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CERTAIN APPLICATIONS OF DIFFERENTIAL SUBORDINATION AND SUPERORDINATION

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Abstract: In the present investigation we obtain sufficient conditions for certain subclass of multivalent analytic functions defined on open unit disk to be subordinated and superordinated by convex univalent functions. As a special case of this we obtain results involving Ruscheweyh derivative, Sălăgean derivative, Carlson-Shaffer operator, Dziok-Srivastava operator and multiplied transformation.

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1. Introduction

Let $\mathcal{A}_p$ denotes the class of functions analytic in open unit disc $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ and of the form

$$f(z) := z^p + \sum_{n=p+1}^{\infty} a_n z^n \tag{1.1}$$

and let $\mathcal{A} := \mathcal{A}_1$. For two functions $f(z)$ as defined in (1.1) and $g(z) := z^p + \sum_{n=p+1}^{\infty} b_n z^n$, Hadamard product or convolution of $f(z)$ and $g(z)$ denoted by $(f * g)(z)$ is defined by

$$(f * g)(z) := z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n.$$

For $\alpha_i \in \mathbb{C}, \quad (i = 1, 2, 3, \ldots)$ and $\beta_j \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}(j = 1, 2, \ldots)$ the Dziok-Srivastava linear operator [4] for functions in $\mathcal{A}_p$ is defined as

$$H_p^{l,m}[\alpha_1]f(z) := z^p + \sum_{n=p+1}^{\infty} \Gamma_n a_n z^n \,,$$

where

$$\Gamma_n := \frac{(\alpha_1)_{n-p}(\alpha_2)_{n-p} \ldots (\alpha_l)_{n-p}}{(\beta_1)_{n-p}(\beta_2)_{n-p} \ldots (\beta_m)_{n-p}(1)_{n-p}} \tag{1.2}$$

and $(\lambda)_n$ is the Pochhammer symbol defined by

$$(\lambda)_n := \begin{cases} 1\,, & n = 0\,, \\ \lambda(\lambda+1)(\lambda+2)....(\lambda+n-1)\,, & n = 1, 2, 3.... \end{cases}$$

By defining $g(z) := z^p + \sum_{n=p+1}^{\infty} \Gamma_n z^n$, we have $(f * g)(z) = H_p^{l,m}[\alpha_1]f(z)$. By taking $p = 1, l = 2, m = 1, \alpha_1 = a, \alpha_2 = 1$, and $\beta_1 = c$ we see that

$$(f * g)(z) := H_1^{2,1}[a]f(z) = L(a, c)f(z)\,,$$

where $L(a, c)f(z)$ denotes the Carlson-Shaffer linear operator, see [3]. On choosing $\frac{z^p}{(1-z)^{\lambda+p}}$; $(\lambda > -p + 1)$ and $z^p + \sum_{n=p+1}^{\infty} \left(\frac{n+\lambda}{p+\lambda}\right)^m a_n z^n$ as $g(z)$ we find $(f * g)(z)$ as $D^{\lambda+p-1}f(z)$ and $I_p(m, \lambda)f(z)$ respectively, where $D^{\lambda+p-1}$ and $I_p(m, \lambda)$ are respectively Ruscheweyh derivative operator of order $\lambda + p - 1$ [8] and Multiplier transformation. Note that $I_p(m, 0)f(z) = \mathcal{D}^m f(z)$ where $\mathcal{D}^m$ denotes the Sălăgean derivative of order m, see [9].

For two analytic functions f and F, we say F is *superordinate* to f, if f is subordinate to F. Let $\mathcal{H}$ denotes the class of functions analytic in open unit disc Δ and $\mathcal{H}[a,n]$ be the subclass $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots$.

Let $p, h \in \mathcal{H}$ and let $\phi(r,s,t;z) : \mathbb{C}^3 \times \Delta \to \mathbb{C}$. If $p(z)$ and $\phi(p(z), zp'(z), z^2p''(z); z)$ are univalent and if p satisfies the second order superordination

$$h(z) \prec \phi(p(z), zp'(z), z^2p''(z); z), \tag{1.3}$$

then $p(z)$ is the solution of the differential superordination (1.3). An analytic function $q(z)$ is called *subordinant*, if $q(z) \prec p(z)$ for all $p(z)$ satisfying (1.3). A univalent subordinant $\tilde{q(z)}$ that satisfies $q(z) \prec \tilde{q(z)}$ for all subordinants $q(z)$ of (1.3), is said to be *best subordinant*.

Recently Miller and Mocanu [6] considered certain first and second order differential superordinations. Using the results of Miller and Mocanu [6], Bulboacă have considered certain classes of first order differential superordinations [2] as well as superordination preserving integral operators [1].

In this present investigation we obtain sufficient conditions for functions in $\mathcal{A}_p$ to satisfy

$$q_1(z) \prec \frac{1}{p}\left(\frac{(f*g)(z)}{z^p}\right)^{\mu} \prec q_2(z),$$

where $q_1(z)$ and $q_2(z)$ are analytic functions defined on Δ with $q_1(0) = q_2(0) = 1$. Interestingly we get several well known results as special cases of our results.

2. Preliminaries

For the present investigation we need the following definition and results.

Definition 2.1. (see [6, Definition 2, p. 817]) Denote by $\mathcal{Q}$, the set of all functions $f(z)$ that are analytic and univalent in $\overline{\Delta} \setminus E(f)$, where

$$E(f) := \{\zeta \in \partial\Delta : \lim_{z\to\zeta} f(z) = \infty\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\Delta \setminus E(f)$.

Theorem 2.1. (cf. Miller and Mocanu [5, Theorem 3.4h, p. 132]) *Let $q(z)$ be univalent in Δ and θ and ϕ be analytic in a domain D containing $q(\Delta)$ with $\phi(w) \neq 0$, when $w \in q(\Delta)$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$. Suppose that:*

(i) $Q(z)$ is starlike univalent in Δ, and

(ii) $\Re\left\{\frac{zh'(z)}{Q(z)}\right\} > 0$ *for* $z \in \Delta$.

If p *is analytic in* Δ *with* $p(\Delta) \subseteq D$ *and*

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \tag{2.1}$$

then

$$p(z) \prec q(z)$$

and $q(z)$ *is the best dominant.*

Theorem 2.2. (see [2]) *Let* $q(z)$ *be univalent in* Δ *and* θ *and* ϕ *be analytic in domain* D *containing* $q(\Delta)$. *Suppose that:*

(i) $\Re\left(\frac{\theta'(q(z))}{\phi(q(z))}\right) \geq 0$ *for* $z \in \Delta$, *and*

(ii) $Q(z) = zq'(z)\phi(q(z))$ *is starlike univalent in* Δ.

If $p \in \mathcal{H}[q(0), 1] \cap \mathcal{Q}$ *with* $p(\Delta) \subseteq D$ *and* $\theta(p(z)) + zp'(z)\phi(p(z))$ *is univalent in* Δ, *and*

$$\theta(q(z)) + zq'(z)\phi(q(z)) \prec \theta(p(z)) + zp'(z)\phi(z), \tag{2.2}$$

then

$$q(z) \prec p(z)$$

and $q(z)$ *is the best subordinant.*

3. Applications of Differential Subordination

Throughout this paper we assume that α, β and γ are complex numbers and $\gamma \neq 0$.

Theorem 3.1. *Let* $q(z)$ *be a convex univalent function defined on* Δ *with* $q(0) = 1$ *and satisfies*

$$\Re\left\{\frac{\beta q(z)}{\gamma} - \frac{zq'(z)}{q(z)}\right\} > 0. \tag{3.1}$$

Let $f \in \mathcal{A}_p$ *and*

$$\psi(z) := \alpha + \frac{\beta}{p}\left(\frac{(f * g)(z)}{z^p}\right)^{\mu} + \gamma\mu\left[\frac{z(f * g)'(z)}{(f * g)(z)} - p\right]. \tag{3.2}$$

If

$$\psi(z) \prec \alpha + \beta q(z) + \frac{\gamma z q'(z)}{q(z)}, \tag{3.3}$$

then

$$\frac{1}{p}\left(\frac{(f*g)(z)}{z^p}\right)^{\mu} \prec q(z)$$

and $q(z)$ is the best dominant.

Proof. Define the function

$$\chi(z) := \frac{1}{p}\left(\frac{(f*g)(z)}{z^p}\right)^{\mu}. \tag{3.4}$$

A simple computation using (3.4) gives that

$$\frac{z\chi'(z)}{\chi(z)} = \frac{z\mu(f*g)'(z)}{(f*g)(z)} - p\mu.$$

Also we find that

$$\psi(z) := \alpha + \frac{\beta}{p}\left(\frac{(f*g)(z)}{z^p}\right)^{\mu} + \gamma\mu\left[\frac{z(f*g)'(z)}{(f*g)(z)} - p\right]$$

$$= \alpha + \beta\chi(z) + \frac{\gamma z\chi'(z)}{\chi(z)}. \tag{3.5}$$

In view of (3.5) the subordination (3.3) becomes

$$\alpha + \beta\chi(z) + \frac{\gamma z\chi'(z)}{\chi(z)} \prec \alpha + \beta q(z) + \frac{\gamma z q'(z)}{q(z)}$$

and this can be written as (2.1) when $\theta(w) := \alpha + \beta w$ and $\phi(w) := \frac{\gamma}{w}$. Note that $\theta(w)$ and $\phi(w)$ are analytic in $\mathbb{C}\setminus\{0\}$. Let the functions $Q(z)$ and $h(z)$ be defined as

$$Q(z) := zq'(z)\phi(q(z)) = \frac{\gamma z q'(z)}{q(z)}, \quad h(z) := \alpha + \beta q(z) + \frac{\gamma z q'(z)}{q(z)}.$$

In light of hypothesis of Theorem 2.1 we see that $Q(z)$ is starlike and

$$\Re\left\{\frac{zh'(z)}{Q(z)}\right\} = \Re\left\{\frac{\beta q(z)}{\gamma} - \frac{zq'(z)}{q(z)} + (1 + \frac{zq''(z)}{q'(z)})\right\} > 0.$$

Hence the result follows as an application of Theorem 2.1. □

By taking $g(z) = \frac{z^p}{1-z}$ in Theorem 3.1 we get the following result of Shanmugam et al [10].

Corollary 3.2. *Let $q(z)$ be convex univalent in Δ with $q(0) = 1$ and satisfies (3.1). Let*

$$\psi(z) := \alpha + \frac{\beta}{p}\left(\frac{f(z)}{z^p}\right)^{\mu} + \gamma\mu\left[\frac{zf'(z)}{f(z)} - p\right]. \tag{3.6}$$

Let $f(z) \in \mathcal{A}_p$ and if

$$\psi(z) \prec \alpha + \beta q(z) + \frac{\gamma z q'(z)}{q(z)}, \quad \textit{then} \quad \frac{1}{p}\left(\frac{f(z)}{z^p}\right)^{\mu} \prec q(z)\,,$$

where $q(z)$ is the best dominant.

By taking $g(z) = \frac{z}{1-z}, \alpha = p = 1, \beta = 0, \gamma = \frac{1}{\mu}$ and $q(z) = e^{\lambda A z}$, in Theorem 3.1 we get the following result obtained by Obradovic and Owa [7].

Corollary 3.3. *Let $f(z) \in \mathcal{A}$. If*

$$\frac{zf'(z)}{f(z)} \prec 1 + Az\,,$$

then

$$\left(\frac{f(z)}{z}\right)^{\mu} \prec e^{\lambda A z},$$

where $e^{\lambda A z}$ is the best dominant.

We remark here that $q(z) = e^{\lambda A z}$ is univalent if and only if $|\lambda A| < \pi$.

For a special case when $q(z) = \frac{1}{(1-z)^{2b}}$ where $b \in \mathbb{C} \setminus \{0\}$, and $g(z) = \frac{z}{1-z}, \alpha = \mu = p = 1, \beta = 0$ and $\gamma = \frac{1}{b}$ in Theorem 3.1, we have the following result obtained by the Srivastava and Lashin [11].

Corollary 3.4. *Let $0 \neq b \in \mathbb{C}$. If $f(z) \in \mathcal{A}$ and*

$$1 + \frac{1}{b}\left[\frac{zf'(z)}{f(z)} - 1\right] \prec \frac{1+z}{1-z}\,,$$

then

$$\frac{f(z)}{z} \prec \frac{1}{(1-z)^{2b}},$$

where $\frac{1}{(1-z)^{2b}}$ is the best dominant.

By taking $q(z) = (1 + Bz)^{\frac{\lambda(A-B)}{B}}, g(z) = \frac{z}{1-z}, \alpha = p = 1, \beta = 0$ and $\gamma = \frac{1}{\mu}$ in Theorem 3.1, then we have the following result of Obradovic and Owa [7].

Corollary 3.5. *Let $-1 \leq B < A \leq 1$. Let μ, A and B satisfy the relation either $\left|\frac{\lambda(A-B)}{B} - 1\right| \leq 1$ or $\left|\frac{\lambda(A-B)}{B} + 1\right| \leq 1$. If $f(z) \in \mathcal{A}$ and*

$$\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}, \quad \text{then} \quad \left(\frac{f(z)}{z}\right)^{\mu} \prec (1+Bz)^{\frac{\lambda(A-B)}{B}}$$

and $(1+Bz)^{\frac{\lambda(A-B)}{B}}$ is the best dominant.

4. Applications of Differential Superordination

Theorem 4.1. *Let $q(z)$ be convex univalent in Δ with $q(0) = 1$ and satisfies*

$$\Re\left\{\frac{\beta q(z)}{\gamma}\right\} > 0. \tag{4.1}$$

Let $\psi(z)$ given by (3.2) be univalent in Δ. If $f \in \mathcal{A}_p$ and $0 \neq \frac{1}{p}\left(\frac{(f*g)(z)}{z^p}\right)^{\mu}$ $\mathcal{H}[1,1] \cap \mathcal{Q}$, then

$$\alpha + \beta q(z) + \frac{\gamma z q'(z)}{q(z)} \prec \psi(z) \tag{4.2}$$

implies

$$q(z) \prec \frac{1}{p}\left(\frac{(f*g)(z)}{z^p}\right)^{\mu},$$

where $q(z)$ is the best subordinant.

Proof. In view of (3.5) the superordination in (4.2) becomes

$$\alpha + \beta q(z) + \frac{\gamma z q'(z)}{q(z)} \prec \alpha + \beta\chi(z) + \frac{\gamma z \chi'(z)}{\chi(z)}$$

and this can be written as (2.2), when $\theta(w) := \alpha + \beta w$ and $\phi(w) := \frac{\gamma}{w}$. Also

$$\Re\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\} = \Re\left\{\frac{\beta q(z)}{\gamma}\right\} > 0.$$

Hence the result follows as an applicaton of Theorem 2.2. □

By taking $g(z) = \frac{z^p}{1-z}$ in Theorem 4.1 we have the following result of Shanmugam et al [10].

Corollary 4.2. *Let $q(z)$ be convex univalent in Δ with $q(0) = 1$ and satisfies (4.1). Let $\psi(z)$ as defined by (3.6) univalent in Δ. If $f(z) \in \mathcal{A}_p, 0 \neq \frac{1}{p}\left(\frac{f(z)}{z^p}\right)^{\mu} \in \mathcal{H}[1,1] \cap \mathcal{Q}$ then*

$$\alpha + \beta q(z) + \frac{\gamma z q'(z)}{q(z)} \prec \psi(z) \quad \textit{implies} \quad q(z) \prec \frac{1}{p}\left(\frac{f(z)}{z^p}\right)^{\mu},$$

where $q(z)$ is the best subordinant.

By taking $g(z) = \frac{z}{1-z}, \alpha = p = 1, \beta = 0, \gamma = \frac{1}{\mu}$ and $q(z) = e^{\lambda A z}$, in Theorem 4.1 we get the following result.

Corollary 4.3. *Let $f \in \mathcal{A}$ and $0 \neq \left(\frac{f(z)}{z}\right)^{\mu} \mathcal{H}[1,1] \cap \mathcal{Q}$. If*

$$1 + Az \prec \frac{zf'(z)}{f(z)} \quad \textit{implies} \quad e^{\lambda A z} \prec \left(\frac{f(z)}{z}\right)^{\mu},$$

where $e^{\lambda A z}$ is the best subordinant.

For a special case when $q(z) = \frac{1}{(1-z)^{2b}}$, where $b \in \mathbb{C} \setminus \{0\}$, and $g(z) = \frac{z}{1-z}$, $\alpha = \mu = p = 1, \beta = 0$ and $\gamma = \frac{1}{b}$ in Theorem 4.1, we have the following result.

Corollary 4.4. *Let $0 \neq b \in \mathbb{C}$. If $f(z) \in \mathcal{A}$ and $0 \neq \frac{f(z)}{z} \in \mathcal{H}[1,1] \cap \mathcal{Q}$, then*

$$\frac{1+z}{1-z} \prec 1 + \frac{1}{b}\left[\frac{zf'(z)}{f(z)} - 1\right] \quad \textit{implies} \quad \frac{1}{(1-z)^{2b}} \prec \frac{f(z)}{z},$$

where $\frac{1}{(1-z)^{2b}}$ is the best subordinant.

By taking $q(z) = (1+Bz)^{\frac{\lambda(A-B)}{B}}, g(z) = \frac{z}{1-z}, \alpha = p = 1, \beta = 0$ and $\gamma = \frac{1}{\mu}$ in Theorem 4.1, then we have the following result.

Corollary 4.5. *Let $-1 \leq B < A \leq 1$. Let μ, A and B satisfy the relation either $\left|\frac{\lambda(A-B)}{B} - 1\right| \leq 1$ or $\left|\frac{\lambda(A-B)}{B} + 1\right| \leq 1$. If $f(z) \in \mathcal{A}$ and $0 \neq \left(\frac{f(z)}{z}\right)^{\mu} \in \mathcal{H}[1,1] \cap \mathcal{Q}$, then*

$$\frac{1+Az}{1+Bz} \prec \frac{zf'(z)}{f(z)} \quad \textit{implies} \quad (1+Bz)^{\frac{\lambda(A-B)}{B}} \prec \left(\frac{f(z)}{z}\right)^{\mu},$$

where $(1+Bz)^{\frac{\lambda(A-B)}{B}}$ is the best subordinant.

5. Sandwich Results

Now by combining Theorem 3.1 and Theorem 4.1 we get the following sandwich type result.

Theorem 5.1. *Let $q_1(z)$ and $q_2(z)$ be convex univalent in Δ with $q_1(0) = q_2(0) = 1$ and $q_1(z)$ satisfying (4.1) and $q_2(z)$ satisfying (3.1). Let $\psi(z)$ as given by (3.2) be univalent in Δ. If $f \in \mathcal{A}_p$ and $0 \neq \frac{1}{p}\left(\frac{(f*g)(z)}{z^p}\right)^\mu \in \mathcal{H}[1,1] \cap \mathcal{Q}$, then*

$$\alpha + \beta q_1(z) + \frac{\gamma z q_1'(z)}{q_1(z)} \prec \psi(z) \prec \alpha + \beta q_2(z) + \frac{\gamma z q_2'(z)}{q_2(z)}$$

$$\text{implies} \;\; q_1(z) \prec \frac{1}{p}\left(\frac{(f*g)(z)}{z^p}\right)^\mu \prec q_2(z)\,,$$

where $q_1(z)$ and $q_2(z)$ are respectively the best subordinant and best dominant.

By taking $g(z) = \frac{z^p}{1-z}$ in Theorem 5.1 we have the following result of Shanmugam et al [10].

Corollary 5.2. *Let $q_1(z)$ and $q_2(z)$ be convex univalent in Δ with $q_1(0) = q_2(0) = 1$ and $q_1(z)$ satisfying (4.1) and $q_2(z)$ satisfying (3.1). Let $\psi(z)$ as defined in (3.6) be univalent in Δ. If $f(z) \in \mathcal{A}_p, 0 \neq \frac{1}{p}\left(\frac{f(z)}{z^p}\right)^\mu \in \mathcal{H}[1,1] \cap \mathcal{Q}$, then*

$$\alpha + \beta q_1(z) + \frac{\gamma z q_1'(z)}{q_1(z)} \prec \psi(z) \prec \alpha + \beta q_2(z) + \frac{\gamma z q_2'(z)}{q_2(z)}$$

$$\text{implies} \;\; q_1(z) \prec \frac{1}{p}\left(\frac{f(z)}{z^p}\right)^\mu \prec q_2(z)\,,$$

where $q_1(z)$ and $q_2(z)$ are respectively the best subordinant and the best dominant.

By taking $g(z) := z^p + \sum_{n=p+1}^{\infty} \Gamma_n z^n$ in Theorem 5.1, where Γ_n is as defined in (1.2), we have the following result which involves Dziok-Srivastava operator.

Corollary 5.3. *Let $q_1(z)$ and $q_2(z)$ be convex univalent in Δ with $q_1(0) = q_2(0) = 1$ and $q_1(z)$ satisfying (4.1) and $q_2(z)$ satisfying (3.1). Let*

$$\psi(z) := \alpha + \frac{\beta}{p}\left(\frac{H_p^{l,m}[\alpha_1]f(z)}{z^p}\right)^\mu + \gamma\mu\left[\frac{\alpha_1 H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)} - \alpha_1\right]$$

be univalent in Δ. *If* $f \in \mathcal{A}_p$ *and* $0 \neq \frac{1}{p}\left(\frac{H_p^{l,m}[\alpha_1]f(z)}{z^p}\right)^{\mu} \in \mathcal{H}[1,1] \cap \mathcal{Q}$, *then*

$$\alpha + \beta q_1(z) + \frac{\gamma z q_1'(z)}{q_1(z)} \prec \psi(z) \prec \alpha + \beta q_2(z) + \frac{\gamma z q_2'(z)}{q_2(z)}$$
$$\text{implies} \;\; q_1(z) \prec \frac{1}{p}\left(\frac{H_p^{l,m}[\alpha_1]f(z)}{z^p}\right)^{\mu} \prec q_2(z)\,,$$

where $q_1(z)$ *and* $q_2(z)$ *are respectively the best subordinant and the best dominant.*

By taking $p = 1, l = 2, m = 1, \alpha_1 = a, \alpha_2 = 1$ and $\beta_1 = c$ in Corollary 5.3 we get the following result involving Carlson-Shaffer linear operator.

Corollary 5.4. *Let* $q_1(z)$ *and* $q_2(z)$ *be convex univalent in* Δ *with* $q_1(0) = q_2(0) = 1$ *and* $q_1(z)$ *satisfying (4.1) and* $q_2(z)$ *satisfying (3.1). Let*

$$\psi(z) := \alpha + \beta\left(\frac{L(a,c)f(z)}{z}\right)^{\mu} + \gamma\mu\left[\frac{aL(a+1,c)f(z)}{L(a,c)f(z)} - a\right]$$

is univalent in Δ.*If* $f \in \mathcal{A}$ *and* $0 \neq \left(\frac{L(a,c)f(z)}{z}\right)^{\mu} \in \mathcal{H}[1,1] \cap \mathcal{Q}$, *then*

$$\alpha + \beta q_1(z) + \frac{\gamma z q_1'(z)}{q_1(z)} \prec \psi(z) \prec \alpha + \beta q_2(z) + \frac{\gamma z q_2'(z)}{q_2(z)}$$
$$\text{implies} \;\; q_1(z) \prec \left(\frac{L(a,c)f(z)}{z}\right)^{\mu} \prec q_2(z)\,,$$

where $q_1(z)$ *and* $q_2(z)$ *are respectively the best subordinant and the best dominant.*

By taking $a = \delta + 1$ and $c = 1$ in Corollary 5.4 we get the following result involving Ruscheweyh derivative.

Corollary 5.5. *Let* $q_1(z)$ *and* $q_2(z)$ *be convex univalent in* Δ *with* $q_1(0) = q_2(0) = 1$ *and* $q_1(z)$ *satisfying (4.1) and* $q_2(z)$ *satisfying (3.1). Let*

$$\psi(z) := \alpha + \beta\left(\frac{D^{\delta}f(z)}{z}\right)^{\mu} + \gamma\mu\left[\frac{z(D^{\delta}f(z))'}{D^{\delta}f(z)} - 1\right]$$

is univalent in UD. *If* $f \in \mathcal{A}$ *and* $0 \neq \left(\frac{D^{\delta}f(z)}{z}\right)^{\mu} \in \mathcal{H}[1,1] \cap \mathcal{Q}$, *then*

$$\alpha + \beta q_1(z) + \frac{\gamma z q_1'(z)}{q_1(z)} \prec \psi(z) \prec \alpha + \beta q_2(z) + \frac{\gamma z q_2'(z)}{q_2(z)}$$

$$\text{implies} \;\; q_1(z) \prec \left(\frac{D^\delta f(z)}{z}\right)^\mu \prec q_2(z)\,,$$

where $q_1(z)$ and $q_2(z)$ are respectively the best subordinant and the best dominant.

By taking $g(z) = z^p + \sum_{n=p+1}^{\infty} \left(\frac{n+\lambda}{p+\lambda}\right)^m z^n$ in Theorem 5.1 we get the following result involving multiplier transformation.

Corollary 5.6. *Let $q_1(z)$ and $q_2(z)$ be convex univalent in Δ with $q_1(0) = q_2(0) = 1$ and $q_1(z)$ satisfying (4.1) and $q_2(z)$ satisfying (3.1). Let*

$$\psi(z) := \alpha + \frac{\beta}{p}\left(\frac{I_p(m,\lambda)f(z)}{z^p}\right)^\mu + \gamma\mu\left[\frac{(p+\lambda)I_p(m+1,\lambda)f(z)}{I_p(m,\lambda)f(z)} - \lambda - p\right]$$

is univalent in Δ. If $f \in \mathcal{A}_p$ and $0 \neq \frac{1}{p}\left(\frac{I_p(m,\lambda)f(z)}{z^p}\right)^\mu \in \mathcal{H}[1,1] \cap \mathcal{Q}$, then

$$\alpha + \beta q_1(z) + \frac{\gamma z q_1'(z)}{q_1(z)} \prec \psi(z) \prec \alpha + \beta q_2(z) + \frac{\gamma z q_2'(z)}{q_2(z)}$$

$$\text{implies} \;\; q_1(z) \prec \frac{1}{p}\left(\frac{I_p(m,\lambda)f(z)}{z^p}\right)^\mu \prec q_2(z)\,,$$

where $q_1(z)$ and $q_2(z)$ are respectively the best subordinant and the best dominant.

By taking $\lambda = 0$ and $p = 1$ in Corollary 5.6 we get the following result involving Sălăgean derivative operator.

Corollary 5.7. *Let $q_1(z)$ and $q_2(z)$ be convex univalent in Δ with $q_1(0) = q_2(0) = 1$ and $q_1(z)$ satisfying (4.1) and $q_2(z)$ satisfying (3.1). Let*

$$\psi(z) := \alpha + \beta\left(\frac{\mathcal{D}^m f(z)}{z}\right)^\mu + \gamma\mu\left[\frac{\mathcal{D}^{m+1} f(z)}{\mathcal{D}^m f(z)} - 1\right]$$

is univalent in Δ.If $f \in \mathcal{A}$ and $0 \neq \left(\frac{\mathcal{D}^m f(z)}{z}\right)^\mu \in \mathcal{H}[1,1] \cap \mathcal{Q}$, then

$$\alpha + \beta q_1(z) + \frac{\gamma z q_1'(z)}{q_1(z)} \prec \psi(z) \prec \alpha + \beta q_2(z) + \frac{\gamma z q_2'(z)}{q_2(z)}$$

$$\text{implies} \;\; q_1(z) \prec \left(\frac{\mathcal{D}^m f(z)}{z}\right)^\mu \prec q_2(z)\,,$$

where $q_1(z)$ and $q_2(z)$ are respectively the best subordinant and the best dominant.

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