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FUNCTIONS COMMUTING WITH ITERATION GROUPS

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Abstract: We prove that if $\mathcal{F} = \{f^t : I \longrightarrow I,\ t \in \mathbb{R}\}$ is a suitable iteration group on an open interval I and a continuous at least at one point function $g : I \longrightarrow I$ commutes with two mappings f^a, $f^b \in \mathcal{F}$ such that $\frac{b}{a}$ is irrational, then $g \in \mathcal{F}$.

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Let I be an open interval. Recall that a family of functions $\{f^t : I \longrightarrow I, t \in \mathbb{R}\}$ (resp., $\{f^t : I \longrightarrow I,\ t > 0\}$) such that for all s, $t \in \mathbb{R}$ (resp., s, $t > 0$),

$$f^s \circ f^t = f^{s+t}$$

is called an *iteration group* (resp., *iteration semigroup*). Such an iteration group/semigroup is said to be *continuous* (resp., *measurable*) if for every $x \in I$ the mapping $t \longmapsto f^t(x)$ is continuous (resp., measurable).

Matkowski [1] proved that if a continuous at least at one point function $g : I \longrightarrow I$ commutes with two mappings f^a, f^b belonging to a measurable iteration group $\mathcal{F} = \{f^t : I \longrightarrow I,\ t \in \mathbb{R}\}$ such that each f^t is continuous and f^1 is a strictly increasing surjection without fixed points and $\frac{b}{a}$ is irrational, then $g \in \mathcal{F}$. The aim of this note is to show that the same assertion holds under another assumptions on the iteration group $\mathcal{F}$.

Let us recall (see Zdun [4]) that a non-trivial iteration group $\mathcal{F} = \{f^t : I \longrightarrow I,\ t \in \mathbb{R}\}$ such that every f^t is a homeomorphism is said to be *strictly*

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disjoint if the fact that $f^t \in \mathcal{F}$ has a fixed point implies $t = 0$.

It is clear that we have the following lemmas.

Lemma 1. *A non-trivial iteration group* $\{f^t : I \longrightarrow I,\ t \in \mathbb{R}\}$ *such that every* f^t *is a homeomorphism is strictly disjoint if and only if for any* $s,\ t \in \mathbb{R}$, $f^s = f^t$ *yields* $s = t$.

Remark. Iteration groups satisfying the above condition are called *effective* (see Tabor [2])).

Lemma 2. *If* $\{f^t : I \longrightarrow I,\ t \in \mathbb{R}\}$ *is a strictly disjoint iteration group, then there exist* $s,\ t \in \mathbb{R}$ *such that* $f^{ns}(x) \neq f^{mt}(x)$ *for* $x \in I$ *and* $n,\ m \in \mathbb{Z}$ *with* $|n| + |m| \neq 0$.

Proof. Suppose, contrary to our claim, that there are $n,\ m \in \mathbb{Z}$ with $|n|+|m| \neq 0$ and $x_0 \in I$ such that $f^{n\sqrt{2}}(x_0) = f^m(x_0)$. Then, $f^{n\sqrt{2}-m}(x_0) = x_0$, and consequently $n\sqrt{2} = m$, which is impossible. □

From our Lemma 2, Proposition 1 and Lemma 2 in Zdun [3] it follows that if $\mathcal{F} = \{f^t : I \longrightarrow I,\ t \in \mathbb{R}\}$ is a strictly disjoint iteration group, then the set $L_{\mathcal{F}}$ of limit points of the orbit $\{f^t(x),\ t \in \mathbb{R}\}$ does not depend on $x \in I$ and either $L_{\mathcal{F}} = \mathrm{cl}I$ or $L_{\mathcal{F}}$ is a perfect and nowhere dense set.

We can now formulate our main result.

Theorem 1. *If* $\mathcal{F} = \{f^t : I \longrightarrow I,\ t \in \mathbb{R}\}$ *is a strictly disjoint iteration group for which* $L_{\mathcal{F}} = \mathrm{cl}I$ *and a continuous at least at one point function* $g : I \longrightarrow I$ *commutes with two mappings* $f^a,\ f^b \in \mathcal{F}$ *such that* $\frac{b}{a}$ *is irrational, then* $g \in \mathcal{F}$.

Proof. By our Lemma 2 and Theorem 2 in Zdun [3], there exist a homeomorphism $\varphi : \mathbb{R} \longrightarrow I$ and an additive function $c : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$f^t(x) = \varphi(c(t) + \varphi^{-1}(x)), \qquad x \in I,\ t \in \mathbb{R}. \tag{1}$$

Since g commutes with f^a, we have

$$\varphi(c(a) + \varphi^{-1}(g(x))) = g(\varphi(c(a) + \varphi^{-1}(x))), \qquad x \in I,$$

and consequently

$$\varphi(c(a) + \varphi^{-1}(g(\varphi(t)))) = g(\varphi(c(a) + t)), \qquad t \in \mathbb{R}.$$

Hence, putting

$$\psi := \varphi^{-1} \circ g \circ \varphi, \tag{2}$$

we get

$$\psi(c(a) + t) = c(a) + \psi(t), \qquad t \in \mathbb{R}. \tag{3}$$

Similarly, from the fact that g commutes with f^b we obtain

$$\psi(c(b)+t) = c(b) + \psi(t), \qquad t \in \mathbb{R}. \tag{4}$$

Next, note that the mapping c is injective. Indeed, if $c(s) = c(t)$, then (1) yields $f^s = f^t$, and Lemma 1 now shows that $s = t$.

Thus, $c(a) \neq 0$. Moreover, we shall show that $\frac{c(b)}{c(a)}$ is irrational. If this assertion were false, then there would be $\frac{c(b)}{c(a)} = \frac{n}{m}$ for some integers n, m, $m \neq 0$, and consequently $c(mb) = c(na)$. Therefore, from the injectivity of c we would have $na = mb$, which is impossible.

Since $\frac{c(b)}{c(a)}$ is irrational and ψ is a continuous at least at one point function satisfying system of equations (3) and (4), Corollary 1 in Matkowski [1] shows that there are c_0, $c_1 \in \mathbb{R}$ such that $\psi(t) = c_0 + c_1 t$ for $t \in \mathbb{R}$. Moreover, from (3) one can conclude that $c_1 = 1$ and (2) together with (1) now yields

$$g(x) = \varphi(c_0 + \varphi^{-1}(x)) = f^{c^{-1}(c_0)}(x), \qquad x \in I,$$

which means that $g \in \mathcal{F}$. □

We shall finally apply Theorem 1 to prove the following result corresponding to Theorem 3 in Matkowski [1].

Theorem 2. *Let $\mathcal{F} = \{f^t : I \longrightarrow I,\ t \in \mathbb{R}\}$ be a strictly disjoint iteration group for which $L_{\mathcal{F}} = \mathrm{cl} I$ and assume that $\mathcal{G} = \{g^t : I \longrightarrow I,\ t > 0\}$ is a continuous iteration semigroup such that each g^t is continuous at least at one point. If*

$$f^t \circ g^t = g^t \circ f^t, \qquad t > 0, \tag{5}$$

then there exists an additive function $C : (0, +\infty) \longrightarrow \mathbb{R}$ such that $g^t = f^{C(t)}$ for $t > 0$.

Proof. By (5) we have

$$f^s \circ g^{ws} = g^{ws} \circ f^s, \qquad s,\ w > 0,\ w \in \mathbb{Q}.$$

Fix s, $t > 0$ and let $(w_n)_{n\in\mathbb{N}}$ be a sequence of positive rationals such that $\lim_{n\longrightarrow+\infty} w_n = \frac{t}{s}$. Then,

$$f^s \circ g^{w_n s} = g^{w_n s} \circ f^s, \qquad n \in \mathbb{N},$$

so the continuity of f^s and the iteration semigroup $\mathcal{G}$ gives $f^s \circ g^t = g^t \circ f^s$. Theorem 1 now shows that there is a $C(t) \in \mathbb{R}$ for which $g^t = f^{C(t)}$. Therefore,

$$f^{C(s+t)} = g^{s+t} = g^s \circ g^t = f^{C(s)+C(t)},$$

and Lemma 1 yields the additivity of C. □

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