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ALL STANDARD FRAMES IN FINITELY OR COUNTABLY HILBERT C^*-MODULES

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Abstract: In this paper we present some methods for generating standard frames of a finitely or countably generated Hilbert C^*-module H over a unital C^*-algebra A by using adjointable module maps on H and Hilbert A-module $l_2(A)$. Finally, we characterize all standard frames of H.

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1. Introduction

Frames in finitely or countably generated Hilbert C^*-modules over unital C^*-algebras as generalization of frames in separable Hilbert spaces, in a very efficient way, was investigated in joint works of M. Frank and D.R. Larson in [3, 4, 5]. These concepts follow the line of researches of D. Han and D.R. Larson in [6]. However, proofs of these concepts are more difficult for Hilbert C^*-modules than separable Hilbert spaces, fortunately, almost all results re-obtained in Hilbert C^*-modules.

Instead of using the concept of frames in Hilbert C^*-modules and C^*-algebras to generalize wavelet, Gabor frame theory and multi-resolution analysis (cf. [2,5]) it has been used in some other fields in C^*-algebras (cf. [4,5]).

The existence of standard frames in arbitrary finitely or countably generated Hilbert C^*-modules was shown in Theorem 3.2 of [3]. In this paper, after introducing some basic properties of frames in a finitely or countably Hilbert

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C^*-module H over unital C^*-algebra A, in Section 3, using adjointable module maps on H and on Hilbert A-module $l_2(A)$, we construct some frames of H. Then, in Section 4, we show how any two standard frames of H are related with each other.

Since the fundamental difference between Hilbert spaces and Hilbert C^*-modules is the more complicated structure of C^*-algebras in comparison with the field of complex numbers $\mathbb{C}$, then the generalization of almost all theorems and proofs is different from Hilbert spaces. We refer to [1] for Hilbert space situations.

2. Preliminaries

Let A be a unital C^*-algebra, H be a finitely or countably generated Hilbert C^*-module, and $\mathbb{J}$ be a finite or countable index subset of $\mathbb{N}$. A sequence $\{x_j : j \in \mathbb{J}\}$ of H is said to be a *standard frame* if there are positive constants C and D such that

$$C\langle x, x\rangle \leq \sum_{j=1}^{\infty} \langle x, x_j\rangle\langle x_j, x\rangle \leq D\langle x, x\rangle\,, \tag{1}$$

for every $x \in H$ and the sum in the middle of the inequality (1) converges in norm. The optimal constants (i.e. maximal for C and minimal for D) are called the *frame bounds*. A frame is *standard normalized tight* if $C = D = 1$.

Suppose that $\{x_j : j \in \mathbb{J}\}$ is a standard frame of H. The corresponding frame transform

$$\theta : H \to l_2(A), \quad \theta(x) = \{\langle x, x_j\rangle\}_{j\in\mathbb{J}}\,, \tag{2}$$

is an ajointable module map (cf. [8], p. 234) that embed H onto an orthogonal summand of $l_2(A)$, such that

$$l_2(A) = \{a = \{a_i\}_{i\in\mathbb{N}} : \text{for every } i \in \mathbb{N}, a_i \in A \text{ and } \sum_{i\in\mathbb{N}} a_i a_i^* \text{ converges in } ||.||_A\}.$$

Moreover, $\theta^*(e_j) = x_j$ holds, for every $j \in \mathbb{J}$, where $\{e_j : j \in \mathbb{J}\}$ is the standard basis for $l_2(A)$ and we have

$$\sum_{j\in\mathbb{J}} \langle x, x_j\rangle\langle x_j, x\rangle = \langle \{\langle x, x_j\rangle\}_{j\in\mathbb{J}}, \{\langle x, x_j\rangle\}_{j\in\mathbb{J}}\rangle = \langle \theta(x), \theta(x)\rangle\,, \tag{3}$$

for every $x \in H$ (cf. [5], Theorem 4.4). Also by Theorem 6.1 of [5], there exists a unique positive, invertible and adjointable module map $S = (\theta^*\theta)^{-1}$ on H, the *frame operator*, such that the reconstruction formula

$$x = \sum_{j \in \mathbb{J}} \langle x, S(x_j) \rangle x_j \tag{4}$$

is valid for every $x \in H$. The sequence $\{S(x_j) : j \in \mathbb{J}\}$ is a standard frame for H and it is called the canonical dual frame of the frame $\{x_j : j \in \mathbb{J}\}$.

Throughout the present paper, we denote the set of all adjointable maps on H by $\mathcal{L}(H)$. If $T \in \mathcal{L}(H)$ then T and T^* are module maps which are bounded (cf. [8], Lemma 15.2.3). Furthermore, we assume that A is a unital C^*-algebra and H is a finitely or countably generated Hilbert A-module.

3. Construction of Standard Frames of H

Let $\{x_j : j \in \mathbb{J}\}$ be a standard frame of H and $T \in \mathcal{L}(H)$. We set $y_j = Tx_j$ for every $j \in \mathbb{J}$. We want to find a condition for T to make $\{y_j : j \in \mathbb{J}\}$ a standard frame for H.

In the following theorem we give a necessary and sufficient condition for $\{y_j = Tx_j : j \in \mathbb{J}\}$ to be a standard frame of $\overline{T(H)}$.

Theorem 1. *Let $\{x_j : j \in \mathbb{J}\}$ be a standard frame of H with bounds $0 < C \leq D$ and T be a module map in $\mathcal{L}(H)$.Then the following conditions are equivalent:*

(i) The sequence $\{y_j : j \in \mathbb{J}\}$ is a standard frame of $\overline{T(H)}$.

(ii) There exists a positive constant k such that T^*, the adjoint of T, satisfies

$$k\langle x, x \rangle \leq \langle T^*x, T^*x \rangle \,, \tag{5}$$

for every $x \in \overline{T(H)}$.

Proof. Suppose that the sequence $\{y_j : j \in \mathbb{J}\}$ is a standard frame for $\overline{T(H)}$ with lower bound C'. Then

$$C'\langle x, x \rangle \leq \sum_{j \in \mathbb{J}} |\langle x, y_j \rangle|^2 = \sum_{j \in \mathbb{J}} |\langle T^*x, x_j \rangle|^2 \leq D\langle T^*x, T^*x \rangle \,, \tag{6}$$

for every $x \in \overline{T(H)}$. From which, condition (5) follows with $k = C'/D$.

The converse conclusion is also a simple calculation. (1), (5) and Proposition 1.2 of [6] imply that

$$k\ C\langle x,x\rangle \leq C\langle T^*x,T^*x\rangle \leq \sum_{j\in\mathbb{J}}\langle T^*x,x_j\rangle\langle x_j,T^*x\rangle = \sum_{j\in\mathbb{J}}\langle x,Tx_j\rangle\langle Tx_j,x\rangle$$
$$\leq D\langle T^*x,T^*x\rangle \leq D||T||^2\langle x,x\rangle\,, \quad (7)$$

for every $x \in \overline{T(H)}$. □

In previous theorem,if $T(H)$ is not dense in H we can find a nonzero element y in $H\setminus\overline{T(H)}$. It is obvious that y is not in the closed A-linear hull of $\{y_j : j \in \mathbb{J}\}$. So, for $\{y_j : j \in \mathbb{J}\}$ to be a standard frame of H it is necessary that $T(H)$ be dense in H and consequently the assumption $\overline{T(H)} = H$ yields the following corollary.

Corollary 2. *Let $\{x_j : j \in \mathbb{J}\}$ be a standard frame of H with bounds $0 < C \leq D$ and T be a map in $\mathcal{L}(H)$ such that $\overline{T(H)} = H$. Then the following conditions are equivalent:*

(i) The sequence $\{y_j : j \in \mathbb{J}\}$ is a standard frame of H.

(ii) There exists a positive constant k such that the adjoint of T, T^*, satisfies

$$k\langle x,x\rangle \leq \langle T^*x,T^*x\rangle\,, \quad (8)$$

for every $x \in H$.

Remark. (1) There exists a Hilbert A-module H and a module map T in $\mathcal{L}(H)$ such that T^* is injective, but $\overline{T(H)} \neq H$ (cf. [8], Exercise 15.F). For this reason, in Corollary 2 we supposed that $\overline{T(H)} = H$. But if $\overline{T(H)}$ is a complemented submodule of H then condition (8) implies that $\overline{T(H)} = H$.

(2) If T is a self adjoint module map in $\mathcal{L}(H)$ and satisfies condition (8), then T is invertible (cf. [7], Lemma 3.1). In particular T is surjective. Then $T(H) = H$.

(3) Suppose that $T \in \mathcal{L}(H)$ and $T(H)$ is closed. Then $T(H)$ is a complemented submodule of H and $T(H) \oplus \ker T^* = H$ (cf. [7], Theorem 3.2). If T^* satisfies condition (8), then $\ker T^* = \{0\}$ and $T(H) = H$.

Corollary 3. *Let $\{x_j : j \in \mathbb{J}\}$ be a standard frame and let H_1 be a closed submodule of H. If T is an adjointable module map from H onto H_1, then the following conditions are equivalent:*

(i) The sequence $\{y_j = Tx_j : j \in \mathbb{J}\}$ is a standard frame for H_1.

(ii) There exists a positive constant k such that

$$k\langle x,x\rangle \leq \langle T^*x,T^*x\rangle\,,$$

for every $x \in H_1$.

By corollary 3, we can construct some standard frames for a closed submodule of H, with a given standard frame.

Now, let $T \in \mathcal{L}(l_2(A))$. $\eta = \{\eta_j : j \in \mathbb{J}\}$ be a standard frame of H with bounds C_η and D_η and frame transform θ_η. We use T to construct the sequence ξ defined by $\xi = \{\xi_j : j \in \mathbb{J}\}$ and

$$\xi_j = \theta_\eta^*(T(e_j)) \quad (j \in \mathbb{J})\,, \tag{9}$$

such that

$$T(e_j) = \sum_{i \in \mathbb{J}} \alpha_{ij} e_i, \quad \{\alpha_{ij}\}_{i \in \mathbb{J}} \in l_2(A).$$

Then

$$\theta_\eta^*(T(e_j)) = \sum_{i \in \mathbb{J}} \alpha_{ij} \theta_\eta^*(e_j) = \sum_{i \in \mathbb{J}} \alpha_{ij} \eta_i.$$

But the sequence $\xi = \{\xi_j : j \in \mathbb{J}\}$ is not always a standard frame for H (e.g. $T = 0$). Now we want to make $\xi = \{\xi_j : j \in \mathbb{J}\}$ a standard frame under an appropriate condition on T.

Theorem 4. *Let $\eta = \{\eta_j : j \in \mathbb{J}\}$ be a standard frame of H with bounds D_η and C_η. If $T \in \mathcal{L}(l_2(A)))$ then the following conditions are equivalent:*

(i) The sequence $\xi = \{\xi_j : j \in \mathbb{J}\}$ is a standard frame of H defined by (9).

(ii) There exists a positive constant k such that

$$k\langle y, y\rangle \leq \langle T^* y, T^* y\rangle\,, \tag{10}$$

for every $y \in \theta_\eta(H)$, where θ_η is the frame transform of $\eta = \{\eta_j : j \in \mathbb{J}\}$.

Proof. Suppose that the sequence $\xi = \{\xi_j : j \in \mathbb{J}\}$ is a standard frame of H defined by (9). Then there are two constants $0 < C_\xi \leq D_\xi$ such that

$$C_\xi \langle x, x\rangle \leq \sum_{j \in \mathbb{J}} \langle x, \xi_j\rangle\langle \xi_j, x\rangle = \langle \theta_\xi(x), \theta_\xi(x)\rangle \leq D_\xi \langle x, x\rangle\,, \tag{11}$$

for every $x \in H$, where θ_ξ is the frame transform of $\xi = \{\xi_j : j \in \mathbb{J}\}$. Also

$$\langle \theta_\xi(x), e_n\rangle \;=\; \langle x, \theta_\xi^*(e_n)\rangle \;=\; \langle x, \xi_n\rangle \;=\; \langle x, \theta_\eta^*(T(e_n))\rangle \;=\; \langle T^*(\theta_\eta(x)), e_n\rangle\,,$$

for every $n \in \mathbb{J}$ and $x \in H$. Since the set of A-linear combinations of $\{e_n\}_{n \in \mathbb{J}}$ is dense in $l_2(A)$, we have

$$\theta_\xi = T^* \theta_\eta. \tag{12}$$

So, by using the left inequality of (11),and (12), we conclude that

$$C_\xi \langle x, x\rangle \leq \langle \theta_\xi(x), \theta_\xi(x)\rangle = \langle T^* \theta_\eta(x), T^* \theta_\eta(x)\rangle\,, \tag{13}$$

for every $x \in H$. Then

$$\frac{C_\xi}{D_\eta}\langle \theta_\eta(x), \theta_\eta(x)\rangle \leq C_\xi \langle x, x\rangle \leq \langle T^*\theta_\eta(x), T^*\theta_\eta(x)\rangle , \tag{14}$$

for every $x \in H$. From which, condition (10) follows with $k = \frac{C_\xi}{D_\eta}$.

Conversely, by Proposition 1.2 of [7], we have

$$\begin{aligned} C_\eta k\langle x, x\rangle &\leq k\langle \theta_\eta(x), \theta_\eta(x)\rangle \\ &\leq \langle T^*\theta_\eta(x), T^*\theta_\eta(x)\rangle \leq ||T||^2\langle \theta_\eta(x), \theta_\eta(x)\rangle \leq ||T||^2 D_\eta\langle x, x\rangle , \end{aligned} \tag{15}$$

for every $x \in H$. Now we define

$$V : H \to l_2(A) \quad \text{by} \quad V(x) = \{\langle x, \xi_j\rangle\}_{j\in\mathbb{J}}$$

and similar to (12), we can show that $V = T^*\theta_\eta$. Therefore, by (15) we have

$$C_\eta k\langle x, x\rangle \leq \langle V(x), V(x)\rangle \leq ||T||^2 D_\eta\langle x, x\rangle , \tag{16}$$

for every $x \in H$. □

4. Characterization of Standard Frames of *H*

The aim of this section is to characterize all standard frames of H. In Theorem 5, we will show how any two standard frames of H are related with each other.

Theorem 5. *Let sequences $\eta = \{\eta_j : j \in \mathbb{J}\}$ and $\xi = \{\xi_j : j \in \mathbb{J}\}$ be two standard frames of H. Then there exists a module map T in $\mathcal{L}(l_2(A))$ such that $\xi_j = \theta_\eta^*(T^*(e_j))$, for every $j \in \mathbb{J}$.*

Proof. Because $\eta = \{\eta_j : j \in \mathbb{J}\}$ is a standard frame of H, there exist constants $0 < C_\eta \leq D_\eta$ such that $C_\eta < x, x >\leq \sum < x, x_j >< x_j, x >\leq D_\eta < x, x >$,

$$\sqrt{C_\eta}||x|| \leq ||\theta_\eta(x)|| \leq \sqrt{D_\eta}||x|| , \tag{17}$$

for every $x \in H$, where θ_η is the frame transform of $\eta = \{\eta_j : j \in \mathbb{J}\}$. Then $\theta_\eta(H)$ is a closed submodule of $l_2(A)$. Let S_η be the frame operator of $\eta = \{\eta_j : j \in \mathbb{J}\}$. We set $V_{ij} = \langle S_\eta\xi_i, \eta_j\rangle$, for every $i, j \in \mathbb{J}$ and define

$$T : \theta_\eta(H) \subseteq l_2(A) \to l_2(A) \quad \textit{by} \quad T(y) = \{\sum_{j\in\mathbb{J}} y_j V_{ij}^*\}_{i\in\mathbb{J}},$$

where $y=\theta_\eta(x)$ and $y_j=\langle x,\eta_j\rangle$, for every $j\in\mathbb{J}$. For every $i\in\mathbb{J}$, we have

$$||\sum_{j=m}^{n} V_{ij}V_{ij}^*|| \;=\; ||\sum_{j=m}^{n}\langle S_\eta\xi_i,\eta_j\rangle\langle\eta_j,S_\eta\xi_i\rangle||.$$

Then $\{V_{ij}\}_{j\in\mathbb{J}}\in l_2(A)$. Consequently,

$$||\sum_{j=m}^{n} y_jV_{ij}^*|| = ||\langle\{y_j\}_{j=m}^{n},\{V_{ij}\}_{j=m}^{n}\rangle|| \leq ||\sum_{j=m}^{n} y_jy_j^*||^{\frac{1}{2}}||\sum_{j=m}^{n} V_{ij}V_{ij}^*||^{\frac{1}{2}}.$$

Then the sum $\sum_{j\in\mathbb{J}} y_jV_{ij}^*$ converges in norm. Also

$$\sum_{j\in\mathbb{J}} y_jV_{ij}^* = \sum_{j\in\mathbb{J}}\langle x,\eta_j\rangle\langle\eta_j,S_\eta\xi_i\rangle = \langle x,\sum_{j\in\mathbb{J}}\langle\xi_i,S_\eta(\eta_j)\rangle\eta_j\rangle = \langle x,\xi_i\rangle. \tag{18}$$

Thus T is well-defined. By (18) we have

$$\begin{aligned}||T(y)||^2 = ||\{\langle x,\xi_i\rangle\}_{i\in\mathbb{J}}||^2 = ||\sum_{i\in\mathbb{J}}\langle x,\xi_i\rangle\langle\xi_i,x\rangle||&\\ \leq D_\xi||x||^2 = D_\xi||\theta_\eta^{-1}(y)||^2 \leq D_\xi||\theta_\eta^{-1}||^2||y||^2,&\end{aligned}$$

for every $y\in\theta_\eta(H)$. So T is bounded. On the other hand, for every $n\in\mathbb{J}$

$$\langle T(y),e_n\rangle = \langle\{\sum_{j\in\mathbb{J}} y_jV_{ij}^*\}_{i\in\mathbb{J}},e_n\rangle = \sum_{j\in\mathbb{J}} y_jV_{nj}^* = \langle\{y_j\}_{j\in\mathbb{J}},\{V_{nj}\}_{j\in\mathbb{J}}\rangle,$$

consequently $T^*(e_n)=\{V_{nj}\}_{j\in\mathbb{J}}$, for every $n\in\mathbb{J}$. Therefore T is an adjointable module map. Since $\theta_\eta(H)$ is a complement-able submodule of $l_2(A)$, the module map T can be extended to all of $l_2(A)$ by defining $Ty=0$ for all $y\in\theta_\eta(H)^\perp$ and we have

$$\theta_\eta^*(T^*(e_n)) \;=\; \theta_\eta^*(\sum_{j\in\mathbb{J}} V_{nj}e_j) \;=\; \sum_{j\in\mathbb{J}} V_{nj}\eta_j \;=\; \sum_{j\in\mathbb{J}}\langle S_\eta\xi_n,\eta_j\rangle\eta_j \;=\; \xi_n\;. \quad \square$$

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