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HYPERBOLIC GEODESICS IN QUASIDISKS

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Abstract: Let D be a Jordan proper subdomain of R^2 whose boundary contains at least three points, $D^* = \overline{R}^2 \setminus \overline{D}$, the exterior of D. In this paper, the authors prove that D is a quasidisk if and only if there exists a constant $c_0 \geq 1$ such that

$$l(\gamma) \leq c_0|z_1 - z_2|$$

for each hyperbolic geodesics γ in D and D^* with endpoints z_1 and z_2.

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1. Introduction

We shall assume throughout this paper that D is a Jordan proper subdomain of $\overline{R}^2$ whose boundary containing at least three points, $D^* = \overline{R}^2 \setminus \overline{D}$. For convenience we shall adopt the notation and terminology as in paper [13]. For $x \in R^2$ and $0 < r < \infty$, let $B^2(x,r) = \{z \in R^2 : |z - x| < r\}$, $\overline{B}^2(x,r)$ be the closure of $B^2(x,r)$, $B^2(r) = B^2(0,r)$ and $B^2 = B^2(1)$.

We say that D is a quasidisk if there exists a $K-$quasiconformal mapping $(K \geq 1)$ $f : \overline{R}^2 \to \overline{R}^2$ such that D is the image of the unit disk B^2 under f.

It is well-known that quasidisks play a very important role in quasiconformal mappings [12], complex dynamics [10], Fuchsian groups [9], Teichmüller space theory [15], etc. A lot of interesting geometric and analytic properties for quasidisks are obtained by many authors.

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The hyperbolic distance between two points x, y in D is given by

$$h_D(x,y) = \inf_{\gamma} \int_{\gamma} \frac{2|f'(z)|}{1-|f'(z)|^2}|dz|, \qquad (1)$$

where the infimum is taken over all rectifiable arcs γ joining x_1 to x_2 in D, and $f(z)$ is a conformal mapping which maps B^2 onto D. A hyperbolic geodesic is an arc γ for which the infimum in (1) is attained.

The hyperbolic geodesic is an important concept in many pure and applied mathematical branches. It has been used extensively in the research fields of Markovian process [2], partially hyperbolic system [6], conformal geometry [3, 14], modular group theory [1], etc.

The main purpose of the present paper is to use the hyperbolic geodesic to depict the geometric characteristics of quasidisks. We obtain the following result.

Theorem. *D is a quasidisk if and only if there exists a constant $c_0 \geq 1$ such that*

$$l(\gamma) \leq c_0|z_1 - z_2| \qquad (2)$$

for each hyperbolic geodesics γ in D and D^* with endpoints z_1 and z_2. Here $l(\gamma)$ is the Euclidean length of γ.

For convenience we need to define and introduce the following important concepts, they will be used in the next section.

Let $c \geq 1$ be a constant. (1) If for any $x_0 \in R^2$ and $0 < r < +\infty$, each pair of points $x, y \in D \cap \overline{B}^2(x_0, r)$ can be joined by an arc γ in $D \cap \overline{B}^2(x_0, cr)$, then we call D is a $c-$inner linearly locally connected domain, denoted by $D \in c - ILC$; (2) If for any $x_0 \in R^2$ and $0 < r < +\infty$, each pair of finite points $x, y \in D \setminus B^2(x_0, r)$ can be joined by an arc γ in $D \setminus B^2(x_0, r/c)$, then we call D is a $c-$outer linearly locally connected domain, denoted by $D \in c - OLC$.

D is called a linearly locally connected domain if $D \in c - ILC$ and $D \in c - OLC$ at the same time for some $c \geq 1$.

The following result was obtained by F.W. Gehring [7] in 1982.

Theorem A. *D is a quasidisk if and only if D is a linearly locally connected domain.*

Let $b > 0$ be a constant, D is called a $b-$cigar domain if each pair of points $x_1, x_2 \in D \setminus \{\infty\}$ can be joined by an arc $\gamma \subseteq D$ such that

$$\min_{j=1,2} \operatorname{diam}(\gamma(x_j, x)) \leq bd(x, \partial D) \qquad \text{for all} \qquad x \in \gamma, \qquad (3)$$

where $\gamma(x_j, x)$ is the part of γ between x_j and x, and $d(x, \partial D)$ is the Euclidean distance from x to ∂D, and $\operatorname{diam}(\gamma)$ is the Euclidean diameter of γ.

We say that D is a cigar domain if D is a $b-$cigar domain for some $b > 0$.

Y.M. Chu, G.D. Wang and X.H. Zhang proved the following two results in [5].

Theorem B. *If D is a b-cigar domain, then $D \in (2b+2)-OLC$.*

Theorem C. *If D^* is a b-cigar domain, then $D \in (16b+21)-ILC$.*

2. Lemmas and the Proof of Theorem

We shall first establish and introduce the following two lemmas, they are the key of the proof of the theorem.

Lemma 1. *If there exists a constant $c_0 \geq 1$ such that (2) holds for each hyperbolic geodesic γ in D with endpoints z_1, z_2 in D, then D^* is a $2a_0(c_0+1)$-cigar domain. In here a_0 is an absolute constant.*

Proof. For any $z, w \in D^* \setminus \{\infty\}$, let γ be the hyperbolic geodesic which joining z and w in D^*. For any $x \in \gamma \setminus \{z, w\}$, making use of Riemann's mapping theorem and the properties of hyperbolic geodesic in [4] we know that there exists conformal mapping f which maps B^2 onto D^* such that $f(0) = x$ and $f^{-1}(\gamma) = \{z : \text{Im}z = 0\}$. [11, Corollary 10.3] implies that there exist $x_1 \in \{z : |z| = 1,\ \text{Im}z > 0\}$ and $x_2 \in \{z : |z| = 1,\ \text{Im}z < 0\}$ such that $\alpha_j = f[0, x_j]$ is a rectifiable arc and

$$l(\alpha_j) < a_0 d(x, \partial D), \quad j = 1, 2, \tag{4}$$

where a_0 is an absolute constant and $[0, x_j]$ is the closed segment which joining 0 and x_j. Taking $y_j = f(x_j)$, then $y_j = \partial D$, $j = 1, 2$, and (4) yields

$$|y_1 - y_2| \leq l(\alpha_1) + l(\alpha_2) < 2a_0 d(x, \partial D). \tag{5}$$

The condition (2) in Lemma 2 and $\partial D = \partial D^*$ is a Jordan curve imply that

$$l(\beta) \leq c_0 |y_1 - y_2|, \tag{6}$$

where β is a hyperbolic geodesic which joining y_1 and y_2 in D.

Let $\alpha = \alpha_1 \cup \beta \cup \alpha_2$, then there exists component γ_j of $\gamma \setminus \{x\}$ such that γ_j is contained in the bounded complement component of α. (4), (5) and (6) yield

$$\begin{aligned} \text{diam}\,(\gamma_j) &\leq \text{diam}\,(\alpha) \leq l(\alpha_1) + l(\alpha_2) + l(\beta) \\ &< 2a_0(c_0+1) d(x, \partial D). \end{aligned} \tag{7}$$

(7) implies that D^* is a $2a_0(c_0+1)$-cigar domain.

Lemma 2. (see [8]) Suppose that D is a simply connected domain of R^2. If γ is a hyperbolic geodesic in D and α is any curve which joins the endpoints of γ in D, then

$$l(\gamma) \leq pl(\alpha), \tag{8}$$

where p is an absolute constant.

Theorem. D is a quasidisk if and only if there exists a constant $c_0 \geq 1$ such that

$$l(\gamma) \leq c_0|z_1 - z_2| \tag{9}$$

for each hyperbolic geodesic γ in D and D^* with endpoints z_1 and z_2.

Proof. ($\Rightarrow$) If D is a quasidisk, then by [7] we know that there exists a constant $a \geq 1$, for any $z_1, z_2 \in D$, there exists a rectifiable arc α which joins z_1 and z_2 in D such that

$$l(\alpha) \leq a|z_1 - z_2|. \tag{10}$$

If γ is a hyperbolic geodesic which joins z_1 and z_2 in D, then Lemma 2 implies

$$l(\gamma) \leq pl(\alpha). \tag{11}$$

(10) and (11) yields

$$l(\alpha) \leq c_0|z_1 - z_2|$$

with $c_0 = pa$ is a constant. Since D is a quasidisk if and only if D^* is a quasidisk, hence the same method as above is valid for D^*.

($\Leftarrow$) (9) and Lemma 1 imply that both D and D^* are $2a_0(c_0+1)$-cigar domain. Making use of Theorem B and Theorem C we know that $D \in (4a_0(c_0+1)+2) - OLC$ and $D \in (32a_0(c_0+1)+21) - ILC$, this and Theorem A imply that D is a quasidisk. □

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