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UNIFORMLY AND ASYMPTOTIC STABILITY FOR TIME VARYING DYNAMIC SYSTEM WITH LINEAR PART ON TIME SCALES

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Abstract: In this article by using the Lyapunov criteria, we study conditions that the solutions of a time varying dynamic system with linear part of the form $x^{\Delta} = A(t)x(t) + F(t, x(t))$ remain stable on certain time scales.

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1. Introduction

First we give some aspects about time scales, then some needed lemmas and theorems are mentioned. The material in this section is drawn mainly from [1] and [6].

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1.1. Time Scales

A *time scale* is any nonempty closed subset of the real numbers **R**. Thus time scale can be any of the usual integer subsets (e.g. **Z** or **N**), the entire real line R, or any combination of discrete points of union with continuous intervals.

The *forward jump operator* of T, $\sigma(t) : T \to T$, is given by $\sigma(t) = \inf_{s\in T}\{s > t\}$. The *backward jump operator* of T, $\rho(t) : T \to T$, is given by $\rho(t) = \sup_{s\in T}\{s < t\}$. The *graininess function* $\mu(t) : T \to T$ is given by $\mu(t) = \sigma(t)-t$. Here we adopt the conventions $\inf \phi = \sup T$ (i.e. $\sigma(t) = t$ if T has a maximum element t), and $\sup \phi = \inf T$ (i.e. $\rho(t) = t$ if T has a minimum element t). For notational purpose, the intersection of a real interval $[a, b]$ with a time scale T is denoted by $[a, b]\cap T := [a, b]_T$. A point $t \in T$ is *right scattered* if $\sigma(t) > t$ and *right dense* if $\sigma(t) = t$. A point $t \in T$ is *left scattered* if $\rho(t) < t$ and *left dense* if $\rho(t) = t$. If t is both left scattered and right scattered, we say t is *isolated*. If t is both left dense and right dense, we say that t is *dense*. The set T^k is defined as follows: if T has a left scattered maximum m, then $T^k = T - \{m\}$; otherwise $T^k = T$. If $f : T \to \mathbf{R}$ is a function, then the composition $f(\sigma(t))$ is often denoted by $f^\sigma(t)$.

For $f : T \to \mathbf{R}$ and $t \in T^k$, define $f^\Delta(t)$ as the number (when it exists), with the property that, for any $\epsilon > 0$, there exists a neighborhood U of t such that

$$\big|[f(\sigma(t)) - f(s)]f^\Delta(t)[\sigma(t) - s]\big| \le \epsilon|\sigma(t) - s|, \quad \forall\ s \in U.$$

The function $f^\Delta : T^k \to \mathbf{R}$ is called the *delta derivative* or the *Hilger derivative* of f on T^k. We say f is delta differentiable on T^k provided $f^\Delta(t)$ exists for all $t \in T^k$.

The following theorem establishes several important observations regarding delta derivatives.

Theorem 1.1. *Suppose $f : T \to \mathbf{R}$ and $t \in T^k$.*

(i) If f is delta differentiable at t, then f is continuous at t.

(ii) If f is continuous at t and t is right scattered, then f is delta differentiable at t and

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

(iii) If t is right dense, then f is delta differentiable at t if and only if $\lim_{s\to t} \frac{f(t)-f(s)}{t-s}$ exists. In this case,

$$f^\Delta(t) = \lim_{s\to t} \frac{f(t) - f(s)}{t - s}.$$

(iv) If f is delta differentiable at t, then $f(\sigma(t)) = f(t) + \mu(t)f^\Delta(t)$.

Note that $f^{\Delta}(t)$ is precisely $f^{'}(t)$ from the usual calculus when $T = \mathbf{R}$. On the other hand, $f^{\Delta} = \Delta f = f(t+1) - f(t)$ (i.e. the forward difference operator) on the time scale $T = \mathbf{R}$. These are but two very special (and rather simple) examples of time scales. Moreover, the realm of differential equations and difference equations can now viewed as but special, particular cases of more general dynamic equations on time scales, i.e. equations involving the delta derivative(s) of some unknown function.

A function $f : T \to \mathbf{R}$ is rd-continuous if f is continuous at every right dense point $t \in T$, and its left hand limit exists at each left dense point. The set of rd-continuous functions $f : T \to \mathbf{R}$ will be denoted by $C_{rd} = C_{rd}(T) = C_{rd}(T, \mathbf{R})$. A function $f : T \to \mathbf{R}$ is called a (delta) antiderivative of $f : T \to \mathbf{R}$ provided $F^{\Delta}(t) = f(t)$ holds for all $t \in T^k$. The Cauchy integral or definite integral is given by $\int_a^b f(t)\Delta t = F(b) - F(a)$, for all $a, b \in T$, where F is any (delta) antiderivative f. Suppose that $\sup T = \infty$. Then the improper integral is defined to by $\int_a^{\infty} f(t)\Delta t = \lim_{b\to\infty} F(t) \mid_a^b$ for all $a \in T$.

Theorem 1.2. (Existence of Antiderivatives) *(i) Every rd-continuous function has an antiderivative. If* $t_0 \in T$, *then* $F(t) = \int_{t_0}^t f(\tau)\Delta\tau, \quad \tau \in T$, *is an antiderivative of* f.

(ii) If $f \in C_{rd}$ *and* $t \in T^k$, *then* $\int_t^{\sigma(t)} f(\tau)\Delta(\tau) = f(t)\mu(t)$.

(iii) Suppose $a, b \in T$ *and* $f \in C_{rd}$.

(a) If $T = \mathbf{R}$, *then* $\int_a^b f(t)\Delta(t) = \int_a^b f(t)dt$ *(the usual Riemann integral).*

(b) If $[a, b]_T$ *consists of only isolated points, then*

$$\int_a^b f(t)\Delta(t) = \begin{cases} \sum_{t\in[a,b)_T} f(t)\mu(t)\,, & a < b\,, \\ 0\,, & a = b\,, \\ -\sum_{t\in[a,b)_T} f(t)\mu(t)\,, & a > b\,. \end{cases}$$

The last result above reveals that in the continuous case,$T = \mathbf{R}$, definite integrals are the usual Riemann integrals from calculus. When $T = \mathbf{Z}$, definite integrals correspond to definite sums from the difference calculus; see [5].

1.2. The Hilger's Complex Plane

For $h > 0$, define the Hilger complex numbers, the Hilger real axis, the Hilger alternating axis, and the Hilger imaginary circle by

$$C_h := \left\{ z \in \mathbf{C} \ : \ z \neq -\frac{1}{h} \right\}, \qquad R_h := \left\{ z \in \mathbf{R} \ : \ z > -\frac{1}{h} \right\},$$

$$A_h := \left\{ z \in \mathbf{R} \ : \ z < -\frac{1}{h} \right\}, \qquad I_h := \left\{ z \in \mathbf{C} \ : \ \left| z + \frac{1}{h} \right| = \frac{1}{h} \right\},$$

Respectively. For $h = 0$, let $C_0 = \mathbf{C},\ R_0 = \mathbf{R},\ A_0 = \emptyset$ and $I_0 = i\mathbf{R}$.

Let $h > 0$ and $z \in C_h$. The Hilger real part of z is defined by $\mathrm{Re}_h(z) := \frac{|zh+1|-1}{h}$, and the Hilger imaginary part of z is defined by $\mathrm{Im}_h(z) := \frac{\mathrm{Arg}(zh+1)}{h}$, where $\mathrm{Arg}(z)$ denotes the principal argument of z (i.e. $-\pi < \mathrm{Arg}(z) \leq \pi$).

For $h > 0$, define the strip $Z_h := \left\{ z \in \mathbf{C} \ : \ -\frac{\pi}{h} < \mathrm{Im}(z) \leq \frac{\pi}{h} \right\}$, and for $h = 0$, set $Z_0 = \mathbf{C}$. Then we can define the cylinder transformation $\xi_h : C_h \to Z_h$ by

$$\xi_h(z) = \frac{1}{h}\mathrm{Log}(1 + zh), \qquad z > 0, \tag{1}$$

where Log is the principal logarithm function. When $h = 0$, we define $\xi_0(z) = z$, for all $z \in \mathbf{C}$. It then follows that the inverse cylinder transformation $\xi_h^{-1} : Z_h \to C_h$ is given by

$$\xi_h^{-1}(z) = \frac{e^{zh} - 1}{h}.$$

Since the graininess may not be constant for a given time scale, we will interchangeably subscript various quantities (such as ξ and ξ^{-1}) with $\mu = \mu(t)$ instead of h to reflect this.

1.3. Generalized Exponential Functions

The function $p : T \to \mathbf{R}$ is *regressive* if $1 + \mu(t)p(t) \neq 0$ for all $t \in T^k$, and this concept motivates the definition of the following sets:

$$\mathcal{R} = \left\{ p : T \to \mathbf{R} \ : \ p \in C_{rd}(T) \text{ and } 1 + \mu(t)p(t) \neq 0,\ \forall t \in T^k \right\},$$
$$\mathcal{R}^+ = \left\{ p \in \mathbf{R} \ : \ 1 + \mu(t)p(t) > 0,\ \forall t \in T^k \right\}.$$

The function $p : t \to \mathbf{R}$ is *uniformly regressive* on T if there exists a positive constant δ such that $0 < \delta^{-1} \leq |1 + \mu(t)p(t)|,\ t \in T^k$. A matrix is regressive if and only if all of its eigenvalues are equivalent, the matrix $A(t)$ is regressive if and only if $I + \mu(t)A(t)$ is invertible for all $t \in T^k$.

If $p \in \mathcal{R}$, then we define the *generalized time scale exponential function* by

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta(\tau) \right) \qquad \text{for all } \ s, t \in T.$$

The following theorem is a compilation of properties of $e_p(t,t_0)$ (some of which are counterintuitive) that we need in the main body of the paper.

Theorem 1.3. *The function has the following properties:*

(i) If $p \in \mathcal{R}$ then $e_p(t,r)e_p(r,s) = e_p(t,s)$ for all $r,s,t \in T$.

(ii) $e_p(\sigma(t),s) = (1+\mu(t)p(t))e_p(t,s)$.

(iii) If $p \in \mathcal{R}^+$, then $e_p(t,t_0) > 0$ for all $t \in T$.

(iv) If $1+\mu(t)p(t) < 0$ for some $t \in T^k$,then $e_p(t,t_0)e_p(\sigma(t),t_0) < 0$.

(v) If $T = \mathbf{R}$ then $e_p(t,s) = e^{\int_s^t p(\tau)d\tau}$. Moreover, if p is constant, then $e_p(t,s) = e^{p(t-s)}$.

(vi) If $T = \mathbf{Z}$, then $e_p(t,s) = \prod_{\tau=s}^{t-1}(1+p(\tau))$. Moreover, if $T = h\mathbf{Z}$, with $h > 0$ and p is constant, then $e_p(t,s) = (1+hp)^{(t-s)/h}$.

If $p \in \mathcal{R}$ is rd-continuous, then the dynamic equation

$$y^{\Delta}(t) = p(t)y(t) \tag{2}$$

is called regressive.

Theorem 1.4. *Let $t_0 = T$ and $y(t_0) = y_0 \in \mathbf{R}$. Then the regressive IVP (2) has a unique solution $y : T \to \mathbf{R}^n$ given by*

$$y(t) = e_p(t,t_0)y_0.$$

If $p \in \mathcal{R}$ and $f : T \to \mathbf{R}$ is rd-continuous, then the dynamic equation

$$y^{\Delta}(t) = p(t)y(t) + f(t) \tag{3}$$

is called regressive.

Theorem 1.5. (Variation of Constants) *Let $t_0 = T$ and $y(t_0) = y_0 \in \mathbf{R}$. Then the regressive IVP (3) has a unique solution $y : T \to \mathbf{R}^n$ given by*

$$y(t) = e_p(t,t_0)y_0 + \int_{t_0}^{t} e_p(t,\sigma(\tau))f(\tau)\Delta\tau.$$

We say the $n \times 1$ -vector-valued system

$$y^{\Delta}(t) = A(t)y(t) + f(t) \tag{4}$$

is regressive provided $A \in \mathcal{R}$ and $f : T \to \mathbf{R}^n$ is a rd-continuous vector-valued function. Let $t_0 \in T$ and assume that A is a $n \times n$-matrix-valued function. The unique matrix-valued solution to the IVP

$$Y^{\Delta}(t) = A(t)Y(t), \quad Y(t_0) = I_n, \tag{5}$$

where I_n is the $n \times n$-identity matrix, is called the transition matrix and it is denoted by $\Phi_A(t, t_0)$. In this paper, we denote the solution to (5) as $\Phi_A(t, t_0)$ when $A(t)$ is time varying and denoted the solution as $e_A(t, t_0) \equiv \Phi_A(t, t_0)$ (the *matrix exponential*), as in [1]) only when $A(t) \equiv A$ is a constant matrix. Also, if $A(t)$ is a function on T and the time scale matrix exponential function is a function on some other time scale $\mathbf{S}$, then $A(t)$ is constant with respect to $e_{A(t)}(\tau, s)$, for all $\tau,\ s \in \mathbf{S}$ and $t \in T$. The following theorem lists some properties of the transition matrix.

Theorem 1.6. *Suppose $A,\ B \in \mathcal{R}$ are matrix-valued function on T.*

(i) Then the semigroup property $\Phi_A(t, r)\Phi_A(r, s) = \Phi_A(t, s)$ is satisfied for all $r, s, t \in T$.

(ii) $\Phi_A(\sigma(t), s) = (I + \mu(t)A(t))\Phi_A(t, s)$.

(iii) If $T = \mathbf{R}$, then A is constant, then $\Phi_A(t, s) = e_A(t, s) = e^{A(t-s)}$.

(iv) If $T = h\mathbf{Z}$, with $h > 0$ and A is constant, then $\Phi_A(t, s) = e_A(t, s) = (1 + hA)^{(t-s)/h}$.

We now present a theorem that guarantees a unique solution to the regressive $n \times 1$-vector-valued dynamic IVP (4).

Theorem 1.7. (Variation of Constants) *Let $t_0 = T$ and $y(t_0) = y_0 \in \mathbf{R}^n$. Then the regressive IVP (4) has a unique solution $y : T \to \mathbf{R}^n$ given by*

$$y(t) = \Phi_A(t, t_0)y_0 + \int_{t_0}^{t} \Phi_A(t, \sigma(\tau))f(\tau)\Delta\tau.$$

1.4. Stability

In this section we start by introducing definitions and notation that will be employed in the sequel. The *Euclidean norm* of an $n \times 1$ vector $x(t)$ is defined to be a real-valued function of t and is denoted by $||x(t)|| = \sqrt{x(t)^T x(t)}$. The *induced norm* of an $m \times n$ matrix A is defined to be $||A|| = \max_{||x||=1} ||Ax||$. The norm of A induced by the Euclidean norm above is equal to the nonnegative square root of the absolute value of the largest eigenvalue of the symmetric matrix $A^T A$. Thus, we define this norm next. The *spectral norm* of an $m \times n$ matrix A is defined to be

$$||A|| = \left[\max_{||x||=1} x^T A^T Ax\right].$$

This will be the matrix norm that is used in the sequel and will be denoted by $||.||$.

A symmetric matrix M is defined to be *positive semidefinite* if for all $n \times 1$ vectors x we have $x^T M x \geq 0$ and it is *positive definite* if $x^T M x > 0$, with equality only when $x = 0$. *Negative semidefiniteness* and *definiteness* are defined in terms of positive definiteness of $-M$. We now define the concepts of uniform stability and uniform exponential stability. These two concepts involve the boundedness of the solutions of the regressive time varying linear dynamic equation

$$x^{\Delta}(t) = A(t)x(t), \qquad x(t_0) = x_0, \quad t_0 \in T. \tag{6}$$

Definition 1.8. The time varying linear dynamic equation (6) is *uniformly stable* if there exists a finite constant $\gamma > 0$ such that for any t_0 and $x(t_0)$, the corresponding solution satisfies

$$||x(t)|| \leq \gamma ||x(t_0)||, \qquad t \geq t_0. \tag{7}$$

For the next definition, we define a stability property that not only concerns the boundedness of a solutions to (6), but also the asymptotic characteristics of the solutions as well. If the solutions to (6) possess the following stability property, then the solutions approach zero exponentially as $t \to \infty$ (i.e. the norms of the solutions are bounded above by a decaying exponential function).

Definition 1.9. The time varying linear dynamic equation (6) is called uniformly exponentially stable if there exist constants $\gamma, \lambda > 0$ with $-\lambda \in \mathcal{R}^+$ such that for any t_0 and $x(t_0)$, the corresponding solution satisfies

$$||x(t)|| \leq \gamma ||x(t_0)|| e_{-\lambda}(t, t_0), \quad t \geq t_0. \tag{8}$$

It is obvious by inspection of the previous definitions that we must have $\lambda \geq 1$. By using the word uniform, it is implied that the choice of γ does not depend on the initial time t_0. The last stability definition given uses a uniformity condition to conclude exponential stability.

Definition 1.10. The linear state equation (6) is defined to be *uniformly asymptotically stable* if it is uniformly stable and given any $\delta > 0$, there exists a $T > 0$ so that for any t_0 and $x(t_0)$, the corresponding solution $x(t)$ satisfies

$$||x(t)|| \leq \delta ||x(t_0)||, \quad t \geq t_0 + T. \tag{9}$$

It is noted that the time T that must pass before the norm of the solution satisfies (9) and the constant $\delta > 0$ is independent of the initial time t_0. We now state and prove four theorems, the first three of which characterize uniform stability and uniform exponential stability in terms of the transition matrix for system (6). The fourth theorem illustrates the relationship between uniform asymptotic stability and uniform exponential stability.

Theorem 1.11. *The time varying linear dynamic equation (6) is uniformly stable if and only if there exists a* $\gamma > 0$ *such that*

$$||\Phi_A(t, t_0)|| \leq \gamma$$

for all $t \geq t_0$ *with* $t, t_0 \in T$.

Theorem 1.12. *The time varying linear dynamic equation (6) is uniformly exponential stable if and only if there exists a* $\gamma, \lambda > 0$ *with* $-\lambda \in \mathcal{R}^+$ *such that*

$$||\Phi_A(t, t_0)|| \leq \gamma e_{-\lambda}(t, t_0)$$

for all $t \geq t_o$ *with* $t, t_0 \in T$.

Theorem 1.13. *Suppose there exists a constant* α *such that for all* $t \in T$, $\|A(t)\| \leq \alpha$. *Then the linear state equation (6) is uniformly exponentially stable if and only if there exists a constant* β *such that*

$$\int_{\tau}^{t} \|\Phi_A(t, \sigma(s))\| \leq \beta \tag{10}$$

for all $t, \tau \in T$ *with* $t \geq \sigma(\tau)$.

Theorem 1.14. *The linear state equation (6) is uniformly exponentially stable if and only if it is uniformly asymptotically stable.*

Now we consider the stability of the regressive time varying linear dynamic system of the form

$$x^{\Delta} = A(t)x(t), \quad x(t_0) = x_0, \quad t_0 \in T. \tag{11}$$

Our goal is to assess the stability of the unforced system by observing the system's total energy as the state of the system evolves in time. If the total energy of the system decreases as the state evolves, then the state vector approaches a constant value (equilibrium point) corresponding to zero energy as time increases. The stability of the system involves the growth characteristics of solutions of the state equation, and these properties can be measured by a suitable (energy-like) scalar function of the state vector. In the following two subsections, we discuss the boundedness properties and asymptotic behavior as $t \to \infty$ of solutions of system (6). The present issue is obtaining a proper scalar function. We assume that the time scale T is unbounded above. To start, we consider conditions that imply all solutions of the linear state equation (6) are such that $\|x(t)\|^2 \to 0$ as $t \to \infty$. For any solution of (6), the delta derivative of the scalar function $\|x(t)\|^2 = x(t)^T x(t)$ with respect to t is:

$$[\|x(t)\|^2]^{\Delta_t} \;=\; x(t)^{T^{\Delta}} x(t) + x(t)^{T^{\sigma}} x^{\Delta}(t)$$

$$\begin{aligned} &= x(t)^T A^T(t)x(t) + x(t)^T(I + \mu(t)A^T(t))A(t)x(t) \\ &= x(t)^T[A^T(t) + A(t) + \mu(t)A^T(t)A(t)]x(t)\,. \end{aligned}$$

So if the quadratic form we obtained is negative definite, i.e. $A^T(t) + A(t) + \mu(t)A^T(t)A(t)$ is negative definite at each t, then $\|x(t)\|^2$ will decrease monotonically as t increases. We later show that if there exists a $\nu > 0$ so that $A^T(t) + A(t) + \mu(t)A^T(t)A(t) \leq -\nu I$ for all t, then $\|x(t)\|^2 \to 0$ as $t \to \infty$. To formalize our discussion, we define time-dependent quadratic forms that are useful for analyzing stability.

We will refer to these quadratic forms as unified time scale quadratic Lyapunov functions. For a symmetric matrix $Q(t) \in C^1_{rd}(T, \mathbf{R}^{n\times n})$ we write the general quadratic Lyapunov function as $x^T(t)Q(t)x(t)$. If $x(t)$ is a solution to (6), and since $x^T(t)Q(t)x(t)$ has a scalar output, our interest lies in the behavior of the quantity $x^T(t)Q(t)x(t)$ for $t > t_0$.

Definition 1.15. Let $Q(t)$ be a symmetric matrix such that $Q(t) \in C^1_{rd}(T, \mathbf{R}^{n\times n})$. A unified time scale *quadratic Lyapunov function* is given by $x^T(t)Q(t)x(t)$ for $t > t_0$. with delta derivative

$$\begin{aligned} [x^T(t)Q(t)x(t)]^{\Delta_t} &= x^T(t)[A^T(t)Q(t) + (I + \mu(t)A^T(t))(Q^\Delta(t) + Q(t)A(t) \\ &\quad + \mu(t)Q^\Delta(t)A(t))] = x(t)[A^T(t)(Q(t) + Q(t)A(t) + \mu(t)A^T(t)Q(t)A(t) \\ &\quad + (I + \mu(t)A^T(t))Q^\Delta(t)(I + \mu(t)A(t))]x(t). \end{aligned}$$

The matrix dynamic equation that is obtained by differentiating quadratic Lyapunov functions with respect to t is given by

$$\begin{aligned} &A^T(t)(Q(t) + Q(t)A(t) + \mu(t)A^T(t)Q(t)A(t) \\ &\quad + (I + \mu(t)A^T(t))Q^\Delta(t)(I + \mu(t)A(t))) = -M, \quad M = M^T\,. \end{aligned}$$

One can easily see that it merges with the familiar continuous matrix differential equation and ($T = \mathbf{R}$) discrete ($T = \mathbf{Z}$) difference (recursive) equation obtained from the respective quadratic Lyapunov functions in $\mathbf{R}$ and $\mathbf{Z}$.

2. Lyapunov Criteria for Uniformly and Exponential Stability

In this section, we mention the criteria for uniform stability of system (6). The criteria introduced in Theorem 2.1 is a generalization of the Lyapunov criteria for uniform stability of discrete and continuous linear systems that can be found in the famous papers in [7, 8]. Uniform stability involves the boundedness of all solutions of system (6) and in the following theorem we derive sufficient conditions for uniform stability of the system. The strategy is to state requirements on the matrix so that the corresponding quadratic form yields uniform stability of the system. Proofs can be found in [6].

Theorem 2.1. *The time varying linear dynamic system (6) is uniformly stable if for all $t \in T$, there exists a symmetric matrix $Q(t) \in C^1_{rd}(T, \mathbf{R}^{n\times n})$ such that:*

(i) $\eta I \leq Q(t) \leq \rho I$,

(ii) $A(t)Q(t) + (I + \mu(t)A^T(t))\left(Q^\Delta(t) + Q(t)A(t) + \mu(t)Q^\Delta(t)A(t)\right) \leq 0$, where $\eta, \rho \in \mathcal{R}^+$.

Theorem 2.2. *The time varying linear dynamic system (6) is uniformly exponentially stable if for all $t \in T$, there exists a symmetric matrix $Q(t) \in C^1_{rd}(T, \mathbf{R}^{n\times n})$ such that:*

(i) $\eta I \leq Q(t) \leq \rho I$,

(ii) $A(t)Q(t) + (I + \mu(t)A^T(t))\left(Q^\Delta(t) + Q(t)A(t) + \mu(t)Q^\Delta(t)A(t)\right) \leq -\nu I$, *where* $\eta, \rho, \nu \in \mathcal{R}^+$ *and* $\frac{-\nu}{\rho} \in \mathcal{R}^+$.

3. Stability in Perturbation Case

It is also useful to consider state equations that are "close" to another linear state equation that is uniformly stable. In [6], if the stability of system (6) has already been determined by an appropriate Lyapunov function, then certain conditions on the perturbation matrix $F(t)$ guarantee stability of the perturbed linear system

$$z^\Delta = [A(t) + F(t)]z(t), \quad z_{(t_0)} = z_0, \quad t \geq t_0\,. \tag{12}$$

Theorem 3.1. *Suppose the linear state equation (6) is uniformly stable. Then the perturbed linear dynamic equation (12) it is uniformly stable if there exist some $\beta \geq 0$ such that for all τ*

$$\int_\tau^\infty |F(s)|\Delta s \leq \beta. \tag{13}$$

Proof. For any t_0 and $z_0 = z(t_0)$, by Theorem 1.6 the solution of (12) satisfies

$$z(t) = \Phi_A(t, t_0)z_0 + \int_{t_0}^t \Phi_A(t, \sigma(\tau))F(\tau)z(\tau)\Delta s, \quad t \geq t_0\,, \tag{14}$$

where $\Phi_A(t,t_0)$ is the transition matrix for system (6). By the uniform stability of (6), there exists a constant $\gamma \geq 0$ such that $\|\Phi_A(t,t_0)\| \leq \gamma$, for all $t,\tau \in T$ with $t \geq \tau$. By taking the norms of both sides of (14), we have

$$\|z(t)\| = \gamma\|z_0\| + \int_{t_0}^{t} \gamma\|F(\tau)\|\|z(\tau)\|\Delta\tau, \quad t \geq t_0.$$

By Gronwall's inequality in [1], a result in [3], and the inequality (13), we obtain

$$\begin{aligned}\|z(t)\| \leq \gamma\|z_0\|e_{\gamma\|F(s)\|}(t,t_0) &= \gamma\|z_0\| \exp\left(\int_{t_0}^{t} \frac{\mathrm{Log}(1+\mu(s)\gamma\|F(s)\|)}{\mu(s)}\Delta s\right)\\ &\leq \gamma\|z_0\| \exp\left(\int_{t_0}^{\infty} \frac{\mathrm{Log}(1+\mu(s)\gamma\|F(s)\|)}{\mu(s)}\Delta s\right)\\ &\leq \gamma\|z_0\| \exp\left(\int_{t_0}^{\infty} \gamma\|F(s)\|\Delta s\right) \leq \gamma\|z_0\|e^{\gamma\beta}, \quad t \geq t_0\,.\end{aligned}$$

Since γ can be used for any t_0 and $z(t_0)$, the state equation (12) is uniformly stable. □

4. Main Results

The treatment of nonlinear systems can be restricted to pseudo-linearwise systems, also in appropriate cases to local behavior. In these situations, the following systems can be considered

$$x^{\Delta} = A(t)x(t) + F(t,x(t)). \tag{15}$$

It is obvious that for any and, by Theorem 1.6, the solution of (15) satisfies

$$x(t) = \Phi_A(t,t_0)x_0 + \int_{t_0}^{t} \Phi_A(t,\sigma(s))F(s,x(s))\Delta s, \quad t \geq t_0\,. \tag{16}$$

Theorem 4.1. *Suppose be bounded rd-continuous and regressive and assume the linear state (11) is uniformly stable. Then (15) is uniformly stable if the following conditions are satisfied:*

(i) $\|F(t,x(t))\| \leq r(t)\|x(t)\|$ *for some positive function* $r : T \to \mathbf{R}$ *and for all* $t \in [t_0,\infty)_T$.

(ii) $\int_{t_0}^{+\infty} r(s)\Delta s \equiv W_0 < \infty$.

Proof. For any t_0 and $x(t_0) = x_0$ by Theorem 1.7 the solution of (15) satisfies (16), where $\Phi_A(t,t_0)$ is transition matrix for system (11). By the uniform stability of (11) there exist a constant $\gamma_1 > 0$ such that $\|\Phi_A(t,\tau)\| \leq \gamma_1$, for all $t,\tau \in T$ with $t \geq \tau$. By taking the norms of both sides of (16) we have

$$\begin{aligned}
\|x(t)\| &\leq \gamma_1\|x_0\| + \int_{t_0}^{t} \gamma_1\|F(s,x(s))\|\Delta s\,, \quad \text{where } t \geq t_0 \\
&\leq \gamma_1\|x_0\| + \int_{t_0}^{t} \gamma_1 r(s)\|x(s)\|\Delta s\,, \quad \text{by Gronwall's inequality} \\
&\leq \gamma_1\|x_0\| e_{\gamma_1 r(s)}(t,t_0) \\
&= \gamma_1\|x_0\| \exp\left(\int_{t_0}^{t} \frac{\operatorname{Log}(1+\mu(s)\gamma_1\, r(s))}{\mu(s)}\Delta s\right) \\
&\leq \gamma_1\|x_0\| \exp\left(\int_{t_0}^{t} \gamma_1 r(s)\Delta s\right) \\
&\leq \gamma_1\|x_0\| e^{\gamma_1 w_0}\,.
\end{aligned}$$

Now if we get $\gamma = \gamma_1 e^{\gamma_1 w_0}$ and since γ can be used for any t_0 and $x(t_0)$ the equation (15) is uniformly stable.

Theorem 4.2. *Let A and F be the same as in Theorem 4.1 and (11) be asymptotically stable, and $\|F(t,x(t))\| = o(\|x(t)\|)$. When $\|x(t)\| \to 0$ uniformly in t,t_0 then (15) is asymptotically stable.*

Proof. The solution of $x(t)$ in (15) with $\|x(t_0)\|$ remains small, can be continued for increasing t so that $\|x(t)\|$ remains small. So long as $\|x(t)\|$ exists, it follows from (15) that

$$\|x(t)\| \leq \|e_A(t,t_0)\|\|x_0\| + \int_{t_0}^{t} \|e_A(t,\sigma(s))\|\|F(s,x(s))\|\Delta s. \tag{17}$$

Equation (5) is asymptotically stable, then there exist $\lambda,\gamma > 0$ with $-\lambda \in \mathcal{R}$ such that

$$\|\Phi_A(t,t_0)\| = \|e_A(t,t_0)\| \leq \gamma e_{-\lambda}(t,t_0) \tag{18}$$

using (17) and (18) yields

$$\|x(t)\| \leq \|\gamma\|x_0\|e_{-\lambda(t,t_0)} + \gamma \int_{t_0}^{t} \|e_{-\lambda}(t,\sigma(s))\|F(s,x(s))\|\Delta s. \tag{19}$$

Given $\varepsilon > 0$, there exist a $\delta > 0$ such that $\|F(s,x(s))\| \leq \varepsilon\frac{\|x(s)\|}{\gamma}$ for $\|x(t)\| < \delta$. Thus because of $\|x(t)\| < \delta$.,if follows from (19) that

$$e_\lambda(t,t_0)\|x(t)\| \leq \gamma\|x(t_0)\| + \varepsilon \int_{t_0}^{t} e_\lambda(s,t_0)\|x(s)\|\Delta s\,.$$

By Gronwall's inequality, we obtain

$$e_{\lambda}(t,t_0)\|x(t)\| \leq \gamma\|x(t_0)\|e_{\varepsilon}(t,t_0) \quad \text{or}$$

$$\|x(t)\| \leq \|\gamma\|x(t_0)\|e_{-(\lambda-\varepsilon)}(t,t_0), \quad \text{where } t \geq t_0 \tag{20}$$

If $\varepsilon > 0$ is chosen so that $\varepsilon < \delta$, Then (20) shows that $\|x(t)\| \leq \gamma\|x(t_0)\|$ so long as $\|x(t)\| \leq \delta$. Thus if $\|x(t)\| \leq \frac{\delta}{\gamma}$, it follows that (20) is valid for all $t \geq t_0$, which completes the proof of Theorem 4.2. □

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