\,. \tag{16}$$

Actually, we can show that the function defined in (9) is the probability density function of the diffusion processes $X(t)$ with the infinitesimal parameters given in Case Ic and Case Id *if* 0 is a reflecting boundary. Indeed, we can check that

$$\left[\frac{1}{2}\frac{\partial}{\partial x}\left\{v(x,t)f(x,t)\right\} - m(x,t)f(x,t)\right]_{x=0} = 0 \tag{17}$$

which is the condition for a reflecting boundary at the origin (see Cox and Miller [2, p. 223]).

2.2. Second Particular Case

Next, we assume that the function $f(x,t)$ is of the form

$$f(x,t) = \frac{\alpha}{\beta}\left(1+\frac{(x+t)}{\beta}\right)^{-\alpha-1}, \tag{18}$$

so that $X(t) \sim \text{GPD}(\alpha, \beta, -t)$. To be more general, we could actually replace t by ct in the function above, where $c \neq 0$ is constant. Moreover, as in the previous particular case, we may write that $X(0) \sim \text{GPD}(\alpha, \beta, 0)$.

Since

$$P[X(t) > 0] = \left(1+\frac{t}{\beta}\right)^{-\alpha} \tag{19}$$

we find that the conditional density function of $X(t)$, given that $X(t) > 0$, is

$$f_{X(t)}(x \mid X(t) > 0) = \frac{\alpha}{\beta}\left(1+\frac{t}{\beta}\right)^{\alpha}\left(1+\frac{(x+t)}{\beta}\right)^{-\alpha-1} \tag{20}$$

for α, β, x and $t > 0$. We may say that $X(t) \mid \{X(t) > 0\}$ has a (more) generalized Pareto distribution.

As previously, we find that $X(t)$ cannot be a time-homogeneous diffusion process and that we cannot have $m(x,t) = m(t)$ and $v(x,t) = v(t)$ at the same time.

Case IIa. If $v(x,t) = v(t) > 0$ and $m(x,t)$ is general, then (4) becomes

$$m_x - \frac{\alpha+1}{\beta+x+t} m = \frac{v}{2} \frac{(\alpha+1)(\alpha+2)}{(\beta+x+t)^2} + \frac{\alpha+1}{\beta+x+t} \tag{21}$$

and has the following particular solution:

$$m(x,t) = -\frac{v(t)(\alpha+1)}{2(\beta+x+t)} - 1\,. \tag{22}$$

In the special case when $v(t)$ is actually a (positive) constant, this formula for the infinitesimal mean of $X(t)$ is quite simple.

Case IIb. If $m(x,t) = m(t)$ and $v(x,t)$ is general, we obtain the o.d.e.

$$v_{xx} - \frac{2(\alpha+1)}{\beta+x+t} v_x + \frac{(\alpha+1)(\alpha+2)}{(\beta+x+t)^2} v = \frac{-2(\alpha+1)(m+1)}{\beta+x+t} \tag{23}$$

which has the particular solution (if $m(t) < -1$ for all $t > 0$)

$$v(x,t) = \frac{-2[m(t)+1]}{\alpha}(\beta+x+t)\,. \tag{24}$$

This time, if $m(t)$ is a constant, then the infinitesimal variance of $X(t)$ is just an affine function of x (and t).

Case IIc. Finally, we present an interesting case in which both m and v depend on x and t. Let $v(x,t) = k(\beta+x+t)^2$ and $m(x,t) = -(\beta+x+t) - 1$. Then (4) has a solution if and only if

$$k = \frac{2}{\alpha-1}\,, \tag{25}$$

where α must be greater than 1.

Remark. Here, we have:

$$\int_{-t}^{\infty} f(x,t)\,dx = 1\,. \tag{26}$$

We can show that the function in (18) corresponds to the case when there is now a reflecting boundary at $-t$ for the diffusion processes $X(t)$ having the infinitesimal parameters given in Cases IIa-c. The condition that must be satisfied is the following:

$$0 = \frac{\partial}{\partial t} \int_{-t}^{\infty} f(x,t)\,dx = \int_{-t}^{\infty} \frac{\partial}{\partial t} f(x,t)\,dx + f(-t,t)\,. \tag{27}$$

Because we have (in this particular case):

$$\begin{aligned} \int_{-t}^{\infty} \frac{\partial}{\partial t} f(x,t)\,dx &= \int_{-t}^{\infty} \frac{\partial}{\partial x} f(x,t)\,dx \\ &= \lim_{x\to\infty} f(x,t) - f(-t,t) = 0 - f(-t,t) \end{aligned} \tag{28}$$

the condition trivially holds true.

3. Conclusion

In this note, we have constructed various diffusion processes for which the random variable $X(t)$ (with t fixed) has a generalized Pareto distribution depending on t. We have considered two particular possibilities for this t-dependent GPD. There are of course other cases that could be considered; for example, the probability density function could be

$$f(x,t) = \frac{\alpha(t+1)}{\beta}\left(1 + \frac{x(t+1)}{\beta}\right)^{-\alpha-1} \quad \text{for } \alpha,\ \beta,\ t \text{ and } x > 0\,, \tag{29}$$

or

$$f(x,t) = \frac{\alpha}{\beta(t+1)}\left(1 + \frac{x}{\beta(t+1)}\right)^{-\alpha-1} \quad \text{for } \alpha,\ \beta,\ t \text{ and } x > 0\,. \tag{30}$$

Note that in these two cases the initial state $X(0)$ of the process still has a generalized Pareto distribution, whereas in the case when

$$f(x,t) = \frac{\alpha}{\beta t}\left(1 + \frac{(x-x_0)}{\beta t}\right)^{-\alpha-1} \quad \text{for } \alpha,\ \beta,\ t > 0 \text{ and } x > x_0$$

we find that

$$\lim_{t\downarrow 0} f(x,t) = \delta(x - x_0)\,, \tag{31}$$

where $\delta(\cdot)$ is the Dirac delta function. That is, we must have $X(0) = x_0$, so that the initial state is deterministic. However, proceeding as above to find particular solutions of the Kolmogorov forward equation by assuming that $m(x,t) = m(t)$ or $v(x,t) = v(t)$, we obtain functions that do not satisfy the conditions (5). It would therefore be interesting to find (if possible) functions m and v depending on x and t such that these conditions are satisfied.

Finally, we could try to construct diffusion processes with probability density functions of other extreme value distributions.

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References

[1] S. Coles, *An Introduction to Statistical Modeling of Extreme Values*, London, Springer (2001).

[2] D.R. Cox, H.D. Miller, *The Theory of Stochastic Processes*, Methuen, London (1965).

[3] S. Karlin, H. Taylor, *A Second Course in Stochastic Processes*, Academic Press, New York (1981).

[4] D. Lamberton, B. Lapeyre, *Introduction au Calcul Stochastique Appliqué à la Finance*, Second Edition, Ellipses, Paris (1997).