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INTEGRAL INEQUALITIES FOR POLYNOMIALS

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Abstract: Let $P(z)$ be a polynomial of degree n which does not vanish in $|z| < k$. For $k = 1$, it is known that

$$||P(Rz)||_r \le \frac{||1 + R^n z^n||_r}{||1 + z^n||_r}||P(z)||_r, \qquad 0 < r < \infty, \qquad R > 1.$$

In this paper we consider the case $k \ge 1$ and obtain a more general result which yields variety of interesting generalizations of some known polynomial inequalities.

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1. Introduction and Statement of Results

Let P_n denote the space of polynomials (over the complex field) of degree n. For $P \in P_n$, define

$$\|P(z)\|_r := \left\{\frac{1}{2\pi}\int_0^{2\pi} |P(e^{\iota\theta})|^r d\theta\right\}^{1/r}, \quad 0 < r < \infty,$$

$$\|P(z)\|_\infty = \max_{|z|=1} |P(z)| \quad \text{and} \quad m(P,k) = \min_{|z|=k} |P(z)|.$$

If $P \in P_n$, then it is known [13], p. 346 that

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$$\|P(Rz)\|_\infty \leq R^n \|P(z)\|_\infty, \quad R > 1. \tag{1}$$

The inequality (1) can be obtained by letting $r \to \infty$ in the inequality (see [12])

$$\|P(Rz)\|_r \leq R^n \|P(z)\|_r, \quad r > 0 \quad \text{and} \quad R > 1. \tag{2}$$

Both these estimates are sharp and equality in (1) and (2) holds for $P(z) = az^n$, $a \neq 0$.

If we restrict ourselves to the class of polynomials $P \in P_n$ having no zero in $|z| < 1$, then inequalities (1) and (2) can be improved. In fact, it was shown by Ankeny and Rivlin [1] that if $P \in P_n$ and $P(z) \neq 0$ in $|z| < 1$, then for $R \geq 1$,

$$\|P(Rz)\|_\infty \leq \frac{R^n + 1}{2} \|P(z)\|_\infty. \tag{3}$$

Boas and Rahman [8] extended inequality (3) to L_r-norm by proving that if $P(z) \neq 0$ for $|z| < 1$, then for $R \geq 1$ and $r \geq 1$,

$$\|P(Rz)\|_r \leq \frac{\|1 + R^n z^n\|_r}{\|1 + z^n\|_r} \|P(z)\|_r. \tag{4}$$

Rahman and Schemeisser [11] showed that the inequality (4) remains valid for $0 < r < 1$ as well. In both (3) and (4) equality holds for $P(z) = az^n + b, |a| = |b|$. As a generalization of inequality (3), Aziz and Mohammad [4] proved that, if $P \in P_n$ and $P(z)$ does not vanish in $|z| < k$, $k \geq 1$, then

$$\|P(Rz)\|_\infty \leq \frac{(R^n + 1)(R + k)^n}{(R + k)^n + (1 + Rk)^n} \|P(z)\|_\infty, \quad R > 1. \tag{5}$$

More recently Aziz and Zargar [7] improved inequality (5) by showing that if $P \in P_n$ and $P(z) \neq 0$ in $|z| < k$, where $k \geq 1$, then for $R \geq 1$,

$$\begin{aligned} \|P(Rz)\|_\infty \leq & \frac{(R + k)^n}{(R + k)^n + (1 + Rk)^n} \\ & \times \left[(R^n + 1)||P(z)||_\infty - \left\{ R^n - \left(\frac{1 + Rk}{R + k} \right)^n \right\} m(P, k) \right]. \end{aligned} \tag{6}$$

In this paper we extend inequality (6) to L_r- mean of $|P(z)|$ and obtain a more general result which not only is a generalization of inequalities (4) and (5) but also yields an interesting refinement of inequality (4). More precisely we prove the following theorem.

Theorem 1. *If $P \in P_n$ and $P(z)$ does not vanish in $|z| < k$, where $k \geq 1$, then for every real or complex number β with $|\beta| \leq 1$, $R > 1$ and $r > 0$,*

$$\begin{aligned} &\left\| P(Rz) + \beta \left(\frac{R^n(R+k)^n - (1+Rk)^n}{(R+k)^n + (1+Rk)^n} \right) m(P,k) \right\|_r \\ &\qquad \leq \frac{\|1 + R^n z^n\|_r}{\left\| \left(\frac{1+Rk}{R+k} \right)^n + z^n \right\|_r} \|P(z)\|_r . \end{aligned} \tag{7}$$

The result is sharp in case $k = 1$ and equality in (7) holds for $P(z) = az^n + b$, $|a| = |b|$.

For $k = 1$, the following result immediately follows from Theorem 1.

Corollary 1. *If $P \in P_n$ and $P(z)$ does not vanish in $|z| < 1$, then for every real or complex number β with $|\beta| \leq 1$, $R > 1$ and each $r > 0$,*

$$\left\| P(Rz) + \beta \left(\frac{R^n - 1}{2} \right) m(P,1) \right\|_r \leq \frac{\|1 + R^n z^n\|_r}{\|1 + z^n\|_r} \|P(z)\|_r . \tag{8}$$

The result is best possible and equality in (8) holds for $P(z) = az^n + b$, $|a| = |b|$.

Remark 1. For $\beta = 0$ inequality (8) reduces to inequality (4) due to Rahman and Schmeisser [10] for each $r > 0$.

Remark 2. Making $r \to \infty$ in (8) and choosing argument of β with $|\beta| = 1$ suitably, it follows that, if $P \in P_n$ and $P(z) \neq 0$ in $|z| < 1$, then for $R > 1$

$$\|P(Rz)\|_\infty \leq \frac{R^n + 1}{2} \|P(z)\|_\infty - \frac{R^n - 1}{2} m(P,1) . \tag{9}$$

The result is sharp and equality in (9) holds for $P(z) = az^n + b$, $|a| = |b|$. This interesting result was also independently proved by Aziz and Dawood [3].

Next we prove the following result which is a generalization of the inequalities (1) and (2).

Theorem 2. *If $P \in P_n$, then for every $R > \rho > 0$,*

$$\rho^n \|P(Rz)\|_r \leq R^n \|P(\rho z)\|_r . \tag{10}$$

The result is sharp and equality holds for $P(z) = az^n$, $|a| \neq 0$.

Remark 3. For $\rho = 1$, Theorem 2 reduces to the inequality (2) and for $R = 1$, it follows from Theorem 2 that if $P \in P_n$, then for every $\rho < 1$,

$$\|P(\rho z)\|_r \geq \rho^n \|P(z)\|_r . \tag{11}$$

The result is sharp.

Remark 4. Making $r \to \infty$ in (10), it follows that, if $P \in P_n$, then for $R > \rho > 0$,

$$\rho^n \|P(Rz)\|_\infty \leq R^n \|P(\rho z)\|_\infty.$$

For $\rho = 1$, it reduces to the inequality (1) and for $R = 1$, it reduces to a result due to Zarantonello and Verga [12].

2. Lemmas

We need the following Lemmas.

Lemma 1. *If $P \in P_n$ and $P(z)$ does not vanish in $|z| < k, k > 0$, then for every $R \geq 1$, $\rho \leq k$ and $0 \leq \theta < 2\pi$*

$$|P(R\rho e^{i\theta})| \leq \left(\frac{R\rho + k}{\rho + Rk}\right)^n \left| R^n P\left(\frac{\rho e^{i\theta}}{R}\right)\right| - \left\{\left(\frac{R\rho + k}{\rho + Rk}\right)^n R^n - 1\right\} m(p, k). \quad (12)$$

Lemma 1 is proved by A. Aziz and B.A. Zargar, see [6].

Lemma 2. *If $P \in P_n$, then for every $R \geq 1$ and α real,*

$$\|P(Rz) + e^{i\alpha} R^n P\left(\frac{z}{R}\right)\|_r \leq |1 + R^n e^{i\alpha}| \|P(z)\|_r, \quad r > 0. \quad (13)$$

This lemma is a special case of a result due to A. Aziz and N.A. Rather, see Lemma 15, with $\beta = 0$.

Lemma 3. *If A,B,C are non negative real numbers such that $B + C \leq A$, then for every real α,*

$$|(A - C)e^{i\alpha} + (B + C)| \leq |Ae^{i\alpha} + B|. \quad (14)$$

This lemma is also established by A. Aziz and N.A. Rather, see [4].

3. Proofs of the Theorems

Proof of Theorem 1. Since all the zeros of the polynomial $P(z)$ lie in $|z| \geq k$ where $k \geq 1$, by Lemma 1 (with $\rho = 1$), we have for $|z| = 1$ and $R \geq 1$,

$$\left(\frac{1 + Rk}{R + k}\right)^n |P(Rz)| \leq \left| R^n P(\frac{z}{R})\right| - \left\{R^n - \left(\frac{1 + Rk}{R + k}\right)^n\right\} m(p, k). \quad (15)$$

This gives for each θ, $0 \leq \theta < 2\pi$ and $R \geq 1$,

$$\begin{aligned}\left(\frac{1+Rk}{R+k}\right)^n &\left\{|P(Re^{i\theta})| + \frac{R^n(R+k)^n - (1+Rk)^n}{(R+k)^n + (1+Rk)^n} m(p,k)\right\} \\ &\leq \left|R^n P(\frac{e^{i\theta}}{R})\right| - \frac{R^n(R+k)^n - (1+Rk)^n}{(R+k)^n + (1+Rk)^n} m(p,k).\end{aligned} \tag{16}$$

Taking $A = |R^n P(\frac{e^{i\theta}}{R})|$, $B = |P(Re^{i\theta})|$ and

$$C = \frac{R^n(R+k)^n - (1+Rk)^n}{(R+k)^n + (1+Rk)^n} m(p,k)$$

in Lemma 3 and noting by (16) that for $R \geq 1$, $k \geq 1$,

$$B + C \leq \left(\frac{1+Rk}{R+k}\right)^n (B+C) \leq A - C \leq A,$$

we get for every real α,

$$\begin{aligned}&\left|\left(|R^n P(\frac{e^{i\theta}}{R})| - \frac{R^n(R+k)^n - (1+Rk)^n}{(R+k)^n + (1+Rk)^n} m(p,k)\right) e^{i\alpha}\right. \\ &\quad \left. + \left\{|P(Re^{i\theta})| + \frac{R^n(R+k)^n - (1+Rk)^n}{(R+k)^n + (1+Rk)^n} m(p,k)\right\}\right| \\ &\qquad \leq \left\| R^n P(e^{i\theta}/R)|e^{i\alpha} + |P(Re^{i\theta})\right\| .\end{aligned} \tag{17}$$

This implies that for each $r > 0$,

$$\int_0^{2\pi} |f(\theta) + e^{i\alpha} g(\theta)|^r d\theta \leq \int_0^{2\pi} \| R^n P(e^{i\theta}/R)|e^{i\alpha} + |P(Re^{i\theta})\|^r d\theta, \tag{18}$$

where

$$f(\theta) = |P(Re^{i\theta})| + \frac{R^n(R+k)^n - (1+Rk)^n}{(R+k)^n + (1+Rk)^n} m(p,k)$$

and

$$g(\theta) = |R^n P(e^{i\theta}/R)| - \frac{R^n(R+k)^n - (1+Rk)^n}{(R+k)^n + (1+Rk)^n} m(p,k).$$

Integrating both sides of (17) with respect to α from 0 to 2π, we get for each $r > 0$, $R \geq 1$ and α real,

$$\int_0^{2\pi}\int_0^{2\pi} |f(\theta) + e^{i\alpha} g(\theta)|^r d\theta d\alpha \leq \int_0^{2\pi}\int_0^{2\pi} \| P(Re^{i\theta})| + e^{i\alpha}|R^n P(e^{i\theta}/R)\|^r d\theta d\alpha$$

$$= \int_0^{2\pi} \left\{ \int_0^{2\pi} \||P(Re^{i\theta})| + e^{i\alpha}|R^n P(e^{i\theta}/R)\||^r d\alpha \right\} d\theta$$
$$= \int_0^{2\pi} \left\{ \int_0^{2\pi} |P(Re^{i\theta}) + e^{i\alpha} R^n P(e^{i\theta}/R)|^r d\alpha \right\} d\theta$$
$$= \int_0^{2\pi} \left\{ \int_0^{2\pi} |P(Re^{i\theta}) + e^{i\alpha} R^n P(e^{i\theta}/R)|^r d\theta \right\} d\alpha.$$

This in conjunction with Lemma 2 yields,

$$\int_0^{2\pi} \int_0^{2\pi} |f(\theta) + e^{i\alpha} g(\theta)|^r d\theta d\alpha \leq \int_0^{2\pi} |R^n e^{i\alpha} + 1|^r d\alpha \int_0^{2\pi} |P(e^{i\theta})|^r d\theta. \quad (19)$$

Now for every real α and $t_1 \geq t_2 \geq 1$,

$$|t_1 + e^{i\alpha}| \geq |t_2 + e^{i\alpha}|, \quad (20)$$

which implies

$$\int_0^{2\pi} |t_1 + e^{i\alpha}|^r d\alpha \geq \int_0^{2\pi} |t_2 + e^{i\alpha}|^r d\alpha, \quad r > 0.$$

If $f(\theta) \neq 0$,we take $t_1 = |g(\theta)/f(\theta)|$ and $t_2 = \left(\frac{1+Rk}{R+k}\right)^n$, then by (16), $t_1 \geq t_2 \geq 1$ and we get

$$\int_0^{2\pi} |f(\theta) + e^{i\alpha} g(\theta)|^r d\alpha = |f(\theta)|^r \int_0^{2\pi} |1 + e^{i\alpha} \frac{g(\theta)}{f(\theta)}|^r d\alpha$$
$$= |f(\theta)|^r \int_0^{2\pi} |\frac{g(\theta)}{f(\theta)} + e^{i\alpha}|^r d\alpha = |f(\theta)|^r \int_0^{2\pi} ||\frac{g(\theta)}{f(\theta)}| + e^{i\alpha}|^r d\alpha$$
$$\geq |f(\theta)|^r \int_0^{2\pi} |\left(\frac{1+Rk}{R+k}\right)^n + e^{i\alpha}|^r d\alpha \quad (\text{by using } (19)).$$

For $f(\theta) = 0$, this inequality is trivially true. Using (20) in (18), we conclude that for $r > 0$, $R \geq 1$ and α real,

$$\int_0^{2\pi} |\left(\frac{1+Rk}{R+k}\right)^n + e^{i\alpha}|^r d\alpha$$
$$\times \int_0^{2\pi} \left| |P(Re^{i\theta})| + \frac{R^n(R+k)^n - (1+Rk)^n}{(R+k)^n + (1+Rk)^n} m(p,k) \right|^r d\theta$$
$$\leq \int_0^{2\pi} |R^n e^{i\alpha} + 1|^r d\alpha . \int_0^{2\pi} |P(e^{i\theta})|^r d\theta.$$

This gives for every real or complex number β with $|\beta| \leq 1$, $r > 0$, $R \geq 1$ and α real,

$$\int_0^{2\pi} |\left(\frac{1+Rk}{R+k}\right)^n + e^{i\alpha}|^r d\alpha \times \int_0^{2\pi} |P(Re^{i\theta}) + \beta \frac{R^n(R+k)^n - (1+Rk)^n}{(R+k)^n + (1+Rk)^n} m(p,k)|^r d\theta \leq \int_0^{2\pi} |R^n e^{i\alpha} + 1|^r d\alpha . \int_0^{2\pi} |P(e^{i\theta})|^r d\theta$$

which immediately leads to (7) and this completes the proof of Theorem 1. □

Proof of Theorem 2. Let $f(z) = P(\rho z)$, then $f(z)$ is a polynomial of degree at most n and hence so is $g(z) = z^n \overline{f(1/\bar{z})} = z^n \overline{P(\rho/\bar{z})}$, that is, an entire function. By Hardy's result [6],

$$\left\{ \frac{1}{2\pi} \int_0^{2\pi} |g(te^{i\theta})|^r d\theta \right\}^{1/r}, \quad r > 0\,,$$

is a nondecreasing function of t ($t > 0$). This gives

$$\left\{ \frac{1}{2\pi} \int_0^{2\pi} |g(te^{i\theta})|^r d\theta \right\}^{1/r} \leq \left\{ \frac{1}{2\pi} \int_0^{2\pi} |g(e^{i\theta})|^r d\theta \right\}^{1/r}. \tag{21}$$

Setting $t = \rho/R < 1$, and replacing $g(z)$ by $z^n \overline{P(\rho/\bar{z})}$ in (21), we immediately get the inequality (10). This completes the proof of Theorem 2. □

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