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ON HOMEOMORPHIC SOLUTIONS OF THE SCHRÖDER EQUATION IN BANACH SPACES

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Abstract: The problem of the existence and uniqueness of continuous and homeomorphic solutions of the Schröder equation defined on Banach spaces is examined.

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1. Introduction

The Schröder equation

$$\varphi(f(x)) = A\varphi(x), \tag{1}$$

one of the fundamental equations of linearization, plays an important role in many branches of mathematics, among others in the theory of dynamical systems. We consider equation (1) in which A is a given bounded linear operator. If A is a hyperbolic operator, then Grobman [2] and Hartman [3]-[6] proved that (1) has exactly one local homeomorphic solution (see also [8; Theorem 2.2]). In this paper we establish conditions, different from that of Grobman and Hartman, under which there exists a homeomorphism φ which solves the Schröder equation. Regarding local smooth solutions see [9; Corollary 1.1] (cf. also [1; Theorem 8.2] and [7; Theorem 8.2.2] in finite dimensional case).

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2. Preliminaries

Let $(X, \|\cdot\|)$ be a Banach space, which is a direct sum of closed linear subspaces $X_1, ..., X_N$ and let $K \subset X$. We will denote by P_i the projection of X onto the space X_i, i.e. $P_i(x) = x_i$ for $x = (x_1, ..., x_N)$, where $x_j \in X_j$, $j = 1, ..., N$. To simplify notations we write φ_i instead of $P_i \circ \varphi$ for any function φ with values in X. Let $\mathcal{B}(X)$ be the set of all bounded linear operators from X to X. By $\mathcal{P}(K)$ we denote the set of all continuous functions $\varphi\colon K \to X$ such that $Id - \varphi$ is bounded and by $\mathcal{P}_i(K)$ we denote the set of all continuous functions $\varphi\colon K \to X_i$ such that $P_i|_K - \varphi$ is bounded. Note that the space $\mathcal{P}_i(K)$ with the metric d_i given by

$$d_i(\varphi, \psi) := \sup_{x \in K} \|\varphi(x) - \psi(x)\|$$

is complete for $i = 1, ..., N$.

We assume that the following conditions hold:

(H1) $A \in \mathcal{B}(X)$ is a bijection such that $A(K) = K$ and

$$A(X_i) = X_i \text{ for } i = 1, ..., N; \tag{2}$$

$$\|A\,|_{X_i}\| < 1 \text{ or } \|A\,|_{X_i}{}^{-1}\| < 1 \text{ for } i = 1, ..., N;$$

(H2) f is a homeomorphism from K onto K and $f - A$ is bounded.

The condition (H1) is in fact (under suitable change of a norm) equivalent to hyperbolicity of operator A. It follows from (2) that

$$(P_i \circ A)x_i = (P_i \circ A)x = Ax_i \text{ for } x = (x_1, ..., x_N) \in X, \ i = 1, ..., N. \tag{3}$$

First we prove that equation (1) has a unique solution in the class $\mathcal{P}(K)$.

Theorem 1. *Under the assumptions* (H1) *and* (H2) *equation* (1) *has a unique solution* $\varphi \in \mathcal{P}(K)$. *This solution* φ *is given by*

$$\varphi = \sum_{i=1}^{N} \varphi_i,$$

where

$$\varphi_i(x) = \begin{cases} \lim_{n\to\infty} A^n P_i(f^{-n}(x)), & \text{if } \|A\,|_{X_i}\| < 1, \\ \lim_{n\to\infty} A^{-n} P_i(f^n(x)), & \text{if } \|A\,|_{X_i}^{-1}\| < 1 \end{cases}$$

for $x \in K$.

Proof. Fix $i \in \{1, ..., N\}$ and consider the operator $\Phi_i : \mathcal{P}_i(K) \to \mathcal{P}_i(K)$ defined by

$$\Phi_i(\varphi) := \begin{cases} A \circ \varphi \circ f^{-1}, & \text{if } \|A|_{X_i}\| < 1, \\ A^{-1} \circ \varphi \circ f, & \text{if } \|A|_{X_i}{}^{-1}\| < 1. \end{cases}$$

It is easy to check that $\operatorname{Lip}(\Phi_i) < 1$. By the Banach Fixed-Point Theorem the operator Φ_i has exactly one fixed point $\varphi_i \in \mathcal{P}_i(K)$, i.e.

$$\varphi_i \circ f = A \circ \varphi_i.$$

Then the function φ given by the formula $\varphi = \sum_{i=1}^{N} \varphi_i$ is a solution of (1) and belongs to $\mathcal{P}(K)$.

Passing to the proof of uniqueness, fix a solution $\psi \in \mathcal{P}(K)$ of (1). For $i = 1, ..., N$ the function ψ_i is a solution of (1) in the class $\mathcal{P}_i(K)$. By the uniqueness in $\mathcal{P}_i(K)$ already proved, we obtain $\psi_i = \varphi_i$ for $i = 1, ..., N$ and then $\psi = \varphi$. □

Now we are interested in solving the equation

$$f(\varphi(x)) = \varphi(g(x)). \tag{4}$$

Let p be a norm in $\mathbb{R}^N$ which is increasing on $[0, \infty)^N$ with respect to each variable and such that

$$p(x) \geq |x_i| \text{ for } i = 1, ..., N, \quad x = (x_1, ..., x_N) \in \mathbb{R}^N. \tag{5}$$

Put

$$\|x\|_{\oplus} := p(\|x_1\|, ..., \|x_N\|)$$

for $x = (x_1, ..., x_N)$ and assume that $c > 0$ is a constant such that

$$\|x\| \leq c\|x\|_{\oplus} \text{ for } x \in X. \tag{6}$$

The space $\mathcal{P}(K)$ with the metric σ given by

$$\sigma(\varphi, \psi) := p(d_1(\varphi_1, \psi_1), ..., d_N(\varphi_N, \psi_N))$$

is complete. Moreover, the following inequalities hold:

$$d_i(\varphi_i, \psi_i) \leq \sigma(\varphi, \psi) \tag{7}$$

for φ, $\psi \in \mathcal{P}(K)$, $i = 1, ..., N$ and

$$d(\varphi, \psi) \leq c\sigma(\varphi, \psi) \tag{8}$$

for φ, $\psi \in \mathcal{P}(K)$, where

$$d(\varphi, \psi) := \sup_{x \in K} \|\varphi(x) - \psi(x)\|.$$

Inequality (7) follows immediately from the definition of σ and from (5). To prove (8) fix φ, $\psi \in \mathcal{P}(K)$. Taking into account (6) and the monotonicity of p we have

$$\begin{aligned}
\|\varphi(x) - \psi(x)\| &\leq c\|\varphi(x) - \psi(x)\|_{\oplus} \\
&= cp(\|\varphi_1(x) - \psi_1(x)\|, ..., \|\varphi_N(x) - \psi_N(x)\|) \\
&\leq cp(d_1(\varphi_1, \psi_1), ..., d_N(\varphi_N, \psi_N)) = c\sigma(\varphi, \psi)
\end{aligned}$$

for $x \in K$, whence we get (8).

We pass to solutions of (4) in $\mathcal{P}(K)$. To obtain the next result we assume:

(H3) $A \in \mathcal{B}(X)$ is bijection such that $A(K) = K$ and A satisfies (2);

(H4) constants $l_1, ..., l_N$ given by

$$l_i := \begin{cases} c \operatorname{Lip}(P_i \circ f), & \text{if } \ i \in I_1, \\ \|A\,|_{X_i}{}^{-1}\|(1 + c \operatorname{Lip}(P_i \circ (f - A))), & \text{if } \ i \in I_2 \end{cases}$$

for $i = 1, ..., N$, where $I_1 \cap I_2 = \emptyset$ and $I_1 \cup I_2 = \{1, ..., N\}$, satisfy

$$p(l_1, ..., l_N) < 1;$$

(H5) g is a homeomorphism from K onto K and $g - A$ is bounded.

Theorem 2. *Under the assumptions* (H2)-(H5) *equation* (4) *has a unique solution* $\varphi \in \mathcal{P}(K)$.

Proof. Let $F := f - A$ and define $\Phi_i \colon \mathcal{P}(K) \to \mathcal{P}_i(K)$ by

$$\Phi_i(\varphi) := \begin{cases} f_i \circ \varphi \circ g^{-1}, & \text{if } \ i \in I_1, \\ A_i^{-1} \circ P_i \circ (\varphi \circ g - F \circ \varphi), & \text{if } \ i \in I_2 \end{cases}$$

for $i = 1, ..., N$. We consider the operator $\Phi \colon \mathcal{P}(X) \to \mathcal{P}(X)$ given by

$$\Phi(\varphi) = \sum_{i=1}^{N} \Phi_i(\varphi)$$

and claim that the function $\varphi \in \mathcal{P}(X)$ is a fixed point of Φ if and only if φ is a solution of equation (1). Indeed, if φ is a fixed point of Φ, then

$$\varphi_i = \Phi(\varphi)_i \ \text{ for } \ i = 1, ..., N.$$

If $i \in I_1$, then

$$f_i \circ \varphi = \varphi_i \circ g. \tag{9}$$

In the second case we have

$$\varphi_i = A_i^{-1} \circ P_i \circ (\varphi \circ g - F \circ \varphi),$$

and then

$$A_i \circ \varphi_i + F_i \circ \varphi = \varphi_i \circ g.$$

According to (3) the function φ is also a solution of (9). Adding (9) for $i = 1, ..., N$ we see that φ satisfies (4).

We will prove that Φ satisfies Lipschitz condition with the constant less than one. Fix $i \in I_1$. Using (8) we have

$$\begin{aligned} d(\Phi(\varphi)_i, \Phi(\psi)_i) \quad &= d(\Phi_i(\varphi), \Phi_i(\psi)) \\ &= \sup_{x\in K} \|f_i \circ \varphi \circ g^{-1}(x) - f_i \circ \psi \circ g^{-1}(x)\| \\ &\leq \mathrm{Lip}\,(f_i) \sup_{x\in K} \|\varphi(x) - \psi(x)\| \\ &= \mathrm{Lip}\,(f_i) d(\varphi, \psi) \leq l_i \sigma(\varphi, \psi) \end{aligned}$$

for $\varphi, \psi \in \mathcal{P}(K)$. If $i \in I_2$ then, by (7) and (8), we obtain the same conclusion as follows

$$\begin{aligned} &d(\Phi(\varphi)_i, \Phi(\psi)_i) \\ &\quad \leq \|A\mid_{X_i}{}^{-1}\| \sup_{x\in K} \|(\varphi_i \circ g - F_i \circ \varphi) - (\psi_i \circ g - F_i \circ \psi)\| \\ &\quad \leq \|A\mid_{X_i}{}^{-1}\| \sup_{x\in K} (\|(\varphi_i \circ g - \psi_i \circ g\| + \|F_i \circ \varphi - F_i \circ \psi)\|) \\ &\quad = \|A\mid_{X_i}{}^{-1}\| (d_i(\varphi_i, \psi_i) + \mathrm{Lip}\,(F_i) d(\varphi, \psi)) \\ &\quad \leq (\|A\mid_{X_i}{}^{-1}\| (1 + c\,\mathrm{Lip}\,(F_i))\sigma(\varphi, \psi) = l_i \sigma(\varphi, \psi) \end{aligned}$$

for φ and $\psi \in \mathcal{P}(X)$. Finally

$$\sigma(\Phi(\varphi), \Phi(\psi)) \leq p(l_1, ..., l_N)\sigma(\varphi, \psi)$$

for $\varphi, \psi \in \mathcal{P}(K)$. By Banach's Theorem the operator Φ has exactly one fixed point. □

3. Main Result

The following result may be proved in the same way as [8, Theorem 2.1].

Theorem 3. *Under the assumptions* (H1), (H2) *and* (H4) *with* $I_1 = \{i : \|A\,|_{X_i}\,\| < 1\}$ *and* $I_2 = \{i : \ \|A\,|_{X_i}\,^{-1}\| < 1\}$, *equation* (1) *has a unique solution* $\varphi \in \mathcal{P}(K)$. *Moreover,* φ *is a homeomorphism.*

Proof. The first part follows from Theorem 1. Let $\varphi \in \mathcal{P}(K)$ be a solution of (1). Applying Theorem 2, let $\psi \in \mathcal{P}(K)$ be a solution of equation (4) with $g = A$, i.e.

$$f \circ \psi = \psi \circ A.$$

Then $\varphi \circ \psi$ and $\psi \circ \varphi$ belong to the class $\mathcal{P}(K)$. In particular they are continuous. Moreover,

$$(\varphi \circ \psi) \circ A = A \circ (\varphi \circ \psi),$$

and

$$(\psi \circ \varphi) \circ f = f \circ (\psi \circ \varphi),$$

i.e. $\phi \circ \psi$ is a solution of

$$\alpha \circ A = A \circ \alpha$$

and $\psi \circ \varphi$ solves

$$\alpha \circ f = f \circ \alpha.$$

On the other hand the identity is a solution of each of these equations. Hence and from the uniqueness part of Theorem 1 and Theorem 2 we get

$$\varphi \circ \psi = \psi \circ \varphi = id,$$

which shows that φ is a homeomorphism. □

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