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STABILITY OF IMPULSIVE FUNCTIONAL DIFFERENTIAL SYSTEMS IN BANACH SPACES

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Abstract: In this note, the stability issues of one class of first-order impulsive functional systems in a Banach space are addressed. Sufficient conditions are presented for stability using comparison principle and inequality analysis. An example is provided to illustrate the theory.

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1. Introduction

Although most dynamical systems are analyzed in either the continuous or discrete-time domain, many systems exhibit both continuous and discrete-time behaviors, see Aeyels [1]. For instance, many evolutionary processes, such as those biological phenomena that involve optimal control models in economics, stimulated neural networks, frequency-modulated systems, and some motions of

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missiles or aircraft, are characterized by that at certain moments of time there are abrupt changes of state. These perturbations usually act instantaneously, that is, in the form of impulses which cannot be well described by a pure continuous-time or pure discrete-time model Bainov et al [2], Deo et al [5]. Moreover, these impulsive phenomena can also be found in other fields such as information science, electronics, automatic control systems, etc., see Gelig et al [6]. Therefore, it is important and necessary to study impulsive dynamical systems. The problem of stability of solutions holds a very significant plan in the theory of impulsive differential equations. In recent years, the study of impulsive control systems has received an increasing interest. There are researches dealing with the fundamental issues such as stability for impulsive systems, see Benzaid et al [3], George et al [7], Lakshmikantham et al [8], Leela et al [9]. In this paper, we investigate the stability for a class of linear impulsive systems.

The paper is organized as follows. In Section 2, the impulsive systems to be dealt with are described and several new results associated with variation of parameters for time-varying impulsive control systems are derived. Sufficient conditions for state stability of such systems are established and there is presented an example for illustration. Finally, some conclusions are drawn in Section 3.

2. Main Result

We are concerned with the following impulsive functional differential system with time delay:

$$\begin{aligned} &\dot{x}(t) = Ax(t) + f(t, x(t), x(t-\tau)), \\ &t \in J = [t_0, T], \quad t \neq t_k, \quad k = 1, 2, \cdots M, \end{aligned} \tag{1}$$

$$\Delta x|_{t=t_k} = D_k x(t_k^-), \quad k = 1, 2, \cdots M, \tag{2}$$

$$x(t) = \varphi(t), \quad t \in [t_0 - \tau, t_0], \tag{3}$$

where the state variable $x(\cdot)$ takes values in Banach space X with the norm $\|\cdot\|$, A is the infinitesimal generator of strongly continuous semigroup of bounded linear operators $T(t)$ in X, and $f(t, x(t), x(t-\tau)) : J \times D \times D \to X$, and $D = \{\varphi : [t_0 - \tau, t_0] \to X, \varphi(t)$ is continuous everywhere except a finite number of points $\tilde{t}$ at which $\varphi(\tilde{t}^-), \varphi(\tilde{t}^+)$ exist and $\varphi(\tilde{t}^-) = \varphi(\tilde{t})\}, I_k : X \to X, \Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$, for all $k = 1, 2, \cdots M$, $t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T, \varphi : [t_0 - \tau, t_0] \to X$.

Define $J_0 = [0, t_1], J_k = (t_k, t_{k+1}], k = 1, 2, \cdots, m$. Let $I \subset R$ be an interval. We define the following classes of functions:

$PC(I, X) = \{x : I \to X : x(t)$ is continuous everywhere except for some t_k at which $x(t_k^-)$ and $x(t_k^+)$ exist and $x(t_k^-) = x(t_k), k = 1, \cdots, m\}$.

For $x \in PC(I, X)$, take $\|x\|_{PC} = \sup_{t\in R} |x(t)|$, then $PC(I, X)$ is a Banach space.

Denote: $\|x\|_\tau = \sup_{0<s<\tau} \|x(t-s)\|$.

First, we give some definitions to formulate the problem concerned in this section clearly.

Definition 1. A solution $x(\cdot) \in PC([-r, b], X)$ is said to be a mild solution of (1)- (3) if $x(t) = \varphi(t)$ on $[-r, 0]; \Delta x|_{t=t_k} = D_k x(t_k^-), k = 1, \cdots, m$. The restriction of $x(\cdot)$ to the interval $J_k (k = 0, \cdots, m)$ is continuous and the following integral equation is verified:

$$x(t) = T(t)\varphi(0) + \int_0^t T(t-s) f(s, x, x_s) ds + \sum_{0<t<t_k} T(t - t_k) D_k x(t_k^-),$$

where $t \in J$.

Definition 2. The system is said to be exponential stable on the interval J if for every initial function $\varphi \in PC([-r, 0], X)$ and $x_1 \in X$, there exists a control $u \in L^2(J, U)$ a positive number $\alpha > 0$, such that the mild solution $x(t)$ of (1)-(3) satisfies $\|x(t)\|_{PC} \le \|x(t)\|_\tau \exp(-\alpha(t - t_0))$.

Now, we will introduce some mathematical preliminaries as the basic tools for the discussion in the remaining parts of the paper.

Lemma 3. *Let A be the infinitesimal generator of strongly continuous semigroup of bounded linear operators $T(t)$, consider the following system:*

$$\dot{x}(t) = Ax(t) + f(t, x(t), x(t-\tau)) , \tag{4}$$

$$x(t) = \varphi(t), \quad t \in [t_0 - \tau, t_0] . \tag{5}$$

Assume $T(t)$ is exponentially stable. If there exists positive numbers $\gamma > 0$ and $M > 0$, continuous functions $\alpha_1(t), \alpha_2(t) \in C(R)$ such that $\|T(t)\| \le M \exp(-\gamma(t - t_0))$ and $\|f(t, x(t), x(t-\tau))\| \le \alpha_1(t) \|x(t)\| + \alpha_2(t) \|x(t-\tau)\|$ and $\gamma > M[\alpha_1(t) \|x(t)\| + \alpha_2(t) \|x(t-\tau)\|]$, then there is an $\alpha > 0$, such that

$$\|x(t)\| \le \|x(t)\|_\tau \exp(-\alpha(t - t_0)).$$

Proof. Since $\gamma > M[\alpha_1(t)\,\|x(t)\| + \alpha_2(t)\,\|x(t-\tau)\|]$ there exists an $\alpha > 0$, such that

$$\gamma - \alpha - M[\alpha_1(t)\,\|x(t)\| + \alpha_2(t)\,\|x(t-\tau)\|] > 0\,.$$

By the method of the variation of parameters, the solution of (4) and (5) is given by

$$x(t) = T(t)x(t_0) + \int_{t_0}^{t} T(t-s)f(s,x(s),x(s-\tau))ds$$

and

$$\begin{aligned}\|x(t)\| \le \|T(t)\|\,\|x(t_0)\| + \int_{t_0}^{t} \|T(t-s)\|\,\|f(s,x(s),x(s-\tau))\|\,ds \\ \le M\exp(-\gamma(t-t_0))\,\|x_{t_0}\|_\tau \\ + \int_{t_0}^{t} M\exp(-\gamma(t-s))[\alpha_1(s)\,\|x(s)\| + \alpha_2(s)\,\|x(s-\tau)\|]ds\,.\end{aligned}$$

Let

$$\begin{aligned}Q(t) = \exp(\alpha(t-t_0))[M\exp(-\gamma(t-t_0))\,\|x_{t_0}\|_\tau \\ + \int_{t_0}^{t} M\exp(-\gamma(t-s))[\alpha_1(s)\,\|x(s)\| + \alpha_2(s)\,\|x(s-\tau)\|]ds]\end{aligned}$$

for $t \ge t_0$, and $Q(t) \le M\,\|x_{t_0}\|_\tau$ for $t \in [t_0-\tau, t_0]$. Then

$$Q(t) \ge \|x(t)\|\,, t \ge t_0 - \tau \tag{6}$$

and

$$\begin{aligned}Q'(t) &\le \alpha Q(t) - \gamma Q(t) \\ &\quad + \exp\alpha(t-t_0)M\exp(-\gamma(t-t_0))[\alpha_1(t)\,\|x(t)\| + \alpha_2(t)\,\|x(t-\tau)\|] \\ &= (\alpha-\gamma)Q(t) + M\exp(-\gamma(t-t_0))[\alpha_1(t)\,\|x(t)\| + \alpha_2(t)\,\|x(t-\tau)\|] \\ &\le (\alpha-\gamma)Q(t) + M[\alpha_1(t)\,\|Q(t)\| + \alpha_2(t)\,\|Q(t-\tau)\|]\exp(\alpha\tau)\end{aligned}$$

in view of (6), for any $k > 1$, we claim that $Q(t) < K\,\|Q_{t_0}\| = L$, $L > 0$, $t \ge t_0 - \tau$.

If not, since $Q(t)$ is continuous, then there exists $t^* > t_0$, such that $Q(t^*) = L$, $Q(t) < L$, $t_0 - \tau \le t \le t^*$, and $Q(t^*) \ge 0$. On the other hand,

$$Q'(t^*) \le (\alpha-\gamma)Q(t^*) + M[\alpha_1(t^*)\,\|Q(t^*)\| + \alpha_2(t^*)\,\|Q(t^*-\tau)\|]\exp(\alpha\tau)$$

$$\leq (\alpha-\gamma)L + M[\alpha_1(t^*)L + \alpha_2(t^*)L]\exp(\alpha\tau)$$
$$\leq [\alpha-\gamma+M(\alpha_1(t^*)+\alpha_2(t^*))\exp(\alpha\tau)]L < 0\,.$$

This contradiction implies that $Q(t) < L$. Let $K \to 1$, then $Q(t) \leq \|Q(t_0)\|$ and

$$\|x(t)\| \leq Q(t)\exp(-\alpha(t-t_0)) \leq \|Q_{t_0}\|\exp(-\alpha(t-t_0))$$
$$= M\,\|x_{t_0}\|_\tau \exp(-\alpha(t-t_0))\,, \quad t \geq t_0-\tau\,.$$

The proof is completed. □

The result is useful when stability is studied since the result gives some estimation of the solution.

Theorem 4. *Assume that* $t_{k+1}-t_k \geq \tau, \gamma > 0$ *and* $M > 0$ *continuous functions* $\alpha_1(t), \alpha_2(t) \in C(X)$ *such that* $\|T(t)\| \leq M\exp(-\gamma(t-t_0))$ *and* $\|f(t,x(t),x(t-\tau))\| \leq \alpha_1(t)\,\|x(t)\| + \alpha_2(t)\,\|x(t-\tau)\|$ *and* $\gamma > M[\alpha_1(t)\,\|x(t)\| + \alpha_2(t)\,\|x(t-\tau)\|]$, *then there is an* $\alpha > 0$, *such that*

$$\|x(t)\| \leq \|x(t)\|_\tau \exp(-\alpha(t-t_0)).$$

Proof. From the above lemma, it follows that for any

$$k, \|x(t)\| \leq M\,\|x(t)\|_\tau \exp(-\alpha(t-t_k))$$

for $t \in [t_k, t_{k+1}]$

Since $x(t_k) = D_k x(t_k^-)$, then

$$\|x(t_k)\|_\tau = \sup_{t_k-\tau\leq t\leq t_k} \|x(t)\| \leq \max\{\sup_{t_k-\tau\leq t\leq t_k} \|x(t)\|\,, \|x(t_k)\|\}$$
$$\leq \max\{\sup_{t_k-\tau\leq t\leq t_k} \|x(t)\|\,, \|D_k\|\,\left\|x(t_k^-)\right\|\}$$
$$\leq \max\{\sup_{t_k-\tau\leq t\leq t_k} M\,\|x(t_{k-1})\|_\tau \exp(-\alpha(t-t_{k-1})),$$

$$M\,\|D_k\|\,\|x(t_{k-1})\|\exp(-\alpha(t_k-t_{k-1}))$$
$$\leq M\,\|x(t_{k-1})\|_\tau \exp(-\alpha(t_k-t_{k-1}))\max\{\exp\alpha\tau, \|D_k\|\}\,.$$

Using a singular argument, we know that for $t_k \leq t < t_{k+1}$:

$$\|x(t_{k-1})\|_\tau \leq M\,\|x(t_{k-2})\|_\tau \exp(-\alpha(t_{k-1}-t_{k-2}))$$
$$\times \max\{\exp\alpha\tau, \|D_{k-1}\|\}(k \in M)\,.$$

As a result, it follows that:

$$\|x(t)\| \leq \|x_{t_k}\|_\tau \exp(-\alpha(t-t_k)) \leq M^2 \|x_{t_k-1}\|_\tau \exp(-\alpha(t-t_k)) \times \max\{\exp \alpha\tau, \|D_k\|\} \leq \cdots \leq \|x_{t_0}\|_\tau M^{k+1} \prod_{j=1}^{k} \max\{\exp \alpha\tau, \|D_j\|\} \exp(-\alpha(t-t_0)) .$$

The proof is completed. □

Example 5. Consider the following partial differential equation of the form $\frac{\partial}{\partial t} z(x,t) = \frac{\partial^2}{\partial^2 t} z(x,t) + q(t, z(x,t-r)), 0 \leq x \leq \pi, t \in J = [0,b], t \neq t_k, k = 1, \cdots, m,\ z(x, t_k^+) - z(x, t_k^-) = D_k z(x, t_k^-), k = 1, \cdots, m,\ z(0,t) = z(\pi, t) = 0, t \geq 0,\ z(x,t) = \varphi(x,t), -r \leq t \leq 0$.

Take $X = L^2[0,\pi]$ and define $w(t) = z(\cdot, t)$. Let $f(t, w_t)(x) = q(t, z(x, t-r)), 0 \leq x \leq \pi$. Define $A : X \to X$ by $Aw = w^n$ with the domain $D(A) = \{w \in X : w, w^{'}$ absolutely continuous, $w'' \in X, w(0) = w(\pi) = 0\}$, then $Aw = \sum_{n=1}^{\infty} n^2 \langle w, w_n\rangle w_n, w \in D(A)$, where $w_n(s) = \sqrt{\frac{2}{\pi}} \sin ns,\ n = 1, 2, 3, \cdots$ is the orthogonal set of eigenvectors of A. It is easy to prove that A is the infinitesimal generator of an analytic semigroup $T(t), t \geq 0$ in X as is given by $T(t)w = \sum_{n=1}^{\infty} \exp(-n^2 t)\langle w, w_n\rangle w_n, w \in X$. Set $\sup_{t \geq t_0} \|f(t, x(t), x(t-\tau))\| < 1$, since the analytic semigroup $T(t)$ is compact there exist constants $M = 1, \gamma = 1$, continuous functions $\alpha_1(t), \alpha_2(t) \in C(R)$ such that

$$\|T(t)\| \leq M \exp(-x(t-t_0))$$

and

$$\|f(t, x(t), x(t-\tau))\| \leq \alpha_1(t) \|x(t)\| + \alpha_2(t) \|x(t-\tau)\|$$

and

$$\gamma > M[\alpha_1(t) \|x(t)\| + \alpha_2(t) \|x(t-\tau)\|].$$

Now, all the conditions of Theorem 4 are satisfied. Hence, system is stable on J.

3. Conclusion

This paper has studied the stability of a class of delay impulsive systems. Sufficient criteria for stability has been established using comparison principle and inequality analysis.

References

[1] D. Aeyels, Local and global controllability for nonlinear systems, *Systems Control Lett.*, **5**, No. 1 (1984), 19-26.

[2] D. Bainov, P.S. Simeonov, *Stability Theory of Differential Equations with Impulse Effects: Theory and Applications*, Ellis Horwood, Chichester (1989).

[3] Z. Benzaid, M. Sznaier, Constrained controllability of linear impulse differential systems, *IEEE Trans. Automat. Control*, **39**, No. 5 (1994), 1064-1066.

[4] D. Chung, C. Park, J.G. Lee, Robustness of controllability and observability of continuous linear time-varying systems with parameters perturbations, *IEEE Trans. Automat. Control*, **44**, No. 10 (1999), 1919-1923.

[5] S. Deo, S.G. Pandit, *Differential Systems Involving Impulses*, Springer, New York (1982).

[6] A. Gelig, A.N. Churilov, *Stability and Oscillations of Nonlinear Pulse-Modulated Systems*, Birkhäuser, Boston (1998).

[7] R. George, A.K. Nandakumaran, A. Arapostathis, A note on controllability ofimpulsive systems, *J. Math. Anal. Appl.*, **241** (2000), 276-283.

[8] V. Lakshmikantham, S. Deo, *Method of Variation of Parameters for Dynamic Systems*, Gordon and Breach Science Publication, London (1998).

[9] S. Leela, F. McRae, S. Sivasundaram, Controllability of n-impulsive differential equations, *J. Math. Anal. Appl.*, **177** (1993), 24-30.