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CHARACTERISATION OF $L^1 - L^p$ MULTIPLIERS FOR THE HEISENBERG GROUP

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Abstract: In this paper, the characterisation of multipliers for $L^1(H^n) \to L^p(H^n)$ is discussed, where H^n denotes the Heisenberg group. This result is also extended to multipliers for the vector valued functions from $L^1(H^n, A)$ into $L^p(H^n, A)$, where A is a commutative Banach algebra with a bounded approximate identity.

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1. Introduction

The characterisation of multipliers for certain classes of function spaces is an interesting problem in harmonic analysis. Fourier multipliers for $L^p(\mathbb{R}^n)$ has been studied by Hörmander [9] in 1960. The classical theorem of Hörmander is stated as follows: Suppose that $m(x)$ is class of C^k in the complement of the origin of $\mathbb{R}^n$, where k is an integer greater than $\frac{n}{2}$. Suppose $m(x)$ also satisfies

$$\left|\frac{\partial^\alpha}{\partial x^\alpha} m(x)\right| \leq B|x|^{-\alpha} \quad \text{whenever} \quad |\alpha| \leq k,$$

where $\alpha = (\alpha_1, \dots \alpha_n)$ with $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$. Then m is a multiplier for $L^p(\mathbb{R}^n)$, $1 < p < \infty$.

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On $\mathbb{R}^n$, the Fourier multipliers are nothing but translation invariant operators. Different types of characterisations of multipliers have been studied by various authors in the literature. In particular, the class of multipliers for $L^1(\mathbb{R}^n)$ is the class of Fourier transforms of finite measures on $\mathbb{R}^n$ and the class of multipliers for $L^2(\mathbb{R}^n)$ is the class of all bounded measurable functions on $\mathbb{R}^n$. However for $1 < p < \infty$, with $p \neq 2$, the sufficient condition is the classical Hörmander Theorem which is mentioned above. But the necessary condition is as follows: If m is a multiplier for $L^p(\mathbb{R}^n)$, then there exists a pseudomeasure σ such that $T_m f = \sigma * f$. These results are also proved for a general locally compact Abelian group G in place of $\mathbb{R}^n$. These are based on the work of Hörmander [9] and Gaudry [7]. In [7] Gaudry obtained a representation theorem through a quasi measure for an (L^p, L^q) multiplier $(1 < p, q < \infty)$. These results can be found in Larsen [12]. In [6], Figa-Talamanca proved that the class of multipliers coincide with the dual space of a function space which is the projective tensor product of L^p and L^q with $1 < p < \infty$. Later characterisations of Fourier multiplier have been extended to Segal algebras in [20].

Hörmander's Multiplier Theorem for Weyl transform has been studied by Mauceri [13] in 1980 and for Heisenberg group Fourier transform by Michele and Mauceri [14] in 1979. In 1998, it was proved by Radha and Thangavelu [16] that Weyl multiplier for $L^p(\mathbb{C}^n)$ corresponds to a twisted convolution operator using a 'pseudomeasure'. In [17], a similar result for an L^p - multiplier is proved for the group Fourier transform on the Heisenberg group H^n in a simple and direct method. The more general abstract result has already been established for amenable groups in [5] and [8]. We also refer to Cowling [3] for further details.

Characterisation of multipliers for vector valued functions has been studied by various authors. Tewari, Dutta and Vaidya studied multipliers for $M(L^1(G, A))$ where G is a locally compact Abelian group. Multipliers for the pair $(L^1(G, A), L^p(G, A))$ for G, a locally compact Abelian group was studied in [15]. In [10] the characterisation of multipliers for $L^1(H^n)$ and $L^1(H^n, A)$ are discussed where A is a commutative Banach algebra with a bounded approximate identity.

In this paper the characterisation of multipliers for $L^1(H^n) \to L^p(H^n)$ is discussed, where H^n denotes the Heisenberg group. This result is also extended to multipliers for the vector valued functions in $(L^1(H^n, A), L^p(H^n, A))$, where A is a commutative Banach algebra with a bounded approximate identity. It is important to mention here that unlike the Fourier transform on $\mathbb{R}^n$, the multiplier associated with the Heisenberg group Fourier transform is vector valued.

We organize our paper as follows. In Section 2, we provide the necessary background. In Section 3, we prove the main result and in Section 4 we extend the results to vector valued functions.

2. Preliminaries

Let H^n denote the Heisenberg group. It is a unimodular nilpotent Lie group whose underlying manifold is $\mathbb{C}^n \times \mathbb{R}$ and the group operation is defined by

$$(z,t)\cdot(w,s) = (z+w, t+s+\frac{1}{2}\mathrm{Im}\,(z\bar{w})).$$

The Haar measure is given by $dzdt$.

By Stone-Von Neumann Theorem, the only infinite dimensional unitary irreducible representations (up to unitary equivalence) are given by $\pi_\lambda,\ \lambda \in \mathbb{R}^*$, where π_λ is defined by

$$\pi_\lambda(z,t)\varphi(\xi) = e^{i\lambda t}e^{i\lambda(x\xi+\frac{1}{2}xy)}\varphi(\xi+y),$$

where $z = x+iy$ and $\varphi \in L^2(\mathbb{R}^n)$.

The group Fourier transform of $f \in L^1(H^n)$ is defined as

$$\hat{f}(\lambda) = \int_{H^n} f(z,t)\pi_\lambda(z,t)dzdt,\ \lambda \in \mathbb{R}^*. \tag{2.1}$$

Notice that $\hat{f}(\lambda) \in \mathcal{B}(L^2(\mathbb{R}^n))$, bounded operators on $L^2(\mathbb{R}^n)$ and $||\hat{f}(\lambda)||_{\mathcal{B}} \leq ||f||_{L^1(H^n)}$. As in the case of $\mathbb{R}^n$, the group Fourier transform $\hat{f}$ satisfies the basic properties, namely if $f \in L^1 \cap L^2(H^n)$, $\hat{f}(\lambda)$ is a Hilbert-Schmidt operator. Further if we define $d\mu(\lambda) = (2\pi)^{-n-1}|\lambda|^n d\lambda$, and if $L^2(\mathbb{R}^*, \mathcal{B}_2, d\mu)$ denotes the collection of square integrable $\mathcal{B}_2$-valued functions on $\mathbb{R}^*$ under the measure $d\mu$, then the group Fourier transform is an isometric isomorphism between $L^2(H^n)$ and $L^2(\mathbb{R}^*, \mathcal{B}_2, d\mu)$. Here $\mathcal{B}_2$ denotes the class of Hilbert-Schmidt operators on $L^2(\mathbb{R}^n)$. The inversion formula is given by

$$f(z,t) = \int tr(\pi_\lambda(z,t)^*\hat{f}(\lambda))\ d\mu(\lambda).$$

If $f, g \in L^1(H^n)$,

$$(f*g)(z,t) = \int f((z,t)(w,s)^{-1})\ g(w,s)\ dwds$$

denotes their convolution, then $(f*g)^\wedge(\lambda)=\hat{f}(\lambda)\hat{g}(\lambda)$. Under this convolution operation $L^1(H^n)$ becomes a non-commutative algebra. For further results of the group Fourier transform we refer to Thangavelu [18].

For $f\in L^p(H^n,A), g\in L^{p'}(H^n,A')$ $(\frac{1}{p}+\frac{1}{p'}=1, 1<p<\infty)$ we define a function $\langle f,g\rangle$ on H^n by setting

$$\langle f,g\rangle(t)=\langle f(t),g(t)\rangle=g(t)(f(t)),\ \forall\ t\in H^n. \tag{2.2}$$

Now $\langle f,g\rangle$ is well defined complex valued function on H^n and if we set L_g by

$$L_g(f)=\int_{H^n}\langle f,g\rangle(s)ds, \tag{2.3}$$

then L_g is a bounded linear functional on $L^p(H^n,A)$ and the mapping

$$g\longmapsto L_g$$

for $g\in L^{p'}(H^n,A')$ is an isometric isomorphism carrying $L^{p'}(H^n,A')$ onto a subspace of $L^p(H^n,A)'$. In other words $L^{p'}(H^n,A')$ is identified as a subspace of $L^p(H^n,A)'$. For a more detailed study of vector valued functions and vector measures, we refer to Diestel and Uhl [4].

3. $L^1\to L^p$ Multipliers for the Heisenberg Group

We first prove the following result.

Theorem 3.1. *If $T: L^1(H^n)\to L^p(H^n)$ is a bounded linear transformation where $1<p<\infty$, then the following statements are equivalent:*

1. T commutes with right translation operators, that is, $TR_s=R_sT$ for each $s\in H^n$, where $R_sf(t)=f(ts)$.

2. $T(f*g)=Tf*g$ for each $f,g\in L^1(H^n)$.

3. There exists a $g\in L^p(H^n)$ such that $Tf=g*f$ for each $f\in L^1(H^n)$. Moreover if $1<p\leq 2$ the above conditions are equivalent to the following

4. There exists a function $\phi\in L^{p'}(\mathbb{R}^*,\mathcal{B},d\mu)$ such that $(Tf)\hat{}\,(\lambda)=\phi(\lambda)\hat{f}(\lambda)$ for each $f\in L^1(H^n)$, $\lambda\in\mathbb{R}^*$ and $\frac{1}{p}+\frac{1}{p'}=1$.

Proof. (1) $\Rightarrow$ (2) In order to show $T(f*g)=Tf*g$ for $f,g\in L^1(H^n)$, it is enough to show $\langle Tf*g,\ h\rangle=\langle T(f*g),\ h\rangle$ for every $h\in L^{p'}(H^n)$. Let $T^*: L^{p'}(H^n)\to L^\infty(H^n)$ denotes the adjoint of T. Let $f,g\in L^1(H^n)$. Using the representation

$$\langle f,h\rangle=\int_{H^n}f(t)h(t^{-1})dt, \tag{3.1}$$

it is easy to show that

$$\langle T(f*g), h\rangle = \int_{H^n}\left[\int_{H^n} f(ts^{-1})g(s)ds\right]T^*h(t^{-1})dt.$$

By applying Fubini's Theorem and the assumption (1), it follows that

$$\langle T(f*g), h\rangle = \int_{H^n} g(s)\langle R_{s^{-1}}Tf, h\rangle ds.$$

Again using equation (2.3) we can show that $\langle Tf*g, h\rangle = \langle T(f*g), h\rangle\ \forall\ h \in L^{p'}(H^n)$.

(2) $\Rightarrow$ (3) Let $f \in L^1(H^n)$. Let $\{f_\alpha\} \subset L^1(H^n)$ be an approximate identity for $L^1(H^n)$ such that $\|f_\alpha\|_1 = 1$. Consider

$$\|Tf_\alpha * f - Tf\|_p = \|T(f_\alpha * f) - Tf\|_p \leq \|T\|\ \|f_\alpha * f - f\|_1 \tag{3.2}$$

tends to zero as $\alpha \to \infty$. Further $\|Tf_\alpha\|_p \leq \|T\|\ \|f_\alpha\|_1 = \|T\|$, $\{Tf_\alpha\}$ is a bounded subset of $L^p(H^n) = [L^{p'}(H^n)]^*$. Then by applying Banach-Alaoglu's Theorem, there exists a subnet $\{Tf_\beta\}$ of $\{Tf_\alpha\}$ and a $\phi \in L^p(H^n)$ such that $Tf_\beta \to \phi$ in the weak* topology. Now let $f, g \in C_C(H^n)$, the space of all complex valued continuous functions defined on H^n having compact support. Then using (3.1)and (3.2) we have

$$\langle Tf, g\rangle = \lim_\beta \langle Tf_\beta * f, g\rangle. \tag{3.3}$$

But it can be easily verified that

$$\begin{aligned}\lim_\beta \langle Tf_\beta * f, g\rangle = \lim_\beta Tf_\beta * f * g(e) = \lim_\beta \langle Tf_\beta, f*g\rangle \\ = \langle \phi, f*g\rangle = \langle \phi * f, g\rangle \quad \forall\ f, g \in C_C(H^n).\end{aligned}$$

As $C_C(H^n)$ is also dense in $L^{p'}(H^n)$ we get

$$\langle Tf, g\rangle = \langle \phi * f, g\rangle\ \ \forall\ \ g \in L^{p'}(H^n).$$

Thus $Tf = \phi * f\quad\forall\ \ f \in C_C(H^n)$. Further $C_C(H^n)$ is dense in $L^1(H^n)$, from which it follows that $Tf = \phi * f\quad\forall\ \ f \in L^1(H^n)$.

(3) $\Rightarrow$ (1) For $f \in L^1(H^n)$, we have $TR_sf = \phi * R_sf$ for some $\phi \in L^{p'}(H^n)$. Now using the definition of convolution and applying change of variables, we get $\phi * R_sf = R_s(\phi * f)$ which in turn shows (1).

(3) $\Rightarrow$ (4) By (3) it follows that there exists a $g \in L^p(H^n)$ such that $Tf = g * f\quad\forall\ \ f \in L^1(H^n)$. Notice here that $1 < p \leq 2$. Hence applying

Fourier transform on both sides, we get $(Tf)\hat{}\,(\lambda) = \hat{g}(\lambda)\hat{f}(\lambda) \quad \forall \;\lambda \in \mathbb{R}^*$. If we take $\phi = \hat{g}$, then applying Hausdorff-Young inequality (see [11]) we see that

$$\|\phi\|_{L^{p'}(\mathbb{R}^*,\mathcal{B},d\mu)} = \left\{ \int_{\mathbb{R}^*} \|\phi(\lambda)\|_{\mathcal{B}}^{p'} d\mu \right\}^{\frac{1}{p'}} = \|\hat{g}\|_{p'} \leq C\|g\|_p < \infty.$$

showing that $\phi \in L^{p'}(\mathbb{R}^*, \mathcal{B}, d\mu)$.

(4) $\Rightarrow$ (1) It follows considering the Fourier transform of TR_sf. By (4) $(TR_sf)\hat{} = \phi(R_sf)\hat{}$. A straightforward calculation shows that $(R_sf)\hat{}\,(\lambda) = \hat{f}(\lambda)\pi_\lambda(s^{-1})$ for each $s \in H^n$. On the other hand,

$$(R_sTf)\hat{}\,(\lambda) = (Tf)\hat{}\,(\lambda)\pi_\lambda(s^{-1}) = \phi(\lambda)\hat{f}(\lambda)\pi_\lambda(s^{-1}),$$

which proves our assertion using uniqueness of Fourier transform. □

Definition 3.2. Let $m \in L^\infty(\mathbb{R}^*, \mathcal{B}, d\mu)$. Define a linear transformation T_m on $L^1 \cap L^p(H^n)$, by $(T_mf)^\wedge = m\hat{f}$. We say that m is an $L^1 - L^p$ multiplier for H^n if $T_mf \in L^1(H^n)$ and T_m is bounded.

Corollary 3.3. *If m is a multiplier for $L^1(H^n)$ into $L^p(H^n)$, $1 < p < \infty$, then T_m is a right translation invariant operator.*

Proof. By using definition of multiplier, it can be easily shown that

$$[T_m(f * g)]\hat{} = [T_mf * g]\hat{} \quad \forall \quad f,\; g \in L^1 \cap L^2(H^n).$$

By uniqueness of Fourier transform we get $T_m(f * g) = T_mf * g \quad \forall \quad f,\; g \in L^1 \cap L^2(H^n)$. As $L^1 \cap L^2(H^n)$ is dense in $L^1(H^n)$, then by density argument we can obtain $T_m(f * g) = T_mf * g \quad \forall \quad f,\; g \in L^1(H^n)$. As m is a multiplier, T_m is bounded. Then it follows from Theorem 3.1 that T_m is right translation invariant. □

Remark 3.4. In case of $\mathbb{R}^n$, a multiplier m is equivalent to saying that T_m is a translation invariant operator. However, in the case of H^n we have proved only one part namely Corollary 3.3.

4. Multipliers for $(L^1(H^n, A), L^p(H^n, A))$

We assume in this section that A is a commutative Banach algebra. Let $L^p(H^n, A)$ denotes the space of all equivalence class of A-valued Bochner integrable functions defined on H^n with

$$\int_{H^n} \|f(s)\|_A^p ds < \infty$$

For $x, y \in A, f \in A'$, we define

$$(xof)(y) = f(yx), \quad \text{then } xof \in A' . \tag{4.1}$$

This o is used to define Aren's product (see [1]).

For $f \in L^1(H^n, A),\ g \in L^q(H^n, A')$ where A' is the dual space of A define

$$f \boxtimes g(x) = \int_{H^n} f(y^{-1}x) o g(y) dy \,,$$

where 'o' is defined in equation (4.1) above. Then

$$\|f \boxtimes g\|_{A'} \leq \int_{H^n} \|f(y^{-1}x)\|_A \|g(y)\|_{A'} dy.$$

By applying Hölders inequality, it is easy to show that

$$\|f \boxtimes g(x)\|_{A'} \leq \|f\|_{L^1(H^n,A)}^{\frac{1}{q'}} \left(\int_{H^n} \|f(y^{-1}x)\|_A \|g(y)\|_{A'}^q dy \right)^{\frac{1}{q}} .$$

Hence

$$\|f \boxtimes g\|_{L^q(H^n,A')}^q = \int_{H^n} \|f \boxtimes g(x)\|_{A'}^q dx$$
$$\leq \|f\|_{L^1(H^n,A)}^{\frac{q}{q'}} \int_{H^n} \|f(y^{-1}x)\|_A dx \int_{H^n} \|g(y)\|_{A'}^q dy = \|f\|_{L^1(H^n,A)}^{1+\frac{q}{q'}} \|g\|_{L^q(H^n,A')}^q$$

by making use of Fubini's Theorem. Thus

$$\|f \boxtimes g\|_{L^q(H^n,A')} \leq \|f\|_{L^1(H^n,A)} \|g\|_{L^q(H^n,A')}, \quad \text{if } f \in L^1(H^n, A), g \in L^q(H^n, A') \,,$$

and so

$$f \boxtimes g \in L^q(H^n, A') \text{ if } f \in L^1(H^n, A),\ g \in L^q(H^n, A'). \tag{4.2}$$

This helps us to extend the definitions of convolution to a function $f \in L^1(H^n, A)$ and a member $\sigma \in [L^{p'}(H^n, A')]'$. This is carried out as follows: For $f \in L^1(H^n, A),\ \sigma \in [L^{p'}(H^n, A')]'$ define

$$\langle h, \sigma \boxdot f\rangle = \sigma(\tilde{f} \boxtimes h), \quad \forall \quad h \in L^{p'}(H^n, A'). \tag{4.3}$$

Here p' is the conjugate exponent of p and $\tilde{f}(x) = f(x^{-1}) \quad \forall \quad x \in H^n$. As $\tilde{f} \in L^1(H^n, A), \tilde{f} \boxtimes h \in L^{p'}(H^n, A')$ by equation (4.2). Thus $\langle h, \sigma \boxdot f\rangle$, the duality relation is meaningful.

Remark 4.1. If $\sigma \in L^p(H^n, A)$ then $\sigma \boxdot f = \sigma * f$ for $f \in L^1(H^n, A)$.

In fact for $h \in L^{p'}(H^n, A')$,

$$\langle h, \sigma \boxdot f\rangle = \sigma(\tilde{f} \boxtimes h) \quad = \quad \int_{H^n} (\tilde{f} \boxtimes h)(x)(\sigma(x))\ dx$$

using duality relation (2.3). Then it follows from equation (4.3) and equation (4.1) that

$$\langle h, \sigma \boxdot f\rangle = \int_{H^n} \int_{H^n} h(y)(g(x)f(x^{-1}y))\ dydx.$$

Now applying change of variable we can see that the R.H.S. reduces to

$$\int_{H^n} h(y)(g * f(y))\ dy.$$

Since this is true for every $h \in L^{p'}(H^n, A')$, we get $\sigma \boxdot f = \sigma * f$ for $\sigma \in L^p(H^n, A)$.

Definition 4.2. Let A be a Banach algebra. Let X be a Banach space. Then X is said to be a Banach module over A if X is a module over A in the algebraic sense and if it satisfies $\|ax\|_X \leq \|a\|_A \|x\|_X, a \in A, x \in X$.

With these definitions, we can obtain the analogue of Theorem 3.1 in the vector version.

Theorem 4.3. *Let A be a commutative Banach Algebra with bounded approximate identity. Let $T : L^1(H^n, A) \to L^p(H^n, A)$ be a bounded linear operator. Then the following conditions are equivalent:*

1. $TR_s f = R_s T f,\ \ T(xf) = xTf \quad \forall \ \ x \in A, \forall f \in L^1(H^n, A)$ and for each $s \in H^n$.

2. $T(f * g) = Tf * g$ for each $f, g \in L^1(H^n, A)$.

3. There exists a $\sigma \in [L^{p'}(H^n, A')]'$ such that $Tf = \sigma \boxdot f$ for each $f \in L^1(H^n, A)$.

In addition if $1 < p \leq 2$ and A is a reflexive Banach space over which $\mathcal{B}(L^2(\mathbb{R}^n))$ is a module , then the above conditions are equivalent to the following:

4. There exists a function $\phi \in L^\infty(\mathbb{R}^*, \mathcal{B}, d\mu)$ such that $\widehat{(Tf)}(\lambda) = \phi(\lambda)\hat{f}(\lambda)$ for each $f \in L^1(H^n, A), \lambda \in \mathbb{R}^*$, and p' is the conjugate exponent of p.

Before proceeding with the proof, we first observe that for $f \in L^1(H^n, A)$, its Fourier transform is defined by

$$\hat{f}(\lambda) = \int_{H^n} f(z,t)\pi_\lambda(z,t)dzdt.$$

This definition is meaningful if A is a Banach space over which $\mathcal{B}(L^2(\mathbb{R}^n))$ is a module.

Proof. (1) $\Rightarrow$ (2) Since $L^1(H^n)\hat{\otimes}A$ is dense in $L^1(H^n, A)$, it is enough the result when $g = x\phi$ with $x \in A, \phi \in L^1(H^n)$. Then

$$T(f * g) = T(f * x\phi) = T(xf * \phi) = T(x(f * \phi)) = xT(f * \phi).$$

If we show that $T(f*\phi) = Tf*\phi$, then $T(f*g) = xTf*\phi = Tf*x\phi = Tf*g$. Thus it remains to show that $T(f * \phi) = Tf * \phi \quad \forall \quad f \in L^1(H^n, A)$. Now for every $h \in L^{p'}(H^n, A')$, consider

$$\begin{aligned}
\int_{H^n} h(t)(Tf * \phi(t))dt &= \int_{H^n} h(t)\left\{\int_{H^n} (\phi(s)Tf(ts^{-1}))ds\right\} dt \\
&= \int_{H^n}\int_{H^n} h(t)(\phi(s)Tf(ts^{-1}))dsdt \\
&= \int_{H^n}\int_{H^n} h(t)(\phi(s)R_{s^{-1}}Tf(t))dsdt \\
&= \int_{H^n} \phi(s)\int_{H^n} h(t)(TR_{s^{-1}}f(t))dtds \\
&= \int_{H^n}\int_{H^n} T^*h(t)(\phi(s)R_{s^{-1}}f(t))dsdt \\
&= \int_{H^n}\int_{H^n} T^*h(t)(\phi(s)R_{s^{-1}}f(t))dsdt \\
&= \int_{H^n} T^*h(t)\left\{\int_{H^n} \phi(s)R_{s^{-1}}f(t)ds\right\} dt \\
&= \int_{H^n} T^*h(t)\left\{\int_{H^n} \phi(s)f(ts^{-1})ds\right\} dt = \int_{H^n} h(t)(T(f*\phi)(t))dt\,.
\end{aligned}$$

Here T^* denotes the adjoint of T. Thus $T(f*\phi) = Tf*\phi\ \forall f \in L^1(H^n, A)$. (2) $\Rightarrow$ (3) Let $f \in L^1(H^n, A)$. Let $\{f_\alpha\} \subset L^1(H^n, A)$ be an approximate identity for $L^1(H^n)$ such that $\|f_\alpha\|_1 = 1$. Then proceeding as in the proof of Theorem 3.1, we obtain $\{Tf_\alpha\}$ to be a norm bounded subset of $L^p(H^n, A) \subseteq L^p(H^n, A'') = [L^{p'}(H^n, A')]'$. Then as an application of Banach-Alouglu's Theorem, we obtain a $\sigma \in [L^{p'}(H^n, A')]'$ such that $Tf_\beta \to \sigma$ in the weak* topology. Let $g \in C_C(H^n, A)$. Then as in (3.3) we get

$$\langle Tf, g\rangle = \lim_\beta \langle Tf_\beta * f, g\rangle\ .$$

On the other hand

$$\langle \sigma \boxdot f, g\rangle = \sigma(\tilde{f} \boxtimes g) = \lim_\beta Tf_\beta(\tilde{f} \boxdot g) = \lim_\beta \langle Tf_\beta \boxdot f, g\rangle.$$

As $Tf_\beta \in L^p(H^n, A)$, using Remark 4.1 we get

$$\langle \sigma \boxdot f, g\rangle = \lim_\beta \langle Tf_\beta * f, g\rangle.$$

We obtain the required result by using the density argument as in Theorem 3.1.

(3) $\Rightarrow$ (1) Let $s \in H^n$ and $f \in L^1(H^1, A)$ then for all $h \in L^{p'}(H^n, A')$, we observe that

$$h(t)(\sigma \boxdot R_s f(t)) = (R_{s^{-1}} h(t))(\sigma \boxdot f(t))$$

and hence

$$\int_{H^n} h(t)(TR_s f(t))dt = \int_{H^n} R_{s^{-1}} h(t)(Tf(t))dt \quad \forall \quad h \in L^{p'}(H^n, A').$$

Then it follows that $TR_s = R_s T\ \ \forall\ \ s \in H^n$ (see Bourbaki [2]). We also have from (4.1) $(xoh)(g(s)) = h(s)(xg(s))\ \ \forall\ \ g \in L^p(H^n, A),\ h \in L^{p'}(H^n, A'),\ x \in A$. Again it can be verified that

$$\tilde{f} \boxtimes (xoh(s)) = x\tilde{f} \boxtimes h\,,$$

which in turn imply

$$h(t)(x(\sigma \boxdot f(t))) = \sigma(x\tilde{f} \boxtimes h(t))$$

and

$$h(t)(x(\sigma \boxdot f(t))) = h(t)(\sigma \boxdot xf(t)).$$

Thus

$$\int_{H^n} h(t)(T(xf)(t))dt = \int_{H^n} h(t)(xT(f)(t))dt\ \forall\ \ h \in L^{p'}(H^n, A').$$

This shows that $T(xf) = xTf \ \forall \ \ x \in A, f \in L^1(H^n, A)$. Since A is assumed to be reflexive we have $[L^{p'}(H^n, A')]' = L^p(H^n, A)$. Then by condition (3) we have $Tf = \sigma * f$. Now define $\hat{\sigma} = \phi$, we obtain (4). As in Theorem 3.1 we can prove (4) $\Rightarrow$ (1). □

By extending the definition of multipliers for vector valued functions, as we mentioned in Remark 3.5, a multiplier for $(L^1(H^n, A), L^p(H^n, A))$ will lead to a translation invariant operator $T_m : L^1(H^n, A) \to L^p(H^n, A)$.

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