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ON THE CONVERGENCE OF THE STRUCTURED PSB UPDATE IN HILBERT SPACE

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Abstract: In this paper we use a weaker version of the Newton-Kantorovich Theorem [1] to provide under weaker hypotheses and the same computational cost as in [6] a finer semilocal convergence analysis for the structured PSB Update in Hilbert space. Moreover the upper bounds on the distances involved are finer and the location of the solution as least as precise as in [6]. Our results extend the applicability of the update algorithm.

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The motivation and the definition of the quantities introduced in the algorithm as well as applications to optimal shape design can be found in the elegant paper by Laumen [6] (see also the references therein). Laumen used Theorem 3.2 given by Dennis in [4, p. 438] to provide his Newton-Kantorovich Type Theorem 2.2 upon which the semilocal convergence of the algorithm was based. In particular he justified the choice of the PSB update (Powell symmetric Broyden update),

$$B_+ = B + \left[(q - Bw) \otimes w + w \otimes (q - Bw)\right] / \langle v, w\rangle \\ - \left[\langle q - Bw, w\rangle\right] w \otimes w / \langle w, w\rangle^2 .$$

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In this paper we are motivated by optimization considerations. Using a recent result of ours [1, Theorem 2] (improving Theorem 3.2 in [4]) about a weaker version of the Newton-Kantorovich we provide under weaker in general hypotheses and the same computational cost as in Theorem 2.2 in [6] a finer semilocal convergence analysis for the above algorithm which enlarges the applicability of it. Moreover we provide finer error bounds on the distances $\|u_n - u_{n-1}\|$, $\|u_n - u^*\|$ $(n \geq 1)$ and at least as precise information on the location of the solution u^*.

1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution u^* of the minimization problem

$$\min_{u \in H} F(u) \tag{1}$$

using the algorithm [6]:

Structured Quasi-Newton method in Hilbert Space H.

Step 1. Given $u \in H$, $E \in L(H)$, $B = C(u) + E \in L(H)$.

Step 2. Compute w as the solution of

$$\langle Bw, v \rangle = \left\langle -F'(u), v \right\rangle, \quad \forall v \in H.$$

Step 3. Set $u_+ = u + w$.

Step 4. Choose $q^{\#}$ approximately.

Step 5. Set $q = C(u_+) + q^{\#}$.

Step 6. Update the quasi-Newton operator

$$E_+ = B\left(E, q^{\#}, w\right),$$

and set

$$B_+ = C(u_+) + E_+.$$

2. Semilocal Convergence Analysis

We state and prove the main semilocal convergence result for the structured PSB update.

Theorem 1. *Let H be a Hilbert space, and let $F'(\cdot) : U \subseteq H \to L(H)$ be Fréchet differentiable. Suppose there exist $u_0 \in U$ and parameters $\delta \in [0, 2)$, $\gamma, \rho, C_0, L^0_{F''}, L_{F''}, L_c$, such that*

$$\begin{aligned}
&B_0^{-1} = [C(u_0) + E_0]^{-1} \in L(H), \\
&\left\|B_0^{-1}\left(B_0 - F''(u_0)\right)\right\| \leq \gamma, \\
&\left\|B_0^{-1} F(u_0)\right\| \leq \rho, \\
&\left\|F''(u) - F''(u_0)\right\| \leq L^0_{F''}\|u - u_0\|, \quad \forall u \in U, \qquad (2)\\
&\left\|F''(u) - F''(w)\right\| \leq L_{F''}\|u - w\|, \quad \forall u, w \in U, \\
&\|C(u) - C(w)\| \leq L_C\|u - w\|, \quad \forall u, w \in U, \\
&\left\|q^{\#} - D(u_+)w\right\| \leq C_0\|w\|^2, \quad \forall u, w \in U, \\
&\left\|B_\eta - F''(u_\eta)\right\| \leq \left\|B_0 - F''(u_0)\right\| + (2C_0 + L_C + L_{F''})\sum_{j=1}^{\eta}\|u_j - u_{j-1}\|, \\
&h_\delta = \left(3L_{F''} + 4C_0 + 2L_C + L^0_{F''}\right)\rho \leq \delta - [2\gamma + (\gamma_0 + \gamma)\delta] \qquad (3)\\
&2\gamma + (\gamma_0 + \gamma)\delta \leq \delta, \qquad (4)
\end{aligned}$$

and

$$U\left(u_0, \frac{2\rho}{2-\delta}\right) = \left\{u : \|u - u_0\| \leq \frac{2\rho}{2-\delta}\right\} \subseteq U.$$

Then, the quasi-Newton method with structured PSB Update is well defined and converges to $u^* \in U\left(u_0, \frac{2\rho}{2-\delta}\right)$, where u^* is the unique solution of $F'(u) = 0$ in $U(u_0, t^*)$, where

$$\begin{aligned}
&t^* = \lim_{n\to\infty} t_n \leq \frac{2\rho}{2-\delta}, \\
&t_0 = 0, \qquad t_1 = \rho, \\
&t_{n+2} = t_{n+1} + \frac{1}{2a_n}\left[L_{F''}(t_{n+1} - t_n) + 2\gamma\right. \\
&\qquad\qquad \left. + 2(2C_0 + L_C + L_{F''})t_n\right](t_{n+1} - t_n)
\end{aligned}$$

and

$$a_n = 1 - \left[\gamma_0 + \gamma + \frac{2\rho}{2-\delta}\left(1 - \left(\frac{\delta}{2}\right)^{n+1}\right)\left(2C_0 + L_C + L_{F''} + L^0_{F''}\right)\right].$$

Moreover, the solution u^* is unique in $U^0(u_0, t_1^*)$, provided that

$$U^0(u_0, t_1^*) \subseteq U,$$

and

$$\frac{L^0_{F''}}{2}\left(t^* + t_1^*\right) \le 1.$$

Furthermore, the following estimates hold for all $n \ge 0$:

$$\|u_{n+1} - u_n\| \le t_{n+1} - t_n,$$

and

$$\|u_{n+1} - u^*\| \le t^* - t_n.$$

Proof. It follows immediately from Lemma 1 and Theorem 2 in Section 4.1 by simply replacing $b_n - \triangle, c_n - a_n, h^n_\delta, K_0, K_1, K, d, q_n$ given in [1] by $\gamma, a, h_\delta = \frac{(2L_{F''}+2C_0+L_C)\rho+2\gamma}{a}, L^0_{F''}, 2C_0+L_C+L_{F''}, \gamma, 1-a$ defined above respectively. □

Remark 2. Lemma 1 and Theorem 2 in [1] were shown under even weaker hypotheses. However in order for us to compare with Theorem 2.2 [6, p. 404] given below is preferred to provide only the above stated results.

Although the results in [6] were not given in affine invariant form we modify and present them here in such a way that they will be comparable to the corresponding ones in our Theorem 1 above, so that an equitable comparison can be made.

Theorem 3. (see [6, p. 404]) *Assume conditions of Theorem 1 but replace (2), (3), t^*, t_1^* by (5), (6) r^*, r_1^**

$$h = \frac{\left(4C_0 + 2L_C + 3L_{F''}\right)\rho}{\left(1-3\gamma\right)^2} \le \frac{1}{2}, \tag{5}$$

$$3\gamma < 1\,, \tag{6}$$

$$r^* = \frac{\left(1 - \sqrt{1-2h}\right)\left(1-3\gamma\right)}{4C_0 + 2L_C + 3L_{F''}}\,,$$

and

$$r_1^* = \frac{\left(1 - \sqrt{1-2h^1}\right)\left(1-\gamma\right)}{L_{F''}},$$

where

$$h^1 = \frac{C L_{F''}}{\left(1-\gamma\right)^2} \le \frac{1}{2},$$

respectively.

Then the conclusions of Theorem 1 hold in this setting.

Note that condition (2) is not used in Theorem 3. This allows a greater flexibility. On one hand Theorem 3 can be reduced to Theorem 1 if $L^0_{F''} = L_{F''}$.

However, in general

$$L^0_{F''} \leq L_{F''}$$

holds and $\frac{L_{F''}}{L^0_{F''}}$ can be arbitrarily large. Moreover, it can easily be seen (simply compare (5) with (3)) that condition (5) $\Longrightarrow$ (3), provided that (4) holds together with

$$\delta \in [\delta_0, 2),$$

and

$$4(\gamma_0 + 2\gamma) + (1 - 3\gamma)^2 < 4,$$

where

$$\delta_0 = \frac{4\gamma + (1 - 3\gamma)^2}{2[1 - (\gamma_0 + \gamma)]},$$

and ρ is sufficiently small.

Note also that in an even more general setting (see Theorem 2, Remark 1 in [1] and Theorem 3.2 in [4]) it was shown in [1] that $t^* \leq r^*$ and upper bounds on the distances $\|u_n - u_{n-1}\|$, $\|u_n - u^*\|$ are finer.

Finally note that all the above advantages are obtained under the same computational cost since in practice the computation of $L_{F''}$ requires that of $L^0_{F''}$.

Hence their usefullness in optimizing the convergence of the structured PSB update has been established.

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